Harmonic functions, crossed products and approximation properties

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In memory of Dimitris Gatzouras

Abstract

The space of harmonic functions on a locally compact group G is the fixed point space of a certain Markov operator. Its 'quantization', the corresponding fixed point space of operators on L^2G, coincides with the weak-* closed bimodule over the group von Neumann algebra generated by this space.

We examine the analogous spaces of jointly harmonic functions and their quantized operator bimodules.

This leads to two different notions of crossed product of operator spaces by actions of G which coincide when G satisfies a certain approximation property.

The talk is a survey of joint work with M. Anoussis *and* I.G. Todorov, *and of more recent work by* D. Andreou.

1 Appetizer: The diagonal problem

Let Γ be a (discrete) group. Any $\phi \in \ell^\infty\Gamma$ defines a multiplication operator $f \mapsto \phi f : \ell^2\Gamma \to \ell^2\Gamma$ which we denote by the same symbol ϕ (here $(\phi f)(s) = \phi(s)f(s)$ for $s \in \Gamma$).

This multiplication operator is 'diagonal' with respect to the orthonormal basis $\{\delta_s : s \in \Gamma\}$ of $\ell^2\Gamma$.

For $r \in G$, if λ_r is the translation operator $\delta_s \mapsto \delta_{rs}$, the operator $\phi\lambda_r \in \mathcal{B}(\ell^2\Gamma)$ 'lives' on the r-th diagonal. Thus the space of operators with finitely many nonzero diagonals is

$$\left\{ \sum_{i=1}^{n} \phi_i \lambda_{r_i} : \phi_i \in \ell^\infty\Gamma, r_i \in \Gamma, n \in \mathbb{N} \right\} .$$

It is not hard to show that this space contains the set of 'matrix units' $\{E_{r,s} : r, s \in \Gamma\}$ and hence is dense in $\mathcal{B}(\ell^2\Gamma)$ for the w*-topology (recall that $\mathcal{B}(\ell^2\Gamma)$ is the dual of the Banach space $\mathcal{S}^1(\ell^2\Gamma)$ of trace class operators). But,

Question 1 *Is it true that every $X \in \mathcal{B}(\ell^2\Gamma)$ is the 'sum of its diagonals'? More precisely, is it true that if $D_r(X) \in \mathcal{B}(\ell^2\Gamma)$ denotes the r-th diagonal of X, then*

$$\lim_{F \subset\subset G} \sum_{r \in F} D_r(X) \to X \quad \textit{in some sense? } (\subset\subset\textit{: finite subset})$$

An incorrect answer appears in more than one classic book...

Question 2 *What if we introduce "multipliers" $u \in \ell^\infty\Gamma$ (more general than χ_F)? Can we find, for each $X \in \mathcal{B}(\ell^2\Gamma)$, a net (u_i) of finitely supported functions on Γ (possibly depending on X), so that*

$$\lim_i \sum_r u_i(r) D_r(X) \to X \text{ ??}$$

2 Harmonic functions

Let $\mu \in M(G)$ be a probability measure on a locally compact group G.

• A function $\phi : G \to \mathbb{C}$ is said to be **μ-harmonic** if

$$\int_G \phi(st) d\mu(t) = \phi(s) . \quad \text{Write } \phi \in \mathcal{H}(\mu) .$$

We studied the notion of μ-harmonic functions, and its connection to random walks, in a seminar [10] with Dimitris Gatzouras and others.

Here, we will limit ourselves to the functional analysis approach.

Thus a μ-harmonic function ϕ is a fixed point of the map P_μ given by

$$(P_\mu \phi)(s) = \int_G \phi(st) d\mu(t) .$$

The map P_μ is positive, unital, w*-continuous on $L^\infty(G)$.

3 The (classical) Poisson boundary

The space $\mathcal{H}(\mu)$ of μ-harmonic functions is the range of a positive unital projection defined on $L^\infty G$. This projection can be obtained by averaging over iterates of P_μ, as follows:

Note that, since $L^\infty G$ is a dual Banach space, so is $\mathcal{B}(L^\infty G)$, and hence its unit ball is w*-compact.

Define

$$E_n := \frac{1}{n}\left(I + P_\mu + (P_\mu \circ P_\mu) + \cdots + P_\mu^{n-1}\right) \in \mathrm{ball}\,\mathcal{B}(L^\infty G)$$

and let $E_\mu \in \mathcal{B}(L^\infty G)$ be a w*-cluster point of $\{E_n\}$.

Then it can be shown that E_μ is a unital positive projection onto $\mathcal{H}(\mu)$.

The space $\mathcal{H}(\mu)$ is not an algebra under pointwise multiplication. But, using the projection E_μ, it can be equipped with an associative multiplication $\diamond$, by defining

$$\phi \diamond \psi := E_\mu(\phi\psi), \quad \phi, \psi \in \mathcal{H}(\mu)\,.$$

Then $(\mathcal{H}(\mu), \diamond)$ becomes a C*-algebra; since it is w*-closed, it is a von Neumann algebra; and since it is abelian, it is an L^∞ space:

Thus there exists a probability space (Ω, ν), called the **Poisson boundary of** μ so that $\mathcal{H}(\mu) \simeq L^\infty(\Omega, \nu)$.

Let us remark that the von Neumann algebra structure on $(\mathcal{H}(\mu), \diamond)$ is independent of the choice of cluster point for $\{E_n\}$ (see section 5).

4 From Harmonic functions to Harmonic operators

Recall that an $\phi \in L^\infty G$ is a μ-harmonic *function* if

$$\int_G \phi(st) d\mu(t) = \phi(s)\,.$$

If we consider $\phi \in L^\infty G$ as a multiplication operator acting on $L^2 G$, we may write the previous equality as an operator-valued integral

$$P_\mu \phi = \int_G \rho_t \phi \rho_t^{-1} d\mu(t) \in \mathcal{B}(L^2 G)$$

which should be interpreted in the 'weak' sense. Here ρ is the right regular representation $G \curvearrowright L^2G$ given by

$$(\rho_r f)(s) = \Delta(r)^{1/2} f(sr), \quad f \in L^2(G),\ s, r \in G$$

where $\Delta : G \to \mathbb{R}^+$ is the modular function, defined by $d(tr) = \Delta(r)dt$.

This interpretation allows us to extend the notion of harmonic functions to operators (quantisation):

- Let us call an operator $T \in \mathcal{B}(L^2G)$ **a μ-harmonic operator** if

$$\int_G \rho_t T \rho_t^{-1} d\mu(t) = T\,. \quad \text{Write } \; T \in \widetilde{\mathcal{H}}(\mu)\,.$$

So μ-harmonic operators are *fixed points* of the map

$$\Theta_\mu : \mathcal{B}(L^2G) \to \mathcal{B}(L^2G) : T \to \int_G \rho_t T \rho_t^{-1} d\mu(t)$$

which is an extension of P_μ and a weak-* continuous unital and *completely positive* map. (Such maps are sometimes called Markov operators.)

Complete positivity means that, not only does Θ_μ map positive operators to positive operators, but for all n its *n-th ampliation* has the same property: If an $n \times n$ matrix $[T_{ij}]$ of operators defines a positive operator on $(L^2G)^{(n)}$, then $[\Theta_\mu(T_{ij})]$ also defines a positive operator on $(L^2G)^{(n)}$.

5 The non-commutative Poisson boundary

The following construction is due to Arveson [5] and Izumi [7]:

Let $\tilde{E}_\mu \in \mathcal{B}(\mathcal{B}(L^2G))$ be a w*-cluster point of $\{\tilde{E}_n\}$, where

$$\tilde{E}_n := \frac{1}{n}\left(I + \Theta_\mu + \cdots + \Theta_\mu^{n-1}\right) : \mathcal{B}(L^2G) \to \mathcal{B}(L^2G)\,.$$

(This uses the fact that $\mathcal{B}(L^2G)$ is a dual Banach space, and hence so is $\mathcal{B}(\mathcal{B}(L^2G))$.) Then $\tilde{E}_\mu$ is a unital completely positive projection onto $\widetilde{\mathcal{H}}(\mu)$.

Using $\tilde{E}_\mu$, we can equip $\widetilde{\mathcal{H}}(\mu)$ with an associative multiplication $\diamond$ by defining

$$T \diamond S := \widetilde{E}_\mu(TS), \quad T, S \in \widetilde{\mathcal{H}}(\mu)\,.$$

Then the space $\mathcal{N}_\mu := (\widetilde{\mathcal{H}}(\mu), \diamond)$ is a (non abelian) von Neumann algebra and $(\mathcal{H}(\mu), \diamond)$ is an abelian *-subalgebra.

As in the classical case, the von Neumann algebra structure on $\mathcal{N}_\mu$ is independent of the choice of cluster point for $\{\tilde{E}_n\}$: indeed every *completely* positive isometric linear isomorphism between von Neumann algebras must be a *-isomorphism.

The algebra $\mathcal{N}_\mu$ is called the **non-commutative Poisson boundary of** μ.

We would like to find a more 'concrete' description. Perhaps the subalgebra $\mathcal{H}(\mu) \simeq L^\infty(\Omega, \nu)$ may provide a 'coordinate representation' for $\mathcal{N}_\mu$.

6 Left Ideals of $L^1(G)$ and $\mathrm{VN}(G)$ bimodules

Observe that the preannihilator J_μ of $\mathcal{H}(\mu)$, given by

$$J_\mu := \mathcal{H}(\mu)_\perp = \left\{ f \in L^1G : \int_G \phi(t) f(t) dt = 0 \,\forall \phi \in \mathcal{H}(\mu) \right\} \subseteq L^1G$$

is invariant under left translations by G (because $\mathcal{H}(\mu)$ is) so J_μ is a left (convolution) ideal.

More generally, consider any closed left ideal $J \subseteq L^1G$.

Then $J^\perp \subseteq L^\infty G \subseteq \mathcal{B}(L^2G)$ is annihilated by the maps Θ_{ν_f} for all $f \in J$ (here $d\nu_f(t) := f(t)dt$); hence $J^\perp$ lies in

$$\begin{aligned} \ker\Theta(J) &= \left\{ T \in \mathcal{B}(L^2(G)) : \int_G \rho_t T \rho_t^{-1} f(t) dt = 0 \;\text{ for all }\; f \in J \right\} \\ &= \bigcap_{f \in J} \ker\Theta_{\nu_f} \,. \end{aligned}$$

But each Θ_{ν_f} commutes with left or right multiplication by left-translation operators $\{\lambda_t,\ t \in G\}$ on $L^2(G)$. Thus $\ker\Theta(J)$ is a *bimodule* over the w*-closed linear span of $\{\lambda_t,\ t \in G\}$, which is known as the *von Neumann algebra* $\mathrm{VN}(G)$ *of* G. It follows that $\ker\Theta(J)$ also contains the w*-closed space

$$\mathrm{Bim}(J^\perp) := \overline{\mathrm{span}}^{w^*}\{\phi\lambda_t : \phi \in J^\perp, t \in G\}$$

which is a bimodule over $\mathrm{VN}(G)$ (since $\lambda_s \phi \lambda_t = \phi_s \lambda_{st}$ where $\phi_s(t) = \phi(s^{-1}t)$).

Thus for every closed left ideal $J \subseteq L^1G$ we have the inclusion

$$\mathrm{Bim}(J^\perp) \subseteq \ker\Theta(J)\,.$$

We think of the elements of $\mathrm{Bim}(J^\perp)$ as w*-limits of 'polynomials' $\sum \phi_i \lambda_{t_i}$ whose coefficients $\phi_i \in L^\infty G$ are annihilated by all $f \in J$. On the other hand, $\ker \Theta(J)$ consists of all operators annihilated by $\{\Theta_{\nu_f}, f \in J\}$.

When can we approximate such operators by suitable polynomials $\sum \phi_i \lambda_{t_i}$?

Theorem 3 *If G has the* Approximation Property AP of Haagerup-Kraus*, then the equality*

$$\mathrm{Bim}(J^\perp) = \ker \Theta(J)\,.$$

holds for every left ideal $J \subseteq L^1(G)$.

(See section 8 for the Approximation Property).

Thie validity of this equality was first proved for G abelian, or compact, or weakly amenable discrete with M. Anoussis and I.G. Todorov [3]. The general case was then proved by J. Crann and M. Neufang [4].

7 Application to jointly harmonic operators

Given a family $\Lambda \subseteq M(G)$ of (complex valued) measures, the set $\mathcal{H}(\Lambda)$ of *jointly harmonic functions* is the set $\bigcap_{\mu\in\Lambda} \mathcal{H}(\mu)$ consisting of all $\phi \in L^\infty G$ which are μ-harmonic for all $\mu \in \Lambda$.
Correspondingly, we define the set of all *jointly harmonic operators* to be

$$\begin{aligned}\widetilde{\mathcal{H}}(\Lambda) &:= \{T \in \mathcal{B}(L^2(G)) : \mu\text{-harmonic for all } \ \mu \in \Lambda\} \\ &= \{T \in \mathcal{B}(L^2(G)) : \Theta(\mu)(T) = T \text{ for all } \ \mu \in \Lambda\}\,.\end{aligned}$$

Clearly, $\widetilde{\mathcal{H}}(\Lambda) \supseteq \mathrm{Bim}(\mathcal{H}(\Lambda))$ where $\mathrm{Bim}(\mathcal{H}(\Lambda))$ is the w*-closed linear space generated by $\{\phi\lambda_t : \phi \in \mathcal{H}(\Lambda), t \in G\}$.

Theorem 3 is applicable not only to ideals $J \subseteq L^1(G)$ which are preannihilators of μ-harmonic functions, but also to preannihilators of jointly Λ-harmonic functions:

Theorem 4 *Suppose G has the Approximation Property.*
For any $\Lambda \subseteq M(G)$,

$$\widetilde{\mathcal{H}}(\Lambda) = \mathrm{Bim}(\mathcal{H}(\Lambda))\,.$$

Remark In the special case of functions which are μ harmonic for a probability measure, the equality $\widetilde{\mathcal{H}}(\mu) = \mathrm{Bim}(\mathcal{H}(\mu))$ holds for all groups. This was shown for discrete groups by M. Izumi [7], and then for general locally compact groups by W. Jaworski and M. Neufang [8] using completely different methods.

The crucial point, in this special case, is that the space $\mathcal{H}(\mu)$ is linearly and covariantly completely isometrically isomorphic to a von Neumann algebra, namely $L^\infty(\Omega,\nu)$ where (Ω,ν) is the Poisson boundary.

8 Interlude: the approximation property AP

Very roughly, a locally compact G has the approximation property AP of Haagerup-Kraus when the Fourier algebra $A(G)$ contains an (unbounded) approximate identity of a weak form. The *Fourier algebra of* G consists of all functions $u : G \to \mathbb{C}$ of the form $u(s) = \langle \lambda_s f, g\rangle$ where $f, g \in L^2G$. Every such fuction defines a bounded mutiplier $M_u : \mathrm{VN}(G) \to \mathrm{VN}(G)$ satisfying $M_u(\lambda_s) = u(s)\lambda_s$ for all $s \in G$.

The following can be taken as the definition: G has the AP if and only if there is a net (u_i) of compactly supported functions in $A(G)$ such that (M_{u_i}) converges in the stable point-weak* topology to the identity, i.e. $(M_{u_i} \otimes \mathrm{id})(a) \to a$ weak* for all $a \in \mathrm{VN}(G)\bar{\otimes}\mathcal{B}(\ell^2)$ [6, Theorem 1.9].

The AP is a weak form of amenability.

Examples *Groups with the AP: Amenable groups, such as abelian or compact groups, but also some non-amenable, such as* $\mathbb{F}_n$.

Groups without the AP: $SL(3,\mathbb{Z})$, $SL(3,\mathbb{R})$.

Under the AP, we can answer Question 2:

Proposition 5 *If* Γ *is a discrete group with the AP, every operator in* $\mathcal{B}(\ell^2\Gamma)$ *can be w*-approximated by linear combinations of its own diagonals.*

Indeed the M_{u_i} mentionned above extend to operators defined on the whole of $\mathcal{B}(L^2G)$ and provide the required multipliers.

9 Change of perspective: The crossed product

Let us return to the space $\mathcal{H}(\Lambda)$ of functions in $L^\infty G$ which are jointly harmonic for a family Λ of complex measures on G, together with its 'quantised' cousin $\widetilde{\mathcal{H}}(\Lambda)$ of jointly harmonic operators. Note that G acts on $\mathcal{H}(\Lambda)$ by left translations. We wish to use $\mathcal{H}(\Lambda)$ together with this action to describe the space $\widetilde{\mathcal{H}}(\Lambda)$.

More generally: Let $\mathcal{V} \subseteq \mathcal{B}(H)$ be a w*-closed linear space of operators on some Hilbert space H (a *dual operator space*) and let $s \mapsto \alpha_s$ be an action of G on $\mathcal{V}$ by weak-* continuous complete isometries.

We wish to represent both G and $\mathcal{V}$ simultaneously and covariantly on the same space. For this, we 'create more space' by enlarging H to accomodate both:

Consider

$$\mathcal{V}\bar{\otimes}L^\infty G \subseteq \mathcal{V}\bar{\otimes}\mathcal{B}(L^2G) \subseteq \mathcal{B}(H \otimes L^2G)$$

(we use $\bar{\otimes}$ for the w*-closure of the algebraic tensor product).

We represent $\mathcal{V}$ on $H \otimes L^2G$ as follows: thinking of $\mathcal{V}\bar{\otimes}L^\infty G$ as consisting of $\mathcal{V}$-valued L^∞ functions on G, we associate to each $v \in \mathcal{V}$ the function $s \mapsto \alpha_s^{-1}(v)$.

More precisely, for each $v \in \mathcal{V}$, we define $\pi_\alpha(v) \in \mathcal{V}\bar{\otimes}L^\infty(G)$ by duality:

$$\langle \pi_\alpha(v), \omega \otimes h\rangle := \int_G \langle \alpha_s^{-1}(v), \omega\rangle h(s)ds, \qquad \omega \in \mathcal{V}_*, h \in L^1(G)\,.$$

(Here $\mathcal{V}_*$ is the space of all w*-continuous linear forms on $\mathcal{V}$, and we are using the fact that the projective tensor product of simple tensors of $\mathcal{V}_*$ and $L^1(G)$ has $\mathcal{V}\bar{\otimes}L^\infty G$ as its dual.)

We also define a map

$$\tilde{\lambda} : G \to \mathcal{B}(H \otimes L^2G) : s \mapsto \tilde{\lambda}_s := \mathrm{Id}_H \otimes \lambda_s\,.$$

So we have the representations

$$\begin{aligned} \pi_\alpha &: \mathcal{V} \to \mathcal{V}\bar{\otimes}\mathcal{B}(L^2G) \subseteq \mathcal{B}(H \otimes L^2G) \\ \tilde{\lambda} &: G \to \mathcal{B}(H \otimes L^2G). \end{aligned}$$

The point is that now the action α becomes 'inner': it is implemented by the unitary group $\tilde{\lambda}$:

$$\pi_\alpha(\alpha_s(v)) = \tilde{\lambda}_s \pi_\alpha(v)\tilde{\lambda}_s^{-1}\,.$$

This setup allows us to define *two versions* of the crossed product:

• The **spatial crossed product** $\mathcal{V} \rtimes_\alpha G$ is defined to be the weak* closed subspace of $\mathcal{V}\bar{\otimes}\mathcal{B}(L^2G)$ generated by all 'polynomials' in $\{\tilde{\lambda}_s : s \in G\}$ with 'coefficients' from $\pi_\alpha(\mathcal{V})$: it is the weak* closed space

$$\mathcal{V} \rtimes_\alpha G := \overline{\mathrm{span}\{\pi_\alpha(v)\tilde{\lambda}_s,\ v \in \mathcal{V}, s \in G\}}^{w*} \subseteq \mathcal{V}\bar{\otimes}\mathcal{B}(L^2G).$$

• The **Fubini crossed product** $\mathcal{V} \rtimes_\alpha^F G$ is defined to be the following fixed point subspace of $\mathcal{V}\bar{\otimes}\mathcal{B}(L^2G)$:

$$\mathcal{V} \rtimes_\alpha^F G := \{T \in \mathcal{V}\bar{\otimes}\mathcal{B}(L^2G) : (\alpha_s \otimes \mathrm{Ad}\rho_s)(T) = T\ \forall s \in G\}$$

Here $\tilde{\alpha}_s := \alpha_s \otimes \mathrm{Ad}\rho_s$ acts on a simple tensor $T = x \otimes y$ as follows: $\tilde{\alpha}_s(T) = \alpha_s(x) \otimes \rho_s y \rho_s^{-1}$.

It is not hard to see that $\pi_\alpha(\mathcal{V})$ and $\tilde{\lambda}(G)$ are both elementwise fixed by the action $\tilde{\alpha}$; hence so is the spatial crossed product generated by them;

$$\mathcal{V} \rtimes_\alpha G \subseteq \mathcal{V} \rtimes_\alpha^F G\,.$$

But do we have equality? In other words, can every $\tilde{\alpha}$-fixed point be w*-approximated by 'polynomials' of the above form?

It is a classical result (see, for example, [9, Corollary X.1.22]) that these two crossed products coincide in case $\mathcal{V}$ is *a von Neumann algebra*. However, for more general dual operator spaces, they can be distinct.

Theorem 6 (D. Andreou, [1]) *The equality $\mathcal{V} \rtimes_\alpha G = \mathcal{V} \rtimes_\alpha^F G$ holds for* all *dual operator spaces $\mathcal{V}$ if and only if the group G has the AP.*

Note that the 'if' direction was also proved by Crann - Neufang [4] using a different approach.

This Theorem can be viewed as a dynamical characterization of the AP.

10 Bimodules and Crossed products

We now apply these concepts to the Kernel-Bimodule problem. The key is the following:

In the special case where $\mathcal{V} = L^\infty G$ and G acts by left translation (we write $G \overset{\alpha_G}{\curvearrowright} L^\infty G$) both crossed products can be represented on $L^2(G)$:

Proposition 7 (D. Andreou, [1]) *There is an isometric normal *-morphism $\Psi : \mathcal{B}(L^2G) \to \mathcal{B}(L^2G)\bar{\otimes}\mathcal{B}(L^2G)$ such that: for any closed left ideal J of $L^1(G)$, we have*

$$\mathrm{Bim}(J^\perp) \overset{\Psi}{\simeq} J^\perp \rtimes_{\alpha_G} G \quad \textit{and} \quad \ker\Theta(J) \overset{\Psi}{\simeq} J^\perp \rtimes_{\alpha_G}^F G.$$

Therefore, applying Theorem 6, we obtain a conceptually different proof of Theorem 3:

Under the AP, the equality $\mathrm{Bim}(J^\perp) = \ker\Theta(J)$ holds for all closed left ideals $J \subseteq L^1G$. In particular, for any $\Lambda \subseteq M(G)$, the space $\widetilde{\mathcal{H}}(\Lambda)$ of jointly harmonic operators is (isomorphic to) the spatial crossed product of the space $\mathcal{H}(\Lambda)$ of jointly harmonic functions by the translation action of G.

Concluding Remarks We have seen that **if** a group G has the AP, **then**

$$J^{\perp} \rtimes_{\alpha_G} G = J^{\perp} \rtimes^{F}_{\alpha_G} G \quad \text{for all closed left ideals } J \text{ of } L^1(G) \qquad (*)$$

but we do not know whether the converse holds:

Question: Is the AP necessary for the validity of $(*)$?

Or is some weaker approximation property sufficient?

Or is $(*)$ valid for all locally compact bgroups G?

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