# Equilibrium threshold joining strategies in partially observable batch service queueing systems

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**Abstract:** We study the strategic customer behavior in queueing systems with batch services under incomplete information. In particular, we assume that arriving customers have the opportunity to observe only the number of waiting batches upon arrival and, afterwards, they make their join/balk decisions. We prove that equilibrium strategies always exist within the legitimate class of threshold strategies, but they may not be unique. We also provide an algorithmic scheme for their computation. Moreover, we compare the strategic behavior under this information level with the corresponding behavior in the complete information case.

**Keywords:** Queueing Theory, Strategic customers, Batch services, Balking, Equilibrium strategies, Partial information, Follow-The-Crowd (FTC) behavior

## 1 Introduction

Our study falls into a stream of research that deals with the strategic customer behavior in queueing systems. This stream goes back to the pioneering work of Naor [22] who studied the join/balk dilemma in the framework of the observable $M/M/1$ queue, i.e., when the customers have the opportunity to observe the queue length and then they decide whether to join or not, based on a natural reward-cost structure. This work was later complemented by Edelson and Hildebrand [14] who studied the same problem for the unobservable $M/M/1$ queue. Since then, there is a growing literature toward this direction (see Hassin and Haviv [18], Hassin [17], and Stidham [24] for comprehensive reviews). However, the vast majority of such studies refers to single-service systems, although batch-service systems occur quite frequently in practice. A prominent example is observed in the tourism sector, since boat cruises and museum guided tours are performed in groups and a group starts a tour only when a certain number of customers has been accumulated. Batch-service systems are also encountered in numerous applications in transportation and supply chain areas, e.g., a truck departs only when the cargo exhausts its capacity (shipment consolidation policies). Relevant examples occur also in hospitality, since customers are often served in batches in receptions.

Notable studies that include some notion of "batch service" are the following: Hassin and Haviv [18] (Section 1.5) considered an airport facility where customers strategically choose between a shuttle and a bus. The shuttle departs whenever it is full, hence it can be considered as a batch service model with zero service times. On the other hand, the bus visits the station according to a renewal process and removes all present customers, hence it can be viewed as a clearing system, i.e., a batch service model with infinite batch size. Clearing systems have been also studied by Economou and Manou [13] and Manou, Economou and Karaesmen [21] under the assumption that a service facility removes all present customers periodically. Moreover, a sequence of interesting papers studies batch service systems with infinite servers, e.g., Calvert [9], Afimeimounga, Solomon and Ziedins [3], [4], Chen, Holmes and Ziedins [10], Afimeimounga [2], and Pai and Cheng [23]. Last but not least, a related thread of research concerns the strategic behavior of customers in queueing systems with catastrophes; see, e.g., Boudali and Economou [5] and [6]. In these models, all present customers at the catastrophe instants are forced to abandon a service system and receive some compensation. Therefore, one can consider the catastrophes as some kind of (unwanted) service completions in which the customers depart in batches.

It is worth noting that the common characteristic of the aforementioned models is that the batches do not queue. Recently, Bountali and Economou [7] studied strategic customer behavior regarding the join/balk dilemma for the $M/M/1$ queue with single arrivals and batch services of fixed size $K$ (also known as the $M/M^K/1$ queue). This is a system with regular batches, in which batches do form a queue. The paper studies two versions of the model with respect to the information that is provided to the customers: The observable version, in which the arriving customers observe the number of customers in the system before making their decisions, and the unobservable version, in which they do not observe anything.

In the present paper, we aim to complement this work by considering the case where the arriving customers observe only the number of complete batches that wait to be served. This is a partial information model that lies between the two extremes considered in Bountali and Economou [7]. The study of partial information models seems interesting both from a theoretical and an applied viewpoint as it provides insights on how much information should be provided to the customers, question that constitutes a recurring theme in the literature of strategic customer behavior in queueing systems. In the context of single service systems, such a topic is addressed in Guo and Zipkin [15], Economou and Kanta [11], and Guo and Zipkin [16]. The overall conclusion of these studies is that more information may help or hurt the customers and/or the administrator of the system, depending on the parameters of the underlying model and the specific reward-cost structure. However, there are no such studies for models with batch services. Hence, our objective is also to shed light on this topic.

Our main contributions are as follows:

1. We compute the basic performance measures of the system, when the customers adopt a threshold joining strategy.

2. We prove several monotonicity properties of the conditional mean sojourn time of a customer given the number of batches observed upon arrival, when the customers follow a threshold joining strategy.

3. We characterize the best response against a threshold joining strategy. We prove that the model is of the Follow-The-Crowd (FTC) type. That is, if the customers modify their joining strategy by adopting a higher threshold, then the best response of a tagged customer will keep the same threshold or will move to a higher one.

4. We characterize the equilibrium threshold joining strategies and we provide an algorithm for the computation of the corresponding thresholds. On this note, we prove that the 'always balk' is an equilibrium strategy, for any parameters of the model. However, multiple equilibrium threshold joining strategies may exist. The non-zero equilibrium thresholds form an interval of consecutive integers.

5. We perform a number of numerical experiments that show the influence of the parameters of the model on the equilibrium strategies and on the social welfare. Our numerical results demonstrate that the lowest and the highest equilibrium thresholds are increasing functions of the arrival rate and the service reward.

6. We study the influence of the information level on the equilibrium social welfare. We show that the maximum equilibrium social welfare over all equilibrium threshold strategies for the partially observable case is always at least as large as the equilibrium social welfare for the observable case. Moreover, the numerical results show that the equilibrium social welfare for the observable case coincides with the equilibrium social welfare under either the highest or the lowest threshold of the partially observable case. Therefore, for some range of the parameters, the equilibrium social welfare when the system is partially observable and the customers adopt the highest equilibrium threshold is strictly larger than the equilibrium social welfare of the observable system.

The remainder of the paper is organized as follows. In Section 2 we describe the model and the associated reward-cost structure. Moreover, we discuss why it is legitimate to limit our search for equilibrium strategies in the class of threshold strategies. In Section 3 we compute the necessary performance measures of the system when the customers follow an arbitrary threshold strategy. In Section 4 we show that an equilibrium threshold strategy always exists and derive an algorithm for the computation of all equilibrium threshold strategies for a given instant of the problem. We also comment on the properties of the equilibrium strategies and compare social welfare under the equilibrium strategies in the partially observable case and under the equilibrium strategy of the (fully) observable case. Finally, in Section 5, we perform a number of numerical experiments and we discuss the corresponding findings.

## 2 The model

We consider a queueing system with Poisson arrival process of customers with rate $\lambda$ and a single server who serves customers in batches of fixed size $K$ ($K > 1$). Successive service times are independent exponentially distributed random variables with rate $\mu$. Starting from an empty system, the server waits until the accumulation of $K$ customers (that will form a complete batch) to start providing service, thus he remains inactive as long as there are less than $K$ customers in the system. Inter-arrival and service times are assumed to be mutually independent.

The customers are strategic and decide whether to join or not, taking into account their expected utility. They valuate service $R$ units, whereas they accumulate cost at rate $C$ per time unit as long as they stay in the system (either in queue or service).

Bountali and Economou [7] characterized the equilibrium customer strategies of this model in two informational cases, the observable one where the arriving customers are informed about the number of customers in the system, $Q(t)$, and the unobservable case where they do not receive any information about $Q(t)$. In the present paper, we consider the partially observable case, where arriving customers are informed about the number $M(t)$ of complete batches that are waiting to be processed. This is a legitimate information level, since arriving customers may not have access to the number of customers that wait to form another complete batch. In terms of $Q(t)$, the customers receive only a partial knowledge of it, since they are informed about $M(t) = \lfloor Q(t)/K \rfloor$ (the integer part of $Q(t)/K$), but do not know $J(t) = Q(t) - KM(t)$ which corresponds to the number of customers in the incomplete batch.

A general mixed customer strategy in this partially observable case is given by an infinite vector $\mathbf{q} = (q_m : m \geq 0)$, where $q_m$ denotes the joining probability for an arriving customer who finds $m$ complete batches in the system. Assume that customers follow such a strategy. Then, to determine the best response of a tagged customer against $\mathbf{q}$, we should compute the conditional mean sojourn time of a customer that decides to enter, given that she observes $m$ complete batches, when the other customers follow the strategy $\mathbf{q} = (q_m : m \geq 0)$. Indeed, if we denote this conditional mean sojourn time by $S^{(p)}(m; \mathbf{q})$, then the tagged customer's best response is to enter if $R - CS^{(p)}(m; \mathbf{q}) > 0$, to balk if $R - CS^{(p)}(m; \mathbf{q}) < 0$, whereas she is indifferent between joining and balking when $R - CS^{(p)}(m; \mathbf{q}) = 0$.

We note here that the derivation of a simple closed formula for the computation of $S^{(p)}(m; \mathbf{q})$, for a general mixed strategy $\mathbf{q}$, does not seem possible. Of course the development of a numerical procedure for the computation of $S^{(p)}(m; \mathbf{q})$ and subsequent numerical assessment of mixed strategies is possible, but it is quite involved. Moreover, in general, $S^{(p)}(m; \mathbf{q})$ is not an increasing function of $m$, for a fixed $\mathbf{q}$. For example, consider a strategy $\mathbf{q} = (q_m : m \geq 0)$ adopted by the population of potential customers, such that $q_n$ is very small, while $q_{n+1}$ is large (i.e., close to 1), for some particular index $n$. Consider now two tagged customers, $C_n$ and $C_{n+1}$ that see upon arrival $n$ and $n + 1$ complete batches in the system. The customer $C_n$ has the advantage over $C_{n+1}$ that she has to wait on average $\frac{1}{\mu}$ time units less, if one focuses on the complete batches that are present upon arrival. However, the information that $C_n$ finds $n$ complete batches implies (for appropriately chosen parameters) that $C_n$ faces on average less customers in the incomplete batch than $C_{n+1}$ (because $q_n$ is very small, while $q_{n+1}$ is large). Then, for sufficiently low values of $\lambda$, the customer $C_n$ will have to wait much longer than $C_{n+1}$ for the completion of her own batch and this disadvantage will outperform her aforementioned advantage. Therefore, it is impossible to establish a monotonicity property for $S^{(p)}(m; \mathbf{q})$, with respect to $m$, under any given strategy $\mathbf{q}$, for all ranges of the parameters. Then, for certain values of the parameters, equilibrium strategies that randomize between joining and balking may exist in several states (proper mixed strategies). One realizes that the computational burden in the general framework is prohibitive and the problem of the characterization and computation of all equilibrium strategies seems too difficult, if not impossible to solve.

Because of these difficulties, we limit our search for equilibrium strategies in the class of pure strategies. Consider, now, a pure strategy $\mathbf{q} = (q_m : m \geq 0)$ (i.e., $q_m = 0$ or 1, for $m \geq 0$) that is adopted by all customers. If $m^*$ is the smallest index for which $q_m = 0$, then no arriving customer will ever observe more than $m^*$ complete batches, thus the best response against the strategy $\mathbf{q}$ is independent of the values of $q_m$ for $m > m^*$. Therefore, we can assume that $q_m = 0$ for $m > m^*$ and proceed to the computation of the best response. This strategy (with $q_m = 1$ for $m < m^*$ and $q_m = 0$ for $m \geq m^*$) will be referred to as the $m^*$-threshold strategy. Thus, we will compute best responses against threshold strategies and will limit our search for equilibrium strategies within the class of threshold strategies. This is also the rule in the literature on partially observable Markovian models (see e.g., [8] and [12]) and other interesting observable Markovian models (see e.g., [1] and [19]).

# 3 Performance measures under a threshold strategy

To simplify the notation, let $S^{(p)}(m;m^*)$ be the conditional mean sojourn time of a tagged customer that decides to enter, given that he observes $m$ complete batches, when the other customers follow the $m^*$-threshold strategy. We can see that the maximum value of $m$ is $m^*$, since the other customers balk if they find more than $m^*-1$ complete batches. Consider , now, a tagged customer that arrives at time $t$ and finds $M(t)=m$ complete batches and let $J(t)$ denote the (unobservable) number of customers in the incomplete batch found by her. Then, conditioning on $J(t)$ yields

$$S^{(p)}(m;m^*) \quad = \quad \sum_{j=0}^{K-1} S(m,j)\pi_{J|M}(j|m;m^*),\ 0\leq m\leq m^*-1, \tag{3.1}$$

where $S(m,j)$ is the mean sojourn time of a joining customer who finds the system at state $(M(t),J(t))=(m,j)$ and $\pi_{J|M}(j|m;m^*)$ is the conditional probability of observing $j$ customers in the incomplete batch at an arrival instant, given that there are $m$ complete batches, when the customers follow the $m^*$-threshold strategy. Moreover, for $m=m^*$, we have that

$$S^{(p)}(m^*;m^*) \quad = \quad \frac{1}{\mu}+S(m^*-1,0), \tag{3.2}$$

simply because if an arriving tagged customer who finds $m^*$ complete batches decides to join, then she knows that she is the first of her batch, i.e., $J(t)=0$ at her arrival instant. Indeed, any previous customer that observed $m^*$ complete batches did not enter, because of the $m^*$-threshold strategy. Moreover, the tagged customer knows that all subsequent customers that arrive at the system till the next service completion will balk, as they will observe $m^*$ complete batches. Therefore, with the join decision of the tagged customer, the system will move to state $(m^*,1)$ and will remain there till the next service completion (with mean duration $\frac{1}{\mu}$). Then, the state of the system will become $(M(t),J(t))=(m^*-1,1)$ and the tagged customer will face the same situation as if she arrived and found the system at state $(m^*-1,0)$.

Using a first-step argument (conditioning on the first event being an arrival or a service completion - see Bountali and Economou [7]) shows that the conditional mean sojourn times $S(m,j)$ can be computed recursively by the scheme

$$S(0,j) \quad = \quad \frac{K-j-1}{\lambda}+\frac{1}{\mu},\ j=0,1,\ldots,K-1, \tag{3.3}$$

$$S(m,K-1) \quad = \quad \frac{m+1}{\mu},\ m=0,1,\ldots, \tag{3.4}$$

$$S(m,j) \quad = \quad \frac{1}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}S(m,j+1)+\frac{\mu}{\lambda+\mu}S(m-1,j), \quad m=1,2,\ldots \text{ and } j=K-2,K-3,\ldots,0. \tag{3.5}$$

Moreover, we can see that

$$S(m,j)=E[\max(Y_m,Z_{K-j-1})]+\frac{1}{\mu},\ m\geq 0, 0\leq j\leq K-1, \tag{3.6}$$

where $Y_m$ and $Z_{K-j-1}$ are Erlang$(m,\mu)$ and Erlang$(K-j-1,\lambda)$ independent random variables, corresponding to the total service time of the present complete batches and the completion time of the current incomplete batch, respectively (with the convention that $Y_0 = Z_0 = 0$). Then, one can compute $S^{(p)}(m^*;m^*)$ using (3.2). To compute $S^{(p)}(m;m^*)$, for $0 \leq m \leq m^*-1$, using (3.1), we need to determine for each possible $(m,j)$ the conditional probability $\pi_{J|M}(j|m;m^*)$ of observing $j$ customers in the incomplete batch at an arrival instant, given that there are $m$ complete batches, when the customers follow the $m^*$-threshold strategy. The Poisson-Arrivals-See-Time-Averages (PASTA) property implies that $\pi_{J|M}(j|m;m^*)$ coincides with the conditional probability of observing $j$ customers in the incomplete batch at an arbitrary instant, given that there are $m$ complete batches, when the customers follow the $m^*$-threshold strategy. Therefore, we have that

$$\pi_{J|M}(j|m;m^*) = \frac{\pi(m,j;m^*)}{\tilde{\pi}(m;m^*)}, \tag{3.7}$$

where $(\pi(m,j;m^*) : (m,j) \in \mathcal{S}^{(p)}_{M,J}(m^*))$ is the stationary distribution of $\{(M(t),J(t))\}$, when the customers follow the $m^*$-threshold strategy and $(\tilde{\pi}(m;m^*) : 0 \leq m \leq m^*)$ is the corresponding marginal distribution of $\{M(t)\}$. Note that the state-space of $\{(M(t),J(t))\}$ under the $m^*$-threshold strategy is $\mathcal{S}^{(p)}_{M,J}(m^*) = \{(m,j) : 0 \leq m \leq m^*-1, 0 \leq j \leq K-1\} \cup \{(m^*,0)\}$. In order to proceed with the computation of $S^{(p)}(m;m^*)$ we need to compute the conditional probabilities $\pi_{J|M}(j|m;m^*)$. We, therefore, provide the following Proposition 3.1.

**Proposition 3.1** *The conditional stationary probabilities $\pi_{J|M}(j|m;m^*)$ are given by*

$$\pi_{J|M}(j|0;m^*) = g_{m^*}(j),\ 0 \leq j \leq K-1, \tag{3.8}$$
$$\pi_{J|M}(j|m;m^*) = f_{m^*-m}(j),\ 1 \leq m \leq m^*, 0 \leq j \leq K-1, \tag{3.9}$$

*where $(g_n(j) : 0 \leq j \leq K-1)$, $n \geq 0$, and $(f_n(j) : 0 \leq j \leq K-1)$, $n \geq 0$, are discrete probability mass functions that are computed recursively. In particular $(f_0(j) : 0 \leq j \leq K-1)$ and $(g_0(j) : 0 \leq j \leq K-1)$ are given by*

$$f_0(j) = g_0(j) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } 1 \leq j \leq K-1, \end{cases} \tag{3.10}$$

*$(f_1(j) : 0 \leq j \leq K-1)$ and $(g_1(j) : 0 \leq j \leq K-1)$ are given by*

$$f_1(j) = \frac{\left(1-\frac{\lambda}{\lambda+\mu}\right)\left(\frac{\lambda}{\lambda+\mu}\right)^j}{1-\left(\frac{\lambda}{\lambda+\mu}\right)^K},\ 0 \leq j \leq K-1, \tag{3.11}$$

*and*

$$g_1(j) = \frac{1}{K},\ 0 \leq j \leq K-1. \tag{3.12}$$

*Given $(f_n(j) : 0 \leq j \leq K-1)$, for a particular $n \geq 0$, the discrete probability mass function $(g_{n+1}(j) : 0 \leq j \leq K-1)$ is computed as the unique stationary distribution of the continuous time Markov chain shown in Figure 1, with transition rates*

$$q_{ij}^{g_{n+1}} = \begin{cases} \lambda & \text{if } 0 \leq i \leq K-2,\ j = i+1, \\ \lambda f_n(j) & \text{if } i = K-1,\ 0 \leq j \leq K-2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

*Likewise, the discrete probability mass function* $(f_{n+1}(j) : 0 \le j \le K-1)$ *is given as the stationary distribution of the continuous time Markov chain shown in Figure 2, with transition rates*

$$q_{ij}^{f_{n+1}} = \begin{cases} \lambda & \textit{if } 0 \le i \le K-2,\ j = i+1, \\ \mu & \textit{if } 1 \le i \le K-2,\ j = 0, \\ \lambda f_n(j) & \textit{if } i = K-1,\ 1 \le j \le K-2, \\ \mu + \lambda f_n(0) & \textit{if } i = K-1,\ j = 0, \\ 0 & \textit{otherwise.} \end{cases} \tag{3.14}$$

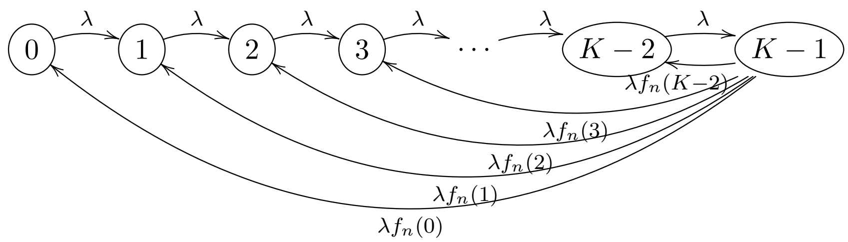


Figure 1: Transition rate diagram of a CTMC with stationary distribution $(g_{n+1}(j))$.

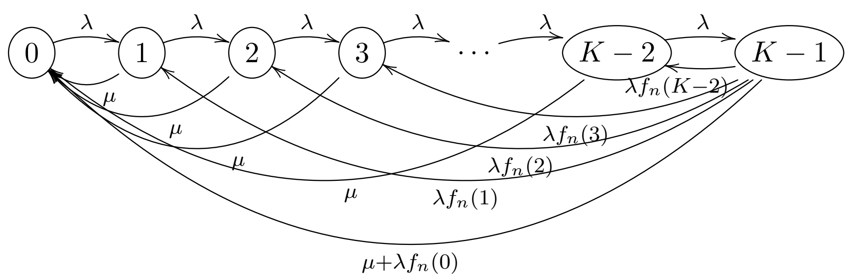


Figure 2: Transition rate diagram of a CTMC with stationary distribution $(f_{n+1}(j))$.

**Proof.** We fix an $m^*$ and observe that under the $m^*$-threshold strategy the process $\{M(t), J(t)\}$ is a CTMC with transition rate diagram given in Figure 3.

We will now proceed with the determination of the stationary probabilities $\pi(m, j; m^*)$, $(m, j) \in \mathcal{S}_{M,J}^{(p)}(m^*)$. We momentarily suppress the dependence on $m^*$ and denote $\pi(m, j; m^*)$ by $\pi(m, j)$. Then, we have the following balance equations:

$$\lambda\pi(0,0) = \mu\pi(1,0), \tag{3.15}$$

$$\lambda\pi(0,j) = \lambda\pi(0,j-1) + \mu\pi(1,j),\ 1 \le j \le K-1, \tag{3.16}$$

$$(\lambda+\mu)\pi(m,0) = \lambda\pi(m-1,K-1) + \mu\pi(m+1,0),\ 1 \le m \le m^*-1, \tag{3.17}$$

$$(\lambda+\mu)\pi(m,j) = \lambda\pi(m,j-1) + \mu\pi(m+1,j),\quad 1 \le m \le m^*-2,\ 1 \le j \le K-1, \tag{3.18}$$

$$(\lambda+\mu)\pi(m^*-1,j) = \lambda\pi(m^*-1,j-1),\ 1 \le j \le K-1, \tag{3.19}$$

$$\mu\pi(m^*,0) = \lambda\pi(m^*-1,K-1). \tag{3.20}$$

For any fixed $m = 0, 1, \ldots, m^*-1$, by equating rates between the sets $\{(m', j) : m' \le m, 0 \le j \le K-1\}$ and $\{(m', j) : m' \ge m+1, 0 \le j \le K-1\}$ we obtain that

$$\lambda\pi(m,K-1) = \mu\tilde{\pi}(m+1),\ 0 \le m \le m^*-1, \tag{3.21}$$

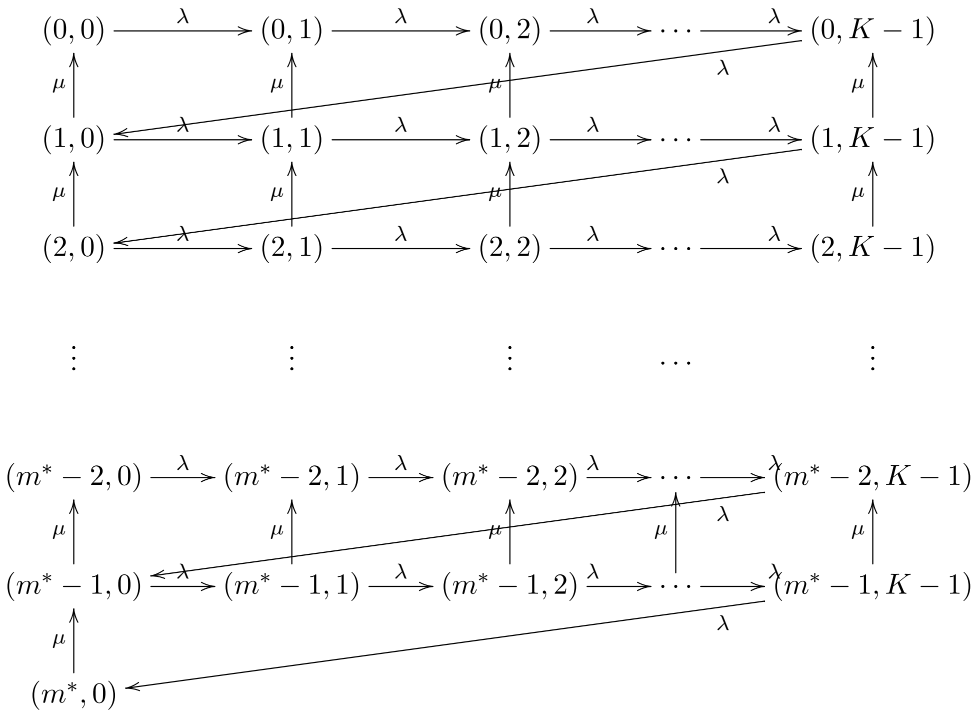

Figure 3: Transition rate diagram of $\{(M(t), J(t))\}$ under the $m^*$-threshold strategy.

where $\tilde{\pi}(m)$ is the marginal stationary probability of $m$ complete batches, i.e.,

$$\begin{aligned} \tilde{\pi}(m) &= \sum_{j=0}^{K-1} \pi(m,j),\ 0 \le m \le m^* - 1, \\ \tilde{\pi}(m^*) &= \pi(m^*, 0). \end{aligned}$$

Dividing the balance equation for the state $(m, j)$ by $\tilde{\pi}(m)$ and denoting the conditional probability $\pi(m, j)/\tilde{\pi}(m)$ by $\pi(j|m)$ we obtain

$$\lambda \pi(0|0) = \mu \frac{\pi(1,0)}{\tilde{\pi}(0)}, \tag{3.22}$$

$$\lambda \pi(j|0) = \lambda \pi(j-1|0) + \mu \frac{\pi(1,j)}{\tilde{\pi}(0)},\ 1 \le j \le K-1, \tag{3.23}$$

$$(\lambda + \mu)\pi(0|m) = \mu + \mu \frac{\pi(m+1,0)}{\tilde{\pi}(m)},\ 1 \le m \le m^* - 1, \tag{3.24}$$

$$\begin{aligned} (\lambda + \mu)\pi(j|m) &= \lambda \pi(j-1|m) + \mu \frac{\pi(m+1,j)}{\tilde{\pi}(m)}, \\ &\quad 1 \le m \le m^* - 2,\ 1 \le j \le K-1, \end{aligned} \tag{3.25}$$

$$(\lambda + \mu)\pi(j|m^* - 1) = \lambda \pi(j-1|m^* - 1),\ 1 \le j \le K-1, \tag{3.26}$$

$$\mu \pi(0|m^*) = \mu, \tag{3.27}$$

where (3.21) has been taken into account to simplify (3.24) and (3.27). Now, (3.27) shows that $\pi(0|m^*) = 1$, i.e.,

$$\pi_{J|M}(j|m^*; m^*) = \pi(j|m^*) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } 1 \le j \le K-1, \end{cases} \tag{3.28}$$

so we obtain (3.9) for $m = m^*$ with $f_0(j)$ given by (3.10). Iteration of (3.26) shows that $\pi(j|m^*-1) = \left(\frac{\lambda}{\lambda+\mu}\right)^j \pi(0|m^*-1)$, $0 \le j \le K-1$. Using the normalization equation $\sum_{j=0}^{K-1} \pi(j|m^*-1) = 1$ yields

$$\pi_{J|M}(j|m^*-1;m^*) = \pi(j|m^*-1) = \frac{\left(1-\frac{\lambda}{\lambda+\mu}\right)\left(\frac{\lambda}{\lambda+\mu}\right)^j}{1-\left(\frac{\lambda}{\lambda+\mu}\right)^K},\ 0 \le j \le K-1, \tag{3.29}$$

so we obtain (3.9) for $m = m^* - 1$ with $f_1(j)$ given by (3.11).

Hence, we have proved the validity of (3.9) for $m = m^*$, $m = m^* - 1$. We will now use 'reverse' induction to prove that (3.9) is valid for $1 \le m \le m^*$, i.e., we will prove that if (3.9) is valid for a certain $m+1$, then it is also valid for $m$. To this end we observe that (3.21) implies that

$$\begin{aligned}\frac{\mu\pi(m+1,j)}{\tilde{\pi}(m)} &= \frac{\mu\tilde{\pi}(m+1)}{\tilde{\pi}(m)} \cdot \frac{\pi(m+1,j)}{\tilde{\pi}(m+1)} = \frac{\lambda\pi(m,K-1)}{\tilde{\pi}(m)} \cdot \frac{\pi(m+1,j)}{\tilde{\pi}(m+1)} \\ &= \lambda\pi(K-1|m)\pi(j|m+1),\ 0 \le m \le m^*-2, 0 \le j \le K-1, \end{aligned} \tag{3.30}$$

and therefore equations (3.22)-(3.25) assume the form

$$\lambda\pi(0|0) = \lambda\pi(0|1)\pi(K-1|0), \tag{3.31}$$

$$\lambda\pi(j|0) = \lambda\pi(j-1|0) + \lambda\pi(j|1)\pi(K-1|0),\ 1 \le j \le K-1, \tag{3.32}$$

$$(\lambda+\mu)\pi(0|m) = \mu + \lambda\pi(0|m+1)\pi(K-1|m),\ 1 \le m \le m^*-1, \tag{3.33}$$

$$\begin{aligned}(\lambda+\mu)\pi(j|m) &= \lambda\pi(j-1|m) + \lambda\pi(j|m+1)\pi(K-1|m), \\ &\quad 1 \le m \le m^*-2,\ 1 \le j \le K-1. \end{aligned} \tag{3.34}$$

Now, the inductive hypothesis (validity of (3.9) for $m+1$) shows that (3.33)-(3.34) can be written as

$$(\lambda+\mu)\pi(0|m) = \mu + \lambda f_{m^*-m-1}(0)\pi(K-1|m), \tag{3.35}$$

$$(\lambda+\mu)\pi(j|m) = \lambda\pi(j-1|m) + \lambda f_{m^*-m-1}(j)\pi(K-1|m), 1 \le j \le K-1. \tag{3.36}$$

These equations can be seen as the balance equations of a CTMC with transition rate diagram given in Figure 2 with $n = m^* - m - 1$ and $(\pi(j|m) : 0 \le j \le K-1)$ is its unique stationary distribution. Therefore, the stationary distribution of this CTMC is given by $(f_{m^*-m}(j) : 0 \le j \le K-1)$ (because of the definition of $(f_{n+1}(j))$). Hence, we have that (3.9) is valid for $m$, so the proof of the inductive step is complete and we have shown (3.9), for all $1 \le m \le m^*$.

Equation (3.8) is proved similarly. Indeed, having proved the validity of (3.9), we can write (3.31)-(3.32) as

$$\lambda\pi(0|0) = \lambda f_{m^*-1}(0)\pi(K-1|0), \tag{3.37}$$

$$\lambda\pi(j|0) = \lambda\pi(j-1|0) + \lambda f_{m^*-1}(j)\pi(K-1|0),\ 1 \le j \le K-1. \tag{3.38}$$

These equations can be seen as the balance equations of a CTMC with transition rate diagram given in Figure 1 with $n = m^* - 1$ and $(\pi(j|0) : 0 \le j \le K-1)$ is its unique stationary distribution. But, the stationary distribution of this CTMC is given by $(g_{m^*}(j) : 0 \le j \le K-1)$ (because of the definition of $(g_{n+1}(j))$). ■

It should be noted here that for given $m^*$ and $m$ the distribution $(\pi_{J|M}(j|m;m^*) : 0 \leq j \leq K-1)$ corresponds to the steady-state distribution of $\{(M(t), J(t))\}$, when one observes $\{(M(t), J(t))\}$ only when $M(t) = m$. Denote by $\{(M^T(t), J^T(t))\}$ the 'restricted' or 'taboo' process that records the state of $\{(M(t), J(t))\}$ only during the time intervals where $M(t) = m$ (i.e., we have $M^T(t) = m$ for all $t$). Then, a transition of $\{(M^T(t), J^T(t))\}$ from $(m, i)$ to $(m, j)$ can occur either because of a direct transition of $\{(M(t), J(t))\}$ from $(m, i)$ to $(m, j)$, or because of a transition of $\{(M(t), J(t))\}$ from $(m, i)$ to some state $(m', k)$ with $m' \neq m$, followed by a sojourn time outside the set $\{(m, 0), (m, 1), \ldots, (m, K-1)\}$ and an entrance to this set in state $(m, j)$.

To interpret, now, the transition diagram in Figure 2, consider an $m > 0$ and focus on Figure 3. Then, for $i = 0, 1, \ldots, K-2$ a transition of $\{(M^T(t), J^T(t))\}$ from $(m, i)$ to $(m, i+1)$ occurs only when a transition of $\{(M(t), J(t))\}$ from $(m, i)$ to $(m, i+1)$ occurs. Thus, the rate from $(m, i)$ to $(m, i+1)$ is $\lambda$ as in the original process. For $i = 1, 2, \ldots, K-2$, a transition of $\{(M^T(t), J^T(t))\}$ from $(m, i)$ to $(m, 0)$ occurs only when a transition of $\{(M(t), J(t))\}$ from $(m, i)$ to $(m-1, i)$ occurs. Indeed, due to the form of the transition diagram in Figure 3, if the process $\{(M(t), J(t))\}$ leaves the set $\{(m, 0), (m, 1), \ldots, (m, K-1)\}$ towards a state $(m-1, i)$, the first re-entrance to the same set will be at state $(m, 0)$. Thus, the rate from $(m, i)$ to $(m, 0)$ of $\{(M^T(t), J^T(t))\}$ is $\mu$, i.e., it is the rate from $(m, i)$ to $(m-1, i)$ in the original process. For $j = 1, 2, \ldots, K-2$, a transition of $\{(M^T(t), J^T(t))\}$ from $(m, K-1)$ to $(m, j)$ occurs only when a transition of $\{(M(t), J(t))\}$ from $(m, K-1)$ to $(m+1, 0)$ occurs. Then, the process $\{(M(t), J(t))\}$ will pass a sojourn time in states $(m', k)$ with $m' > m$. The first re-entrance of $\{(M(t), J(t))\}$ to $\{(m, 0), (m, 1), \ldots, (m, K-1)\}$ will occur due to some service completion, i.e., due to a transition from some state $(m+1, j)$ to $(m, j)$. Since all such transitions have the same rate $\mu$, the probability that the first re-entrance to $\{(m, 0), (m, 1), \ldots, (m, K-1)\}$ occurs at state $(m, j)$ is proportional to the steady-state probability of $(m+1, j)$. Thus, the rate from $(m, K-1)$ to $(m, j)$ is the transition rate from $(m, K-1)$ to $(m+1, 0)$ times the normalized steady-state probability of $(m+1, j)$. This justifies the rates $\lambda f_n(j)$ in the transition rate diagram in Figure 2. Finally, a transition of $\{(M^T(t), J^T(t))\}$ from $(m, K-1)$ to $(m, 0)$ occurs either because of a transition of $\{(M(t), J(t))\}$ from $(m, K-1)$ to $(m-1, K-1)$ or because of a transition from $(m, K-1)$ to $(m+1, 0)$ . Using the same reasoning, we conclude with the rate $\mu + \lambda f_n(0)$. This intuitive reasoning gives an interpretation of the transition diagram in Figure 2 as a diagram of a 'taboo' process. The above argument can be made completely rigorous, following the existing theory (see e.g., Chapter 5 in Latouche and Ramaswami [20], in particular Theorem 5.5.3). The same argument can be applied for the interpretation of the transition diagram in Figure 1. The only difference is that there are no service completions in this case (i.e., when $m = 0$) and, hence, the rates $\mu$ disappear.

The expressions (3.1)-(3.2) in combination with the recursive scheme (3.3)-(3.5) and Proposition 3.1 enable the computation of $S^{(p)}(m; m^*)$, for any desired $m$ and $m^*$.

The social welfare per time unit under the $m^*$-threshold strategy is given as

$$\mathcal{B}^{(p)}(m^*) = \lambda R(1 - \pi(m^*, 0; m^*)) - C \sum_{(m,j) \in \mathcal{S}^{(p)}_{M,J}(m^*)} (mK + j)\pi(m, j; m^*), \qquad (3.39)$$

where $(\pi(m, j; m^*) : (m, j) \in \mathcal{S}^{(p)}_{M,J}(m^*))$ is the stationary distribution of $\{(M(t), J(t))\}$ given that the customers enter according to the $m^*$-threshold strategy. Indeed, the first summand in (3.39) corresponds to the effective arrival rate at the system times the reward $R$, while the second corresponds to the mean number of customers in the system times the waiting cost per customer and time unit $C$. For the stationary probabilities $\pi(m, j; m^*)$ we have that

$\pi(m,j;m^*) = \tilde{\pi}(m;m^*)\pi_{J|M}(j|m;m^*)$, where $\tilde{\pi}(m;m^*)$ is the marginal stationary probability of $m$ complete batches, under the $m^*$-threshold strategy. Taking into account Proposition 3.1 (in particular (3.8) and (3.9)) we have that

$$\pi(0,j;m^*) = \tilde{\pi}(0;m^*)g_{m^*}(j),\ 0 \le j \le K-1, \tag{3.40}$$
$$\pi(m,j;m^*) = \tilde{\pi}(m;m^*)f_{m^*-m}(j),\ 1 \le m \le m^*,\ 0 \le j \le K-1. \tag{3.41}$$

So, we need to compute the marginal stationary distribution $(\tilde{\pi}(m) : 0 \le m \le m^*)$ of the number of complete batches. Equation (3.21), in combination with (3.40) and (3.41), yields

$$\lambda\tilde{\pi}(0;m^*)g_{m^*}(K-1) = \mu\tilde{\pi}(1;m^*), \tag{3.42}$$
$$\lambda\tilde{\pi}(m;m^*)f_{m^*-m}(K-1) = \mu\tilde{\pi}(m+1;m^*),\ 1 \le m \le m^*-1. \tag{3.43}$$

Therefore, we conclude that the probabilities $\tilde{\pi}(m;m^*)$ can be computed by the 'birth-death' formula

$$\tilde{\pi}(m;m^*) = \frac{\lambda^n g_{m^*}(K-1)f_{m^*-1}(K-1)f_{m^*-2}(K-1)\cdots f_{m^*-(m-1)}(K-1)}{\mu^n}\tilde{\pi}(0;m^*), \tag{3.44}$$

where $\tilde{\pi}(0;m^*)$ is computed from the normalization equation $\sum_{m=0}^{m^*}\tilde{\pi}(m;m^*) = 1$. In a nutshell, the social welfare per time unit under a threshold strategy can be computed easily from (3.39), using (3.40), (3.41) and (3.44). Then, a numerical procedure can be used for the determination of socially optimal strategies, since an explicit formula does not seem possible.

# 4 Equilibrium threshold joining strategies

In this section we determine all equilibrium joining strategies of threshold type. Proposition 3.1 reveals a very special structure of the stationary distributions of the chains $\{(M(t), J(t))\}$ as $m^*$ varies: For $m > 0$, the conditional stationary distribution $(\pi_{J|M}(j|m;m^*) : 0 \le j \le K-1)$ depends on $m$ and $m^*$ only through their difference $m^* - m$. This fact plays an important role in proving monotonicity properties for $S^{(p)}(m;m^*)$ that are crucial in finding the equilibrium strategies. We first need to establish some stochastic comparison results regarding the distributions $(f_n(j))$ and $(g_n(j))$ that appear in Proposition 3.1.

**Lemma 4.1** *The discrete probability mass functions $(f_n(j))$, $(g_n(j))$ satisfy the following stochastic order relationships:*

*(i)* $(g_n(j)) \le_{st} (g_{n+1}(j))$, $n \ge 0$,

*(ii)* $(f_n(j)) \le_{st} (f_{n+1}(j))$, $n \ge 0$,

*(iii)* $(f_n(j)) \le_{st} (g_n(j))$, $n \ge 0$,

*where $\le_{st}$ denotes the strong stochastic order between respectively distributed random variables, i.e., $(f(j)) \le_{st} (g(j))$ if and only if $\sum_{k\ge j} f(k) \le \sum_{k\ge j} g(k)$, for all $k$.*

**Proof.** We use induction on $n$ to prove the three statements. For $n = 0$, note that $(g_0(j)) \le_{st} (g_1(j))$ is clearly valid from (3.10) and (3.12). Similarly, $(f_0(j)) \le_{st} (f_1(j))$ because of (3.10) and (3.11). Moreover, we have trivially $(g_0(j)) \le_{st} (f_0(j))$, because of (3.10).

Assume that (i)-(iii) are valid for a certain $n$. To prove (i) for $n+1$, i.e., $(g_{n+1}(j)) \le_{st} (g_{n+2}(j))$, we construct two CTMCs $\{Z(t)\}$ and $\{Z'(t)\}$ with transition rates $q_{ij}^{g_{n+1}}$ and $q_{ij}^{g_{n+2}}$

respectively on the same probability space (i.e., we couple them) in the following way: Both start from state 0 at time 0. The dynamics of $\{Z(t)\}$ is specified by the transition rates $q_{ij}^{g_{n+1}}$ given by (3.13). Whenever an event occurs in $\{Z(t)\}$ with rate $\lambda$ ('arrival'), the same event occurs in $\{Z'(t)\}$ (i.e., with rate $\lambda$ both processes move one step to the right). Moreover, when $\{Z(t)\}$ leaves $K-1$ and makes a jump to the left, to a state $i$, according to $(f_n(j))$, the next time that $\{Z'(t)\}$ leaves $K-1$ will make a jump to the left according to $(f_{n+1}(j))$, to a state $i' \geq i$ (we can couple the realizations of $(f_n(j))$ and $(f_{n+1}(j))$ using the inductive hypothesis). Therefore, the realizations of the two processes $\{Z(t)\}$ and $\{Z'(t)\}$ have been constructed in such a way that $\{Z'(t)\}$ follows the dynamics given by (3.13) with $(f_n(j))$ replaced by $(f_{n+1}(j))$. Moreover, for any time $t$ and state $j$, the process $\{Z'(t)\}$ spends more time than $\{Z(t)\}$ on the right of $j$ in the interval $[0,t]$. Thus the long-run fraction of time that the process $\{Z'(t)\}$ spends on the right of $j$ exceeds the corresponding fraction of time for the process $\{Z(t)\}$. By the ergodic theorem of CTMCs, these fractions correspond to the stationary probabilities for exceeding $j$ and we conclude that the stationary distribution of $\{Z(t)\}$ is stochastically smaller than the stationary distribution of $\{Z'(t)\}$, i.e. $(g_{n+1}(j)) \leq_{st} (g_{n+2}(j))$ and therefore (i) is valid for $n+1$. The same inductive argument can be used to prove (ii) for $n+1$. In this case the coupling is done as follows: The dynamics of $\{Z(t)\}$ is specified by the transition rates $q_{ij}^{f_{n+1}}$ given by (3.14). Whenever an event occurs in $\{Z(t)\}$ with rate $\lambda$ ('arrival') or $\mu$ ('catastrophe'), the same event occurs in $\{Z'(t)\}$ (i.e., with rate $\lambda$ both processes move one step to the right and with rate $\mu$ both processes move to 0). Moreover, when $\{Z(t)\}$ leaves $K-1$ and makes a jump to the left, to a state $i$, according to $(f_n(j))$, the next time that $\{Z'(t)\}$ leaves $K-1$ will make a jump to the left according to $(f_{n+1}(j))$, to a state $i' \geq i$. Using the same argument, we conclude that $(f_{n+1}(j)) \leq_{st} (f_{n+2}(j))$ and therefore (ii) is valid for $n+1$. Statement (iii) is proved along the same lines with the obvious adjustments. ■

We can now establish several monotonicity properties for the conditional mean sojourn time of a customer, given the number of observed complete batches upon arrival.

**Proposition 4.1** *The following monotonicity properties are satisfied for* $S^{(p)}(m;m^*)$*:*

*(i)* $S^{(p)}(m;m^*) > \frac{m+1}{\mu}$, $0 \leq m \leq m^*$.

*(ii)* $S^{(p)}(m^*;m^*) > S^{(p)}(m^*-1;m^*)$, $m^* \geq 1$.

*(iii) For any fixed* $m \geq 0$, $S^{(p)}(m^*-m;m^*)$ *is an increasing function of* $m^*$*, for* $m^* \geq \max(m,1)$.

*(iv) For any fixed* $m \geq 0$, $S^{(p)}(m;m^*)$ *is a decreasing function of* $m^*$*, for* $m^* \geq m$.

*(v) For any fixed* $m^* \geq 0$, $S^{(p)}(m;m^*)$ *is an increasing function of* $m$*, for* $0 \leq m \leq m^*$.

**Proof.** (i) Equation (3.6) shows that

$$S(m,j) \geq E[Y_m] + \frac{1}{\mu} = \frac{m+1}{\mu},\ m \geq 0,\ 0 \leq j \leq K-1,$$

with strict inequality for $j \neq K-1$. Now, (3.1) yields $S^{(p)}(m;m^*) > \frac{m+1}{\mu}$, for $0 \leq m \leq m^*-1$. Moreover, (3.2) implies that $S^{(p)}(m^*;m^*) > \frac{m^*+1}{\mu}$.

(ii) For any fixed $m$, $S(m,j)$ is a strictly decreasing function of $j$, because of equation (3.6). Therefore equation (3.2) yields

$$\begin{aligned} S^{(p)}(m^*;m^*) &= \frac{1}{\mu} + S(m^*-1,0) \\ &> \sum_{j=0}^{K-1} S(m^*-1,0)\pi_{J|M}(j|m^*-1;m^*) \\ &> \sum_{j=0}^{K-1} S(m^*-1,j)\pi_{J|M}(j|m^*-1;m^*) \\ &= S^{(p)}(m^*-1;m^*),\ m^* \geq 1. \end{aligned}$$

(iii) For $m=0$, we have to prove that $S^{(p)}(m^*;m^*)$ is increasing in $m^*$, for $m^* \geq 1$. Indeed, we have

$$S^{(p)}(m^*;m^*) = \frac{1}{\mu} + S(m^*-1,0) < \frac{1}{\mu} + S(m^*,0) = S^{(p)}(m^*+1;m^*+1),$$

where the equalities are justified by (3.2) and the inequality by the monotonicity of $S(m,j)$ with respect to $m$ (because of equation (3.6), we have that $S(m,j)$ is increasing in $m$ for every fixed $j$).

For $m \geq 1$ and $m^* = m$, we have that

$$\begin{aligned} S^{(p)}(m^*-m;m^*) &= S^{(p)}(0;m^*) \\ &= \sum_{j=0}^{K-1} S(0,j)g_{m^*}(j) \\ &\leq \sum_{j=0}^{K-1} S(0,j)f_{m^*}(j) \\ &\leq \sum_{j=0}^{K-1} S(1,j)f_{m^*}(j) \\ &= S^{(p)}(1;m^*+1) \\ &= S^{(p)}(m^*+1-m;m^*+1). \end{aligned}$$

Here, the second equality is justified from (3.1) and (3.8), while the third equality is justified by (3.1) and (3.9). On the other hand, the first inequality is justified from the stochastic order relation $(f_{m^*}(j)) \leq_{st} (g_{m^*}(j))$ (Lemma 4.1(iii)) and the monotonicity of $S(m,j)$ with respect to $j$. Finally, the second inequality is obvious by the monotonicity of $S(m,j)$ with respect to $m$.

For $m \geq 1$ and $m^* \geq m+1$, we have that

$$\begin{aligned} S^{(p)}(m^*-m;m^*) &= \sum_{j=0}^{K-1} S(m^*-m,j)f_m(j) \\ &\leq \sum_{j=0}^{K-1} S(m^*+1-m,j)f_m(j) \\ &= S^{(p)}(m^*+1-m;m^*+1), \end{aligned}$$

where the equalities are justified by equations (3.1) and (3.9), while the inequality is valid because of the monotonicity of of $S(m,j)$ with respect to $m$.

(iv) Consider an $m \geq 0$. We need to show that $S^{(p)}(m;m^*) \geq S^{(p)}(m;m^*+1)$, for all $m^* \geq m$. For $m=0$ we have that

$$S^{(p)}(0;m^*) = \sum_{j=0}^{K-1} S(0,j) g_{m^*}(j) \geq \sum_{j=0}^{K-1} S(0,j) g_{m^*+1}(j) = S^{(p)}(0;m^*+1).$$

The equalities are valid because of (3.1) and (3.8), while the inequality is justified from the stochastic order relation $(g_{m^*}(j)) \leq_{st} (g_{m^*+1}(j))$ (Lemma 4.1(i)) and the monotonicity of $S(m,j)$ with respect to $j$.

For $m \geq 1$ and $m^* = m$, we need to show that $S^{(p)}(m^*;m^*) \geq S^{(p)}(m^*;m^*+1)$. Indeed we have

$$\begin{aligned} S^{(p)}(m^*;m^*) &= \frac{1}{\mu} + S(m^*-1,0) \\ &\geq S(m^*,0) \\ &\geq \sum_{j=0}^{K-1} S(m^*,j) f_1(j) \\ &= S^{(p)}(m^*;m^*+1). \end{aligned}$$

The first equality is justified by equation (3.2), while the last equality is valid in light of (3.1) and (3.9). The first inequality can be easily deduced by (3.6) and the second inequality is clear from the monotonicity of $S(m,j)$ with respect to $j$.

For $m \geq 1$ and $m^* \geq m+1$, we have similarly

$$S^{(p)}(m;m^*) = \sum_{j=0}^{K-1} S(m,j) f_{m^*-m}(j) \geq \sum_{j=0}^{K-1} S(m,j) f_{m^*+1-m}(j) = S^{(p)}(m;m^*+1).$$

(v) Successive applications of (iv) and (iii) gives the monotonicity of $S(m;m^*)$ with respect to $m$. Indeed we have that

$$S(0;m^*) \leq S(0;m^*-1) \leq S(1;m^*) \leq S(1;m^*-1) \leq S(2;m^*) \leq \ldots.$$

■

Using the monotonicity properties of $S^{(p)}(m;m^*)$, we can now easily see that the best response against an $m^*$-threshold strategy is a non-decreasing function of the threshold $m^*$, i.e., the model exhibits a Follow-The-Crowd (FTC) behavior (see Hassin and Haviv [18], subsection 1.1.6). Indeed, the $n$-threshold strategy is a best response against the $m^*$-threshold strategy, if and only if $S^{(p)}(m;m^*) \leq \frac{R}{C}$, for $m = 0,1,\ldots,n-1$ and $S^{(p)}(n;m^*) \geq \frac{R}{C}$. But in light of the monotonicity of $S^{(p)}(m;m^*)$ with respect to $m$ (Proposition 4.1-(v)), this is equivalent to $S^{(p)}(n-1;m^*) \leq \frac{R}{C}$ and $S^{(p)}(n;m^*) \geq \frac{R}{C}$. To ensure uniqueness of the best response when the equality occurs in the last inequality, we may assume that customers do enter when they are indifferent between joining and balking. Then, the $n$-threshold strategy is the best response against the $m^*$-threshold strategy, and we will use the notation $n = BR(m^*)$, if and only if

$$S^{(p)}(n;m^*) > \frac{R}{C} \geq S^{(p)}(n-1;m^*),$$

i.e.,

$$BR(m^*) = \min\left\{n : S^{(p)}(n; m^*) > \frac{R}{C}\right\}.$$

Consider, now, two thresholds $m_1^*$ and $m_2^*$ with $m_1^* < m_2^*$. Then, using Proposition 4.1(iv), we have that $S^{(p)}(n; m_1^*) \geq S^{(p)}(n; m_2^*)$, for all $n$, thus

$$\left\{n : S^{(p)}(n; m_2^*) > \frac{R}{C}\right\} \subseteq \left\{n : S^{(p)}(n; m_1^*) > \frac{R}{C}\right\}.$$

We, therefore, conclude that

$$BR(m_2^*) = \min\left\{n : S^{(p)}(n; m_2^*) > \frac{R}{C}\right\} \geq \min\left\{n : S^{(p)}(n; m_1^*) > \frac{R}{C}\right\} = BR(m_1^*),$$

which shows that $BR(m^*)$ is an increasing function of $m^*$.

The fact that we have an FTC situation seems counterintuitive at first glance. Indeed, the more customers enter the system, the higher is the congestion. As a result, a tagged customer is expected to be discouraged from joining the system if the other customers assume a higher threshold (i.e., they enter more easily). However, the system is partially observable, hence a tagged customer has the opportunity to observe the number of complete batches waiting to be served. Given this information, the increase of the arrival rate (which is a consequence of a higher threshold adopted by the other customers) increases the expected number of customers in the incomplete batch, so it has a positive effect on the tagged customer. This intuitively justifies the fact that we have an FTC situation.

An FTC situation is known to be associated with possibly multiple equilibrium strategies (see Hassin and Haviv [18]). Moreover, notice that the 0-threshold strategy is always an equilibrium strategy since $S^{(p)}(0; 0) = \infty > \frac{R}{C}$ (if no one enters, then the system will be continuously empty and if a tagged customer enters he will wait forever). The following theorem shows that the set $M^*$ of the equilibrium thresholds is the union of the set $\{0\}$ with a finite interval of consecutive integers. Moreover, it characterizes the maximum and the minimum non-zero equilibrium thresholds and provides an easily implemented algorithmic procedure for their computation.

**Theorem 4.1** *Define the sequences* $(h_1(m) : m \geq 1)$ *and* $(h_2(m) : m \geq 1)$ *by*

$$h_1(m) = S^{(p)}(m-1; m),\ m \geq 1,\ h_2(m) = S^{(p)}(m; m),\ m \geq 1. \tag{4.1}$$

(*i*) *For* $m^* \geq 1$, *the* $m^*$*-threshold strategy is an equilibrium joining strategy if and only if* $h_1(m^*) \leq \frac{R}{C} \leq h_2(m^*)$.

(*ii*) *Let* $M^*$ *be the set of equilibrium thresholds. If* $h_1(1) > \frac{R}{C}$ *then* $M^* = \{0\}$, *otherwise* $M^* = \{0\} \cup \{m^* : m_L \leq m^* \leq m_U\}$, *where*

$$m_U = \max\left\{m^* \geq 1 : h_1(m^*) \leq \frac{R}{C}\right\}, \tag{4.2}$$

$$m_L = \min\left\{m^* \geq 1 : m^* \leq m_U \text{ and } h_2(m^*) \geq \frac{R}{C}\right\}. \tag{4.3}$$

**Proof.** Note first that the sequences $(h_1(m) : m \geq 1)$ and $(h_2(m) : m \geq 1)$ are increasing, in light of Proposition 4.1(iii). Moreover, $h_1(m) \leq h_2(m)$, $m \geq 1$, because of Proposition 4.1(v).

(i) For $m^* \geq 1$, an $m^*$-threshold strategy is an equilibrium strategy, if and only if it is best response against itself, which is equivalent to $S^{(p)}(m^*-1; m^*) \leq \frac{R}{C}$ and $S^{(p)}(m^*; m^*) \geq \frac{R}{C}$, i.e., $h_1(m^*) \leq \frac{R}{C}$ and $h_2(m^*) \geq \frac{R}{C}$.

(ii) If $h_1(1) > \frac{R}{C}$, then no $m^*$-threshold strategy with $m^* \geq 1$ is an equilibrium, since $h_1(m) \geq h_1(1) > \frac{R}{C}$. Therefore, in this case we have $M^* = \{0\}$.

If $h_1(1) \leq \frac{R}{C}$ then the set $\left\{m^* \geq 1 : h_1(m^*) \leq \frac{R}{C}\right\}$ is not empty. Moreover, the sequence $(h_1(m) : m \geq 1)$ is unbounded since $h_1(m) = S^{(p)}(m-1; m) > \frac{m}{\mu}$ (since $S(m,j) \geq \frac{m+1}{\mu}$, for all $m \geq 0$, $0 \leq j \leq K-1$). We conclude that the set $\left\{m^* \geq 1 : h_1(m^*) \leq \frac{R}{C}\right\}$ has a maximum, so $m_U$ is well-defined. Moreover, we have that $h_2(m_U) = S^{(p)}(m_U; m_U) > S^{(p)}(m_U; m_U + 1) = h_1(m_U+1) > \frac{R}{C}$, where the first inequality is justified by Proposition 4.1-(iv) and the second by the definition of $m_U$ as the maximum $m$ for which $h_1(m) \leq \frac{R}{C}$. Therefore, the set $\left\{m^* \geq 1 : m^* \leq m_U \text{ and } h_2(m^*) \geq \frac{R}{C}\right\}$ in non-empty and finite, so $m_L$ is well-defined.

Now, for any $m^* \geq 1$ with $m_L \leq m^* \leq m_U$, we have that $h_1(m^*) \leq \frac{R}{C} \leq h_2(m^*)$, so the $m^*$-threshold strategy is an equilibrium strategy (because of (i)). An $m^*$ with $m^* > m_U$ cannot be an equilibrium strategy since $h_1(m^*) > \frac{R}{C}$. Similarly, an $m^* \geq 1$ with $m^* < m_L$ cannot be an equilibrium strategy since $h_2(m^*) < \frac{R}{C}$. We conclude that $M^* = \{0\} \cup \{m^* : m_L \leq m^* \leq m_U\}$. ■

Theorem 4.1 suggests an algorithm for the identification of all non-zero threshold equilibrium joining strategies. One has to start computing $(h_1(m))$ up to the first term that exceeds $\frac{R}{C}$. This 'forward' phase of the algorithm yields the highest equilibrium threshold $m_U$ (of course, if the first term of $(h_1(m))$ exceeds $\frac{R}{C}$, then the algorithm terminates immediately and gives 'always balking' (the 0-threshold strategy) as the only equilibrium threshold strategy). Then, one has to start computing $(h_2(m))$, starting from $h_2(m_U)$ and going down to 1, till the first term that is strictly below $\frac{R}{C}$. This 'backward' phase yields the lowest equilibrium threshold $m_L$.

Up to now we characterized the customer equilibrium behavior, when the strategies of the customers are limited to the class of pure threshold strategies. We now discuss a number of extensions.

First of all, it is worthwhile noting that the established equilibrium pure threshold strategies in Theorem 4.1 are equilibrium strategies for the unrestricted game as well. That is, if we consider a tagged customer and the other customers use an $m^*$-threshold strategy for some $m^* \in M^*$ (with $M^*$ given in Theorem 4.1(ii)), then the best response of the tagged customer *among all mixed strategies* is the $m^*$-threshold strategy. Indeed, for $m^* = 0$ it is obvious that the 0-threshold strategy is best response among all mixed strategies, when the others follow the 0-threshold strategy (i.e., they always balk). For $m^* > 0$, the fact that $m^*$ specifies an equilibrium strategy for the restricted game, where only pure threshold strategies are allowed for the customers, implies that $R - CS^{(p)}(m; m^*) \geq 0$, for $m = 0, 1, \ldots, m^*-1$ and $R - CS^{(p)}(m; m^*) \leq 0$, for $m = m^*, m^*+1, \ldots$. Consider now a general mixed strategy $\mathbf{q} = (q_m : m \geq 0)$ adopted by the tagged customer. Then, his utility if he uses $\mathbf{q}$ when the others follow the $m^*$-threshold strategy, $\mathbf{m}^*$, is

$$\mathcal{U}(\mathbf{q}, \mathbf{m}^*) = \sum_{m=0}^{\infty} q_m (R - CS^{(p)}(m; m^*)). \tag{4.4}$$

To optimize this function with respect to $\mathbf{q}$, the tagged customer should use $q_m = 0$ whenever $R - CS^{(p)}(m; m^*) < 0$, $q_m = 1$ whenever $R - CS^{(p)}(m; m^*) > 0$ and any $q_m \in [0, 1]$ whenever $R - CS^{(p)}(m; m^*) = 0$. In particular the function is optimized for $q_m = 1$, for $m = 0, 1, \ldots, m^*-1$, and $q_m = 0$, for $m = m^*+1, m^*+2, \ldots$, which corresponds to the $m^*$-threshold strategy.

It seems probable that the analysis can be extended to identify equilibrium mixed threshold strategies as the ones considered in Hassin and Haviv [19] (randomization between consecutive pure threshold strategies) and the conjecture is that there exists an equilibrium genuine mixed strategy between any two consecutive equilibrium pure threshold strategies. However, the technicalities seem quite involved, because the transition rate diagram of the process $\{(M(t), J(t))\}$ given in Figure 3 is a bit different, as the corresponding penultimate row will have different arrival rates from the other rows. This happens because the randomization between the thresholds influences the subsequent analysis in a non-trivial way. Thus, we do not elaborate further on this issue.

Another interesting question concerns the comparison of the equilibrium strategies and the corresponding social welfares of this partially observable model with the corresponding observable model studied by Bountali and Economou [7] (i.e., the comparison of the almost observable and the fully observable cases in the terminology of Burnetas and Economou [8]). Economou and Bountali [7] showed that when both the number of complete batches, $M(t)$, and the number of customers in the incomplete batch, $J(t)$, are observable, then there exists a unique equilibrium strategy of threshold type. More specifically, a customer observing $(M(t), J(t)) = (m, j)$ upon arrival, enters if $m \leq m_j^e$, where $m_0^e \leq m_1^e \leq \ldots \leq m_{K-1}^e$ (see Theorem 4.1 in [7]). Under this strategy, the Markov chain $\{(M(t), J(t))\}$ is absorbed in the set $\mathcal{S}_{M,J}^{(o)}(m_o^*) = \{(m, j) : 0 \leq m \leq m_o^* - 1, 0 \leq j \leq K-1\} \cup \{(m_o^*, 0)\}$, where $m_o^* = m_0^e + 1$ and the social welfare per time unit under the equilibrium strategy of the observable model is given from formula (3.39) with $m^*$ replaced by $m_o^*$, i.e.,

$$\mathcal{B}^{(o)}(m_o^*) = \lambda R(1 - \pi(m_o^*, 0; m_o^*)) - C \sum_{(m,j)\in\mathcal{S}_{M,J}^{(o)}(m_o^*)} (mK + j)\pi(m, j; m_o^*). \tag{4.5}$$

The following proposition presents the relationship of $m_o^*$ with $m_U$ and $m_L$.

**Proposition 4.2** *(i) The equilibrium threshold $m_o^*$ of the observable case (for customers who observe an empty incomplete batch upon arrival) belongs to the set $M^*$ of the equilibrium thresholds of the partially observable model given in Theorem 4.1(ii). Moreover, in the non-trivial case where $m_o^* \neq 0$, we have that*

$$m_L \leq m_o^* \leq m_U.$$

*(ii) The maximum equilibrium social welfare over all equilibrium strategies for the partially observable case is always at least as large as the equilibrium social welfare for the (fully) observable case.*

**Proof.** We have that $m_o^*$ can be characterized as

$$m_o^* = \min\left\{m^* : S(m, 0) > \frac{R}{C}\right\}.$$

Therefore we have that

$$S(m_o^* - 1, 0) \leq \frac{R}{C} < S(m_o^*, 0). \tag{4.6}$$

To prove that $m_o^*$ is an equilibrium threshold for the partially observable case, we have to show that $h_1(m_o^*) \leq \frac{R}{C} \leq h_2(m_o^*)$ (according to the characterization of Theorem 4.1(i)), i.e., that

$$S^{(p)}(m_o^* - 1; m_o^*) \leq \frac{R}{C} \leq S^{(p)}(m_o^* - 1; m_o^*).$$

In light of (4.6), it suffices to show that

$$S^{(p)}(m_o^*-1;m_o^*) \quad \leq \quad S(m_o^*-1,0), \tag{4.7}$$

$$S^{(p)}(m_o^*;m_o^*) \quad \geq \quad S(m_o^*,0). \tag{4.8}$$

Indeed, we have

$$\begin{aligned} S^{(p)}(m_o^*-1;m_o^*) &= \sum_{j=0}^{K-1} S(m_o^*-1,j)\pi_{J|M}(j|m_o^*-1;m_o^*) \\ &= \sum_{j=0}^{K-1} S(m_o^*-1,j)f_1(j) \\ &\leq S(m_o^*-1,0)\sum_{j=0}^{K-1} f_1(j) \\ &= S(m_o^*-1,0), \end{aligned}$$

where we have used (3.1), (3.9) and the fact that $S(m,j)$ is decreasing in $j$ for every fixed $m$. Thus, (4.7) is valid.

For proving (4.8), note that $S^{(p)}(m_o^*;m_o^*) = \frac{1}{\mu} + S(m_o^*-1,0)$, because of (3.2). Therefore, we should prove that $\frac{1}{\mu} + S(m_o^*-1,0) \geq S(m_o^*,0)$. In light of (3.6), we should prove that

$$\frac{1}{\mu} + E[\max(Y_{m_o^*-1}, Z_{K-1})] \geq E[\max(Y_{m_o^*}, Z_{K-1})], \tag{4.9}$$

where $Y_m$ and $Z_{K-1}$ are Erlang$(m,\mu)$ and Erlang$(K-1,\lambda)$ independent random variables. For any real numbers $a$, $b$ and $c$ with $a \geq 0$, it is easy to check that $a + \max(b,c) \geq \max(a+b,c)$. Substituting $Y_1$, $Y_{m_o^*-1}$ and $Z_{K-1}$ for $a$, $b$ and $c$ respectively and taking expectations yields

$$E[Y_1 + \max(Y_{m_o^*-1}, Z_{K-1})] \geq E[\max(Y_1 + Y_{m_o^*-1}, Z_{K-1})]$$

that proves (4.9). This completes the proof of the statement (i) of the proposition.

Since $m_o^*$ lies between $m_L$ and $m_U$ and the social welfare functions of the observable and partially observable cases coincide (i.e., $\mathcal{B}^{(o)}(m) = \mathcal{B}^{(p)}(m)$), we conclude the statement (ii) of the proposition. ∎

# 5 Numerical conclusions

In this section, we summarize the findings of the numerical experiments that we have performed by exploiting the theoretical results of the earlier sections, regarding the effect of several system parameters on (1) the equilibrium threshold strategies and (2) the equilibrium social welfare. To address these issues, we conducted a number of numerical experiments, studying a wide rande of parameters. We found that the qualitative results are similar, regardless of the choice of the parameters. For the sake of concreteness, we present the typical behavior below. For brevity and illustrative purposes, the results are shown with reference to a concrete numerical scenario for each set of experiments.

The first set of experiments illustrates the effect of various parameters of the model on the equilibrium thresholds. In Figure 4, we consider a numerical scenario with $\mu = 2$, $K = 6$, $R = 4.9$, $C = 1$, where the arrival rate $\lambda$ varies in $[0, 2.5]$. We present the graphs of the lowest and highest equilibrium thresholds as functions of $\lambda$. We observe that both functions are increasing and bounded functions of $\lambda$. Indeed, starting from zero arrival rate, the higher the arrival rate, the more willing the customers become to enter the system, as they hope that their batch will be completed soon. After a point, however, the arrival rate is large enough to ensure that the batch of an entering customer will be completed rapidly. Then, the only crucial factor for the mean sojourn time of a tagged customer is the number of waiting batches in the system and the customer behaves similarly to a customer in the classical Naor [22] model of the observable $M/M/1$ queue. Therefore, for low values of $\lambda$ only the 0-threshold strategy is an equilibrium, while for high values of $\lambda$ the 0-threshold strategy and Naor's threshold strategy are the only equilibrium strategies. However, for intermediate values of $\lambda$ there are many non-trivial equilibrium threshold strategies. In Figure 5, we consider a numerical scenario with $\lambda = 1.8$, $\mu = 2$, $K = 6$, and $C = 1$, where the service reward $R$ varies in $[0, 10]$ and we provide the graphs of the lowest and highest equilibrium thresholds as functions of $R$. The functions are increasing in $R$, but they are not bounded. For very low values of $R$, the only equilibrium threshold strategy is the 'always balk' strategy. Then, as $R$ increases, many equilibrium threshold strategies exist. But after a point there exists a unique non-trivial equilibrium threshold strategy that corresponds to Naor's threshold.

The second set of numerical experiments investigates the effect of various parameters of the model on the equilibrium social welfare. In particular, we are interested in studying whether it is advantageous to reveal the number of customers in the incomplete batch to the arriving customers or not. To this end, in Figure 6 we consider a numerical scenario with $\mu = 2$, $K = 6$, $R = 4.9$ and $C = 1$, where $\lambda$ varies in $[0, 14]$. In the upper part of Figure 6 we present the graphs of the equilibrium social welfare in the partially observable case of the present paper when the customers follow the lowest and the highest equilibrium thresholds. Moreover, in the same figure we present the graph of the equilibrium social welfare in the observable case, where the customers also know the number of customers in the incomplete batch upon arrival. We show that all functions are constantly zero for low arrival rates and then begin to grow almost linearly up to a point where they start to decrease. They all coincide for low and large values of $\lambda$. There is only a small range of values of $\lambda$, in $[0.5, 1.4]$, where they do differ. To make the differences more visible, we provide the graphs of the functions for $\lambda \in [0, 2.5]$ in the lower part of Figure 6. We see that the equilibrium social welfare for the partially observable model when the highest possible equilibrium threshold is adopted exceeds the corresponding equilibrium social welfare for the fully observable model. Thus, for this range of $\lambda$ it seems advantageous to conceal the number of customers in the incomplete batch from arriving customers. A similar behavior is observed with respect to $R$. We provide the graphs for a scenario with $\lambda = 1.8$, $\mu = 2$, $K = 6$ and $C = 1$ in Figure 7. The upper part corresponds to $R \in [0, 20]$, whereas the lower part corresponds to $R \in [0, 5]$ (and shows more clearly what happens in the critical range of $R$ where the three graphs differ).

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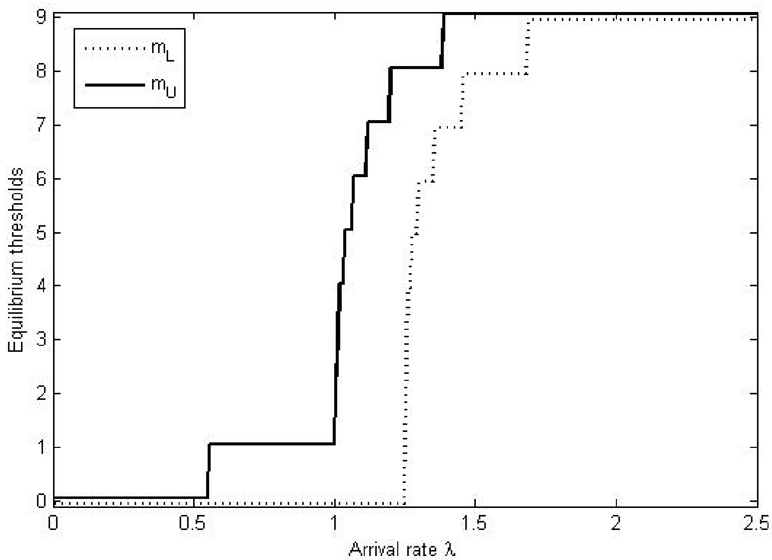



Figure 4: Equilibrium thresholds with respect to $\lambda$ for $\lambda \in [0, 2.5]$, $\mu = 2$, $K = 6$, $R = 4.9$ and $C = 1$.

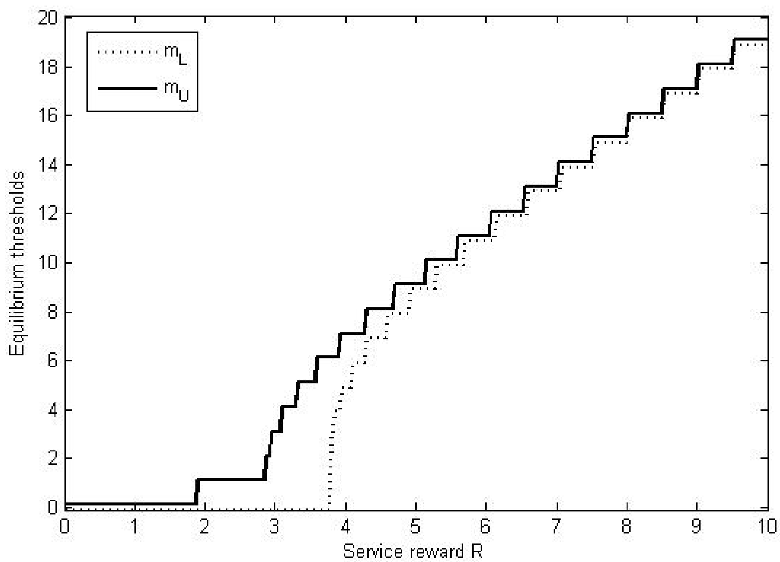
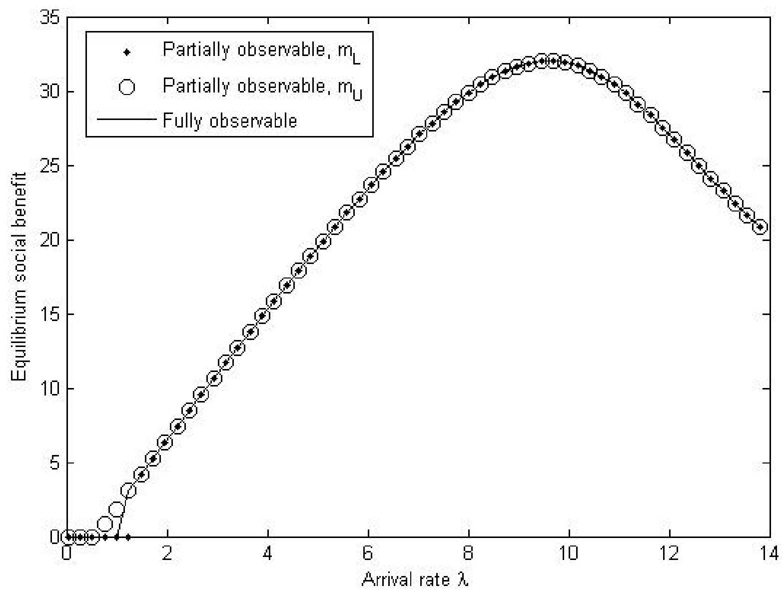



Figure 5: Equilibrium thresholds with respect to $R$ for $\lambda = 1.8$, $\mu = 2$, $K = 6$, $R \in [0, 10]$ and $C = 1$.



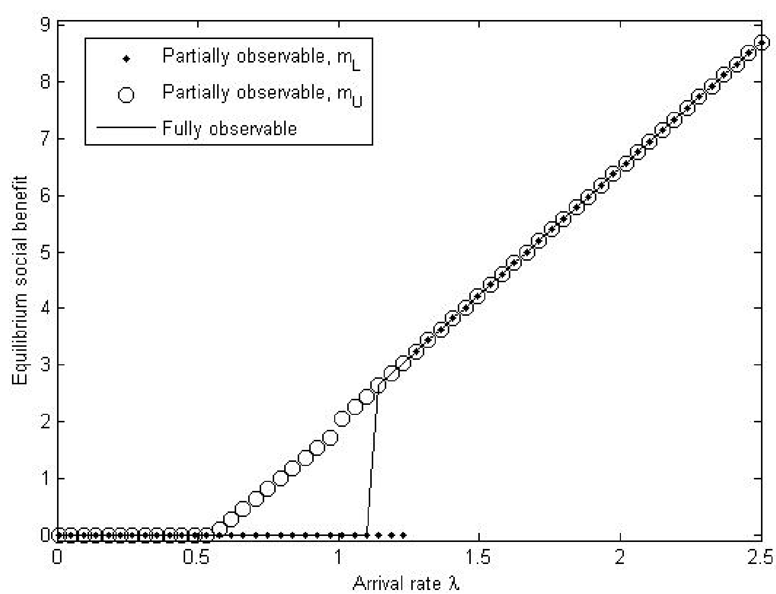
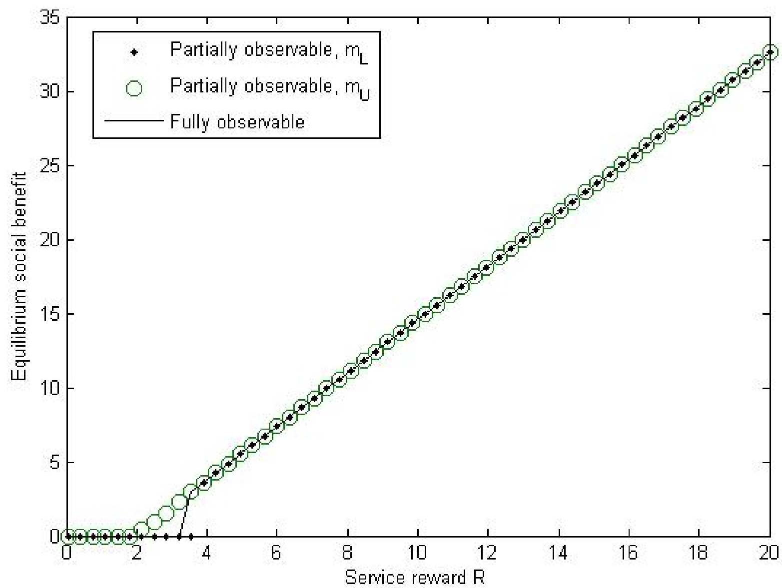




Figure 6: Equilibrium social benefit with respect to $\lambda$ for $\lambda \in [0, 14]$, $\mu = 2$, $K = 6$, $R = 4.9$ and $C = 1$.



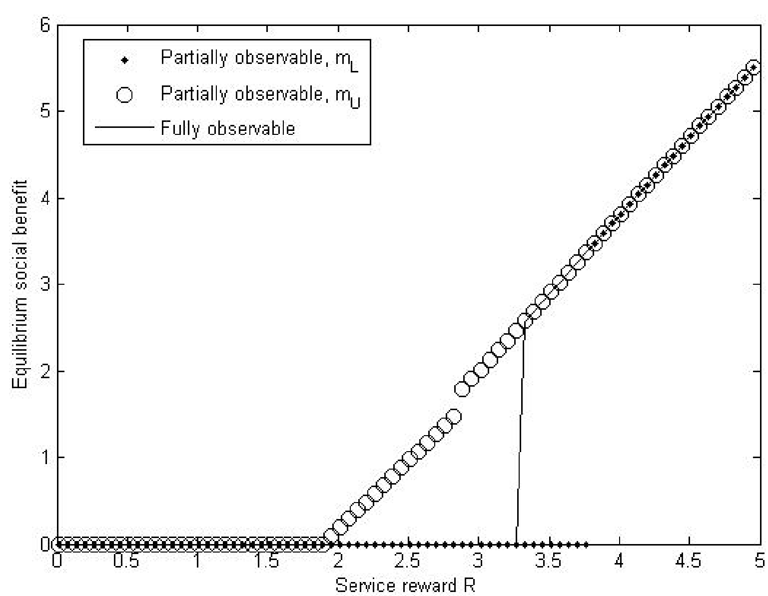




Figure 7: Equilibrium social benefit with respect to $R$ for $\lambda = 1.8$, $\mu = 2$, $K = 6$, $R \in [0, 20]$ and $C = 1$.