Homomorphisms between Fourier algebras

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The Wiener or Fourier algebra of the circle group $\mathbb{T}$

This is the set $A(\mathbb{T})$ of (autom. continuous) functions $f : \mathbb{T} \to \mathbb{C}$ whose Fourier series $\sum \hat{f}(k)e^{ikt}$ converges absolutely (to f, of course).

• It is an algebra under pointwise multiplication.
Qu: If $f \in A(\mathbb{T})$ never vanishes, is $1/f$ in $A(\mathbb{T})$?

• $A(\mathbb{T})$ is a Banach algebra with the norm

$$\|f\|_A = \sum |\hat{f}(k)| = \|\hat{f}\|_{\ell^1}$$

•... and its character space is (homeo to) the group $\mathbb{T}$. So, YES!

The Fourier algebra of an (abelian) locally compact group G

Let Γ be a l.c. abelian group with dual group G. Let

$$A(G) := \{\hat{f} : f \in L^1(\Gamma)\} \subseteq C_0(G), \quad \|\hat{f}\|_A := \|f\|_{L^1(\Gamma)}\,.$$

Getting rid of Γ: For $f \in L^1(\Gamma)$, can write $f = \xi\bar{\eta}$ with $\xi, \eta \in L^2(\Gamma)$ and

$$\hat{f}(s) = \int_\Gamma (\xi\bar{\eta})(\chi)\chi(s)d\chi = (\phi_s\xi, \eta)_{L^2(\Gamma)}$$

where $\phi_s(\chi) = \chi(s)$ $(\chi \in \Gamma)$, so after Fourier transform :

$$\hat{f}(s) = (\phi_s\xi, \eta)_{L^2(\Gamma)} = (\lambda_s\hat{\xi}, \hat{\eta})_{L^2(G)}\,, \quad s \in G\,.$$

where λ_s is left translation by s on $L^2(G)$: $(\lambda_s g)(t) := g(s^{-1}t)$.

The Fourier algebra of a locally compact group G

The Fourier algebra $A(G)$ [Eym64] of a locally compact group G is the space of all functions $u : G \to \mathbb{C}$ of the form

$$u(s) = (\lambda_s \xi, \eta)$$

where λ is the left regular representation of G and ξ, η are in $L^2(G)$.

The Fourier-Stieltjes algebra of an (abelian) locally compact group G

Let Γ be a l.c. abelian group with dual group G. Let

$$B(G) := \{\hat{\mu} : \mu \in M(\Gamma)\} \subseteq C_b(G), \quad \|\hat{\mu}\|_B := \|\mu\|_{M(\Gamma)}.$$

Getting rid of Γ:
Recall Bochner's Theorem: If $u \in C(G)$ is of positive type, i.e. the matrix $[u(s_j^{-1}s_i)] \succcurlyeq 0$ for all n and all $(s_i) \in G^n$, then (and only then) there exists $\mu \in M^+(\Gamma)$ such that $u = \hat{\mu}$.
Thus $B(G) = \operatorname{span} P(G)$ (=continuous functions of positive type).

The Fourier and Fourier-Stieltjes algebras

The Fourier-Stieltjes algebra $B(G)$ (Eymard, 1964 [Eym64]) of a locally compact group G is the set of all complex-linear combinations of continuous, functions $u : G \to \mathbb{C}$ *of positive type.*
But note that each function $u \in P(G)$ defines, via GNS, a unitary cyclic representation (π, ξ, H) of G such that $u(s) = (\pi(s)\xi, \xi)$.
Hence equivalently
• $B(G)$ is the space of all functions $u : G \to \mathbb{C}$ of the form

$$u(x) = (\pi(x)\xi, \eta)$$

where π is a unitary representation of G and ξ, η are vectors in the space of the representation.
It is an algebra under pointwise multiplication and is a Banach algebra with norm

$$\|u\| = \inf\{\|\xi\|.\|\eta\| : u(\cdot) = (\pi(\cdot)\xi, \eta)\}\,.$$

The magic of the Fourier algebra

• The Fourier algebra $A(G)$ is a closed ideal of the Fourier-Stieltjes algebra $B(G)$. In fact $B(G) \cap C_c(G)$ is (densely) contained in $A(G)$.

• The Fourier algebra $A(G)$ is the predual of the von Neumann algebra $vN(G) := \{\lambda_s : s \in G\}'' \subseteq \mathcal{B}(L^2(G))$ of G for the duality

$$\langle u, \lambda_s \rangle = u(s)$$

(Thus $u(s) = (\lambda_s \xi, \eta)$ uniquely defines the w*-cts linear form $T \mapsto (T\xi, \eta),\ T \in vN(G)$.)

• The spectrum (=max. ideal space) of the Banach algebra $A(G)$ is homeomorphic to G.

The Fourier and Fourier-Stieltjes algebras remember the group

Theorem (Walter, 1972 [Wal72])

Let G and H be locally compact groups. The following are equivalent:

(1) $B(G)$ *and* $B(H)$ *are isometrically isomorphic as Banach algebras,*

(2) $A(G)$ *and* $A(H)$ *are isometrically isomorphic as Banach algebras,*

(3) G *and* H *are isomorphic as topological groups.*

[Spr14]

Homomorphisms

- Cohen (1960, [Coh60b]) characterized the homomorphisms from $A(H)$ into $B(G)$ in terms of piecewise affine maps when H, G are locally compact abelian groups. To obtain his result he proved a characterization of idempotents in $B(G)$ [Coh60a].
- Host (1986, [Hos86]) extended the characterization of idempotents in $B(G)$ to general locally compact groups.
- Ilie and Spronk (2005, [IS05]) characterized *completely bounded* homomorphisms from $A(H)$ into $B(G)$ for locally compact groups H and G (with H amenable) in terms of piecewise affine maps.
- The duals of $A(H)$ and $B(G)$ are C* algebras so they can be considered (isometrically) as spaces of bdd operators on some Hilbert space.
  A linear map $\phi : A(H) \to B(G)$ is completely bounded (cb) if its dual map $\phi^* : B(G)^* \to A(H)^*$ is cb, i.e. if

$$\mathrm{id} \otimes \phi^* : \mathcal{K}(\ell^2) \otimes B(G)^* \to \mathcal{K}(\ell^2) \otimes A(H)^*$$

  is bounded.

Idempotents in $B(G)$

The open coset ring $\Omega_0(G)$ of G is the ring generated by the open cosets (:translates of open subgroups, so clopen) of the group G.

Theorem (Host, [Hos86])

For a locally compact group G and $F \subseteq G$ the idempotent χ_F is in $B(G)$ iff $F \in \Omega_0(G)$.

$$\{u \in B(G) : u = u^2\} = \{\chi_F : F \in \Omega_0(G)\}$$

Homomorphisms I

Observation: Let

$$\rho : A(H) \to B(G)$$

be a bounded homomorphism. Given $s \in G$, the map

$$\rho_s : A(H) \to \mathbb{C} : u \mapsto \rho(u)(s)$$

is multiplicative, but maybe zero.
Let $Y := \{s \in G : \rho_s \neq 0\}$. If $s \in Y$ then ρ_s is a character of $A(H)$ so there is $\alpha(s) \in H$ s.t. $\rho(u)(s) = u(\alpha(s))$ for all $u \in A(H)$. Thus have map $\alpha : Y \to H$ s.t.

$$\rho(u) = \chi_Y(u \circ \alpha) \quad u \in A(H).$$

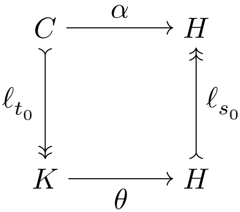
The point is to determine the structure of α.

Homomorphisms II

Let G, H be locally compact groups, K be a subgroup of G and $C = t_0^{-1}K$ a left coset of K in G. A map $\alpha : C \to H$ is called affine if there exists a *continuous* homomorphism $\theta : K \to H$ and $s_0 \in H$ such that

$$\alpha(t) = s_0\theta(t_0 t),$$

for all $t \in C$.



Homomorphisms III

A map $\alpha : Y \to H$ is called *piecewise affine* if

• $Y \subseteq G$ is a disjoint union $Y = \bigcup_{i=1}^{m} Y_i$, where each Y_i belongs to the open coset ring $\Omega_0(G)$

• each restriction $\alpha|_{Y_i}$ extends to an affine map $\alpha_i : C_i \to H$ defined on an open coset $C_i \supseteq Y_i$.

Homomorphisms, cb homomorphisms

Let $\alpha : Y \to H$ be a piecewise affine map where $Y \subseteq G$ as before. Define $\rho : A(H) \to B(G)$ by

$$\rho(u)(t) = \begin{cases} (u \circ \alpha)(t), & t \in Y \\ 0, & t \in G \backslash Y \end{cases}$$

Theorem (Cohen, [Coh60b])

Consider G, H abelian groups. Then ρ is a bounded homomorphism. Every bounded homomorphism $A(H) \to B(G)$ is of this form.

Theorem (Ilie-Spronk, [IS05])

Consider G, H general locally compact groups.
The map ρ is a completely bounded (cb) homomorphism.
If H is amenable, every cb $A(H) \to B(G)$ is of this form.

H is amenable iff $\exists\, m : L^\infty(H) \to \mathbb{C}$ left invariant state.
Result fails when H contains $\mathbb{F}_2$.

Homomorphisms and idempotents: an example

Let $F = \{-k, \ldots, k\} \subseteq \mathbb{Z}$. For $u \in A(\mathbb{Z})$, let

$$u_F(j) := \begin{cases} u(0) & \text{if } j \in F \\ 0 & \text{if } j \notin F \end{cases}$$

Then the map

$$\rho_F : A(\mathbb{Z}) \to A(\mathbb{Z}) : u \mapsto u_F$$

is a well defined bounded homomorphism.

It is of the form $u \mapsto \chi_F(u \circ \alpha)$ where α is the piecewise affine map whose each 'component' α_n is defined on the coset $\{n\} \subset F$ of the subgroup $\{0\}$ and translates n to 0.

Homomorphisms and idempotents: an example (continued)

Consider the function $u_0 : \mathbb{Z} \to \mathbb{Z}$ given by $u_0(i) = \delta_{i,0}$. Then $u_0 \in A(\mathbb{Z})$ and $\|u_0\|_{A(\mathbb{Z})} = 1$.
Since $\rho_F(u_0) = \chi_F$, its Fourier transform is $\widehat{\chi_F}(e^{it}) = \sum\limits_{n\in F} e^{-int}$ and so

$$\|\rho_F\| \geq \|\rho_F(u_0)\|_{A(\mathbb{Z})} = \|\chi_F\|_{A(\mathbb{Z})} = \|\widehat{\chi_F}\|_{L^1(\mathbb{T})}$$

$$= \int_0^{2\pi} |D_k(t)| \frac{dt}{2\pi}$$

where D_k is the Dirichlet kernel, and it is known that the L^1 norm of D_k grows like $\log k$.

Homomorphisms and idempotents

Conclusion: The existence of idempotents in $A(\mathbb{Z})$ of large norm imply the existence of homomorphisms $A(\mathbb{Z}) \to A(\mathbb{Z})$ with large norm.

Questions:

- for which locally compact groups G do there exist idempotents of arbitrarily large norm in the Fourier-Stieltjes algebra $B(G)$?
- how is the existence of 'large' idempotents related to the existence of homomorphisms of arbitrarily large norm between Fourier algebras?

Norms of idempotents

Proposition

Let G be an infinite discrete group. Then

$$\sup\{\|\chi_F\|_{B(G)} : F \subseteq G \text{ finite}\} = +\infty.$$

(NB. If $F \subseteq G$ is finite, then χ_F is in $B(G)$ - in fact, in $A(G)$.)
For the following we use a result of Leiderman, Morris and Tkachenko [LMT19] for totally disconnected groups.

Theorem

Let G be an infinite totally disconnected group. Then

$$\sup\{\|\chi_F\|_{B(G)} : F \subseteq G,\ \chi_F \in B(G)\} = +\infty.$$

(better: true for the $M_{cb}(B(G))$ norm here, which is no larger.)

Tools

We also use the following

Theorem (Eymard, [Eym64])

Let G be a locally compact group and H a closed, normal subgroup of G. Let $q : G \to G/H$ be the quotient map. The map

$$j_q : B(G/H) \to B(G) : u \mapsto u \circ q$$

is an isometry. Moreover, if H is compact, then $j_q(A(G/H)) \subseteq A(G)$.

(again, can use cb, adding Nico)

Proposition

Let G be a locally compact group and G_e be the connected component of $e \in G$. The coset rings $\Omega_0(G)$ and $\Omega_0(G/G_e)$ are isomorphic as rings (via the map $Y \mapsto q(Y)$ (where $q : G \to G/G_e$ is the quotient map)).

(so if G/G_e is finite, there are finitely many $\chi_F in B(G)$)

Norms of idempotents in $B(G)$

Theorem

Let G be a locally compact group and G_e be the connected component of $e \in G$. The following are equivalent

1. *The quotient G/G_e is infinite.*
2. $\sup\{\|\chi_F\|_{B(G)} : \chi_F \in B(G)\} = +\infty$.

Proof. If G/G_e is infinite, since it is totally disconnected,

$$\sup\{\|\chi_F\|_{B(G/G_e)} : F \subseteq G/G_e, \chi_F \in B(G/G_e)\} = +\infty.$$

Let $q : G \to G/G_e$ be the quotient map. Since $\chi_{q^{-1}(F)} = \chi_F \circ q$, it follows that

$$\sup\{\|\chi_F\|_{B(G)} : F \subseteq G, \chi_F \in B(G)\} = +\infty.$$

□

Norms of idempotents in $A(G)$

Theorem

Let G be a locally compact group and G_e be the connected component of $e \in G$. The following are equivalent

1. *The quotient G/G_e is infinite and G_e is compact*
2. $\sup\{\|\chi_F\|_{A(G)} : \chi_F \in A(G)\} = +\infty.$

Norms of homomorphisms

Proposition

Let G, H be locally compact groups and $F \in \Omega_0(G)$. For $u \in A(H)$ we define

$$\rho_F(u)(t) = \begin{cases} u(e), & t \in F \\ 0, & t \notin F \end{cases}$$

Then the map ρ_F is a bounded homomorphism $A(H) \to B(G)$ and

$$\|\rho_F\| = \|\chi_F\|_{B(G)}.$$

Proof. There exists $u \in A(H)$ such that $u(e) = 1$ and $\|u\|_{A(H)} = 1$. Then

$$\|\rho_F\| \geq \|\rho_F(u)\|_{B(G)} = \|u(e)\chi_F\|_{B(G)} = \|\chi_F\|_{B(G)}.$$

On the other hand, for any $v \in A(H)$,

$$\|\rho_F(v)\|_{B(G)} = \|v(e)\chi_F\|_{B(G)} = |v(e)|\|\chi_F\|_{B(G)} \leq \|v\|_{A(H)}\|\chi_F\|_{B(G)},$$

and hence

$$\|\rho_F\| = \|\chi_F\|_{B(G)}.$$

□

NB. Actually, ρ_F is completely bounded.

Norms of homomorphisms

Theorem

Let G be a locally compact group and G_e be the connected component of $e \in G$. The following are equivalent:

1. *The group G/G_e is infinite.*
2. $\sup\{\|\chi_F\|_{B(G)} : F \in \Omega_0(G)\} = +\infty$.
3. *For every locally compact group H,*
$$\sup\{\|\rho : A(H) \to B(G)\| : \rho \text{ is a cb homomorphism}\} = +\infty.$$
4. *There is an amenable locally compact group H such that*
$$\sup\{\|\rho : A(H) \to B(G)\| : \rho \text{ is a cb homomorphism}\} = +\infty.$$

A constructive approach to large idempotents

Let G be a compact group and χ_F an idempotent in $B(G)$.

- $\|\chi_F\|_{B(G)} = 1$ if and only if F is a coset of an open subgroup of G (Ilie-Spronk [IS05]).
- Stan [Sta09] and Forrest and Runde [FR11] proved that if $\|\chi_F\|_{B(G)} < \frac{2}{\sqrt{3}}$ then F is a coset of an open subgroup of G.
- Let G be a finite group with $|G| \geq 3$. Then there exists $A \subseteq G$ such that A is not a coset of a subgroup of G (for example, take A such that $|A|$ does not divide $|G|$). Then, since $A \in \Omega_0(G)$, it follows that $\|\chi_A\|_{B(G)} \geq \frac{2}{\sqrt{3}}$.

A constructive approach to large idempotents (Continued)

- Now consider finite groups $G_1, G_2, \dots, G_n$ with all $|G_i| \geq 3$ and set $G = \prod_{i=1}^{n} G_i$. Choose $A_i \in G_i$ with $\|\chi_{A_i}\|_{B(G_i)} \geq \frac{2}{\sqrt{3}}$ and set $A = A_1 \times A_2 \times \cdots \times A_n$.
  Then
$$\|\chi_A\|_{B(G)} \geq \left(\frac{2}{\sqrt{3}}\right)^n.$$

A constructive approach to large idempotents (Continued)

- Now let $G = \prod_{i=1}^{\infty} G_i$, an infinite product of finite groups G_i. We may assume that $|G_i| \geq 3$ for all $i \in \mathbb{N}$.
- Let $H_n = \prod_{i=n+1}^{\infty} G_i$ and q_n the quotient map $G \to G/H_n$.

  Since $G/H_n \simeq \prod_{i=1}^{n} G_i$, we can choose $A \subseteq G/H_n$ such that

$$\|\chi_A\|_{B(G/H_n)} \geq \left(\frac{2}{\sqrt{3}}\right)^n .$$

- Setting $F_n = q_n^{-1}(A)$, since the map $B(G/H_n) \to B(G) : u \mapsto u \circ q_n$ is isometric, we obtain

$$\|\chi_{F_n}\|_{B(G)} \geq \left(\frac{2}{\sqrt{3}}\right)^n$$

A constructive approach to large idempotents (Continued)

Theorem

Let G be an infinite product of finite groups. Then

$$\sup\{\|\chi_F\|_{B(G)} : F \subseteq G, \chi_F \in B(G)\} = +\infty.$$

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Ευχαριστώ!

Спасибо!