# Masa-bimodules of Toeplitz type

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This is a report of work in progress with M. Anoussis (Aegean) and I.G. Todorov (QUB).

# Supports of operators and masa bimodules

Say $T : L^2(X,\mu) \to L^2(Y,\nu)$ is vanishes in a Borel rectangle $A \times B$ whenever $P(B)TP(A) = 0$.
Then $T$ is supported in every set $\Omega \subseteq X \times Y$ which 'almost misses' $A \times B$.
This means $\Omega \cap (A \times B) \subseteq M \times Y \cup X \times N$ where $\mu(M) = 0 = \nu(N)$.
Write $\Omega \cap (A \times B) =_\omega \emptyset$.

Fix $\Omega \subseteq X \times Y$. The set $\mathcal{M} = \mathcal{M}_{max}(\Omega)$ of all $T$ which are supported in $\Omega$ is
(a) w*-closed
(b) a masa bimodule: $\mathcal{D}_x \mathcal{M} \mathcal{D}_y \subseteq \mathcal{M}$
(c) reflexive, i.e. $\mathcal{M} = \mathcal{R}^\perp$ for a set $\mathcal{R}$ of *rank ones*.
(cf Loginov-Shulman)

# Supports of operators and masa bimodules

Given a w*-closed masa bimodule $\mathcal{M}$, 'the support' ought to be the complement of the union of all Borel rectangles on which every $T \in \mathcal{M}$ vanishes. Measurability?

There is a countable family $\mathcal{E}$ of Borel rectangles whose union $\omega$-contains every Borel $A \times B$ s.t. $P(B)\mathcal{M}P(A) = \{0\}$.
The complement of this union (such a set is called $\omega$-closed) is called the $\omega$-support $\operatorname{supp}\mathcal{M}$ of $\mathcal{M}$.

Every *reflexive* masa bimodule is of the form $\mathcal{M}_{max}(\Omega)$; and if $\Omega$ is $\omega$-closed, then it is unique (mod. marg. null sets) (Erdos - K - Shulman, 1998).

# Supports of operators and masa bimodules

Arveson (1974) defines his $\mathrm{supp}_A$ to be closed; so, two different reflexive masa bimodules can have the same closed support.

*Example*: (for $H = L^2([0,1])$), let $\mathcal{M} := \mathcal{D} + E\mathcal{B}(H)E$ where $E = E(A)$ with $A$ and $A^c$ meeting every open set nontrivially. This has the same $\mathrm{supp}_A$ with $\mathcal{B}(H)$.

Starting with a reflexive masa bimodule, he *defines* the topology using it.
We just fix the masas, and represent all masa bimodules simultaneously.

# Failure of operator synthesis: Arveson's example

If $\Omega \subseteq X \times Y$ is $\omega$-closed, and $\mathcal{M}$ is a w*-closed masa bimodule with $\mathrm{supp}\, M = \Omega$, does it follow that $\mathcal{M} = \mathcal{M}_{max}(\Omega)$?
Equivalently, is every w*-closed masa bimodule automatically reflexive?

Arveson (1974): No! Take $\mathbb{S}^2 \subseteq \mathbb{R}^3$.
Then let $\Omega = \{(s, t) \in \mathbb{R}^3 \times \mathbb{R}^3 : t - s \in \mathbb{S}^2\}$

# Predual formulation

(Shulman-Turowska, 2004)
Go to $\|\cdot\|_t$-closed subspaces of *predual*
$T(X, Y) := L^2(X)\hat{\otimes}L^2(Y)$.
This can be identified with the space of all functions of the form

$$h(x, y) = \sum_i f_i(x)g_i(y)$$

where $f_i, g_i \in L^2$ and $\sum_i \|f_i\|_2 \|g_i\|_2 < \infty$ (identify functions differing on a marginally null set).

# Operator Synthesis

(Take $X = Y = G$, locally compact 2nd countable group now)
Let $F \subseteq G \times G$ be closed (or just $\omega$-closed). Define

$$\Phi(F) = \{u \in T(G) : u \text{ vanishes m.a.e. on } F\}$$

$$\Psi(F) = \{u \in T(G) : u \text{ vanishes } \omega\text{-near } F\}$$

$$\Phi_0(F) = \overline{\Psi(F)}^{\|\cdot\|_T}.$$

The subspace $\Phi(F)$ is $\|\cdot\|_T$-closed and contains $\Psi(F)$.
The latter consists of all $u : G \times G \to \mathbb{C}$ vanishing m.a.e. *on some countable cover of $F$ by Borel rectangles*.
Both are invariant under left and right multiplications by elements of $L^\infty(G)$ (masa bimodules).

# Operator Synthesis

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Say that $F$ satisfies **operator synthesis** if $\Phi_0(F) = \Phi(F)$.

That is, if every $u \in T(G)$ vanishing *m.a.e. on $F$* can be approximated in the norm of $T(G)$ by a sequence $(u_n)$ of elements which vanish *"$\omega$-near" $F$*.

# The Fourier algebra $A(G)$ for non abelian groups

Represent $G$ on $L^2(G)$ by $(\lambda_s f)(t) = f(s^{-1}t),\ f \in L^2(G)$.

**Definition (Eymard [2], 1964)**

*The Fourier algebra $A(G)$ is the set of all functions $u : G \to \mathbb{C}$ of the form $u(s) = (\lambda_s f, g)$ with $f, g \in L^2(G)$.*

- This is a linear space, in fact an algebra of functions on $G$, complete in the norm is given by $\|u\|_A = \inf \|f\|_2 \|g\|_2$.
- Its dual is (isom. & w*-homeo.) to the von Neumann algebra of $G$:

$$\mathrm{VN}(G) = \text{w-* span}\{\lambda_s : s \in G\}.$$

Duality: $\langle \lambda_s, u \rangle_a := u(s) = (\lambda_s f, g)$.

# Synthesis in $A(G)$

For $E \subseteq G$ closed, define

$$I(E) = \{g \in A(G) : g|_E = 0\}$$

and its subset

$$J(E) = \overline{\{g \in A(G) : \operatorname{supp} g \cap E = \emptyset\}}^{\|\cdot\|_A}.$$

Then $E$ is called **a set of spectral synthesis** if $J(E) = I(E)$.

If $G$ is discrete, then $J(E) = I(E)$ for all $E$. [1]
If not, there always exists $E \subseteq G$ s.t. $J(E) \subsetneq I(E)$ (Malliavin [1], 1959).

[1] $E$ is a nhd of $E$

# *Synthesis* and $\Sigma\upsilon\nu\theta\epsilon\sigma\iota\varsigma$

### Theorem

*Let G be (locally) compact second countable. Assume $A(G)$ has an approximate unit. A closed set $E \subseteq G$ is synthetic if and only if the set*

$$E^* = \{(s,t) \in G \times G : ts^{-1} \in E\}$$

*is operator synthetic.*

Due to : Froelich ([1], 1988) for abelian $G$,
Spronk-Turowska ([2], 2002) for compact $G$,
Ludwig-Turowska ([3], 2006) for general $G$ but with local synthesis.

- Are there any groups s.t. $A(G)$ has no approximate unit?

# Our approach (w. Anoussis & Todorov)

$$\begin{array}{ccccccc} A(G) & \supseteq & J & \xrightarrow{\perp} & J^{\perp} & \subseteq & \mathrm{VN}(G) \\ & & \downarrow & & \downarrow & & \\ T(G) & \supseteq & \mathrm{Sat}(J) & \xrightarrow{\perp} & \mathrm{Bim}(J^{\perp}) & \subseteq & \mathcal{B}(L^2(G)) \end{array}$$

# From ideals to invariant $L^\infty(G)$-bimodules

Let $J \subseteq A(G)$ be a closed ideal. Suppose momentarily that $G$ is compact. • First, horizontally: form the annihilator $J^\perp$ in $A(G)^*$:

$$J \to J^\perp \subseteq A(G)^*$$

Recall $A(G)^* = \mathrm{VN}(G) \subseteq \mathcal{B}(L^2(G))$. So can saturate $J^\perp$ on left and right by multiplication operators to form the smallest w*-closed $L^\infty(G)$-bimodule $\mathrm{Bim}(J^\perp)$ of $\mathcal{B}(L^2(G))$ containing $J^\perp$:

$$J^\perp \to \mathrm{Bim}(J^\perp) \subseteq \mathcal{B}(L^2(G)).$$

• Now, go vertically: first embed $J \to N(J)$ in $T(G)$ isometrically via $(Nu)(s,t) = u(ts^{-1})$ (Bożejko-Fendler), form the smallest $\|\cdot\|_T$-closed $L^\infty(G)$-bimodule $\mathrm{Sat}(J)$ of $T(G)$ containing $N(J)$, then form its annihilator in $\mathcal{B}(L^2(G))$ :

$$J \to N(J) \to \mathrm{Sat}(J) \to (\mathrm{Sat}(J))^\perp.$$

These two procedures give the same result:

$$\mathrm{Bim}(J^\perp) = (\mathrm{Sat}(J))^\perp.$$

# From ideals to invariant $L^\infty(G)$-bimodules

To form $\mathrm{Sat}\, J$ for $J \subseteq A(G)$ when $G$ is is not compact:

For $u \in J \lhd A(G)$, the function $N(u)$ is locally (i.e. when restricted to compact sets) in $T(G)$.
Define $\mathrm{Sat}(J)$ to be the $\|\cdot\|_t$-closed $L^\infty(G)$-bimodule generated by these localised functions.

Alternatively, given that $N$ takes $A(G)$ to *(completely bounded) multipliers* of $T(G)$, we can show

$$\mathrm{Sat}(J) = \overline{[N(J)T(G)]}^{\|\cdot\|_t}.$$

# From ideals to invariant $L^\infty(G)$-bimodules

**Theorem (2013)**

*Let $J \subseteq A(G)$ be a closed ideal. Then $(\mathrm{Sat}\, J)^\perp = \mathrm{Bim}(J^\perp)$.*

The proof use ideas of Ludwig – Spronk – Turowska.
Express Schur multipliers in terms of *invariant* Schur multipliers and vice versa.
The resulting series converge in the appropriate topology when restricted to compact sets.

# Schur multipliers

Call a function $\varphi \in L^\infty(G \times G)$ a *Schur multiplier* if

$$\varphi T(G) \subseteq T(G) \quad \text{(pointwise multiplication).}$$

Easy to see that the map $m_\varphi : T(G) \to T(G)$ given by $m_\varphi h = \varphi h$ is bounded.

Recall that $T(G)^* \simeq \mathcal{B}(L^2(G))$.

Let $S_\varphi = m_\varphi^* : \mathcal{B}(L^2(G)) \to \mathcal{B}(L^2(G))$.

Denote the space of Schur multipliers by $\mathfrak{S}(G)$.

# Jointly invariant subspaces

Recall the adjoint right action of $G$ on $\mathcal{B}(L^2(G))$ given by $\mathrm{Ad}\rho_r(T) = \rho_r T \rho_r^*$, $(r \in G)$ (so if $T = T_k$ is Hilbert-Schmidt then $\mathrm{Ad}\rho_r(T_k)$ has kernel $k_r(s,t) = k(sr,tr)$).

This integrates to a representation of the measure algebra $M(G)$ as operators on $\mathcal{B}(L^2(G))$ :
For $\mu \in M(G)$, define $\Gamma(\mu) : \mathcal{B}(L^2(G)) \to \mathcal{B}(L^2(G))$ by

$$\Gamma(\mu)(T) = \int_G \rho_r T \rho_r^* d\mu(r), \quad T \in \mathcal{B}(L^2(G)).$$

studied by E. Størmer, F. Ghahramani, M. Neufang, Zh.-J. Ruan and N. Spronk.

Note that a weak-* closed subspace $\mathcal{U} \subseteq \mathcal{B}(L^2(G))$ is a bimodule over $L^\infty(G)$ iff it is invariant under all Schur multipliers.

We characterize subspaces of $\mathcal{B}(L^2(G))$ which are invariant under *both actions*.

# Jointly invariant subspaces

### Theorem

*Let $\mathcal{U} \subseteq \mathcal{B}(L^2(G))$ be a weak* closed subspace. The following are equivalent:*
*(i) the space $\mathcal{U}$ is invariant under all mappings $S_w$ and $\Gamma(\mu)$,*
*(ii) the space $\mathcal{U}$ is invariant under all Schur multipliers $S_w$ and all $\mathrm{Ad}\rho_r : T \to \rho_r T \rho_r^*$, $r \in G$;*
*(ii) there exists a closed ideal $J \subseteq A(G)$ such that $\mathcal{U} = \mathrm{Bim}(J^\perp)$.*

The range of $\Gamma$ consists of all $\mathrm{VN}(G)$-bimodule maps leaving the multiplication masa $\mathcal{D}$ invariant.
The range of $w \to S_w$ consists of all $\mathcal{D}$-bimodule maps leaving $\mathrm{VN}(G)$ invariant.

*Idea of proof*: Show that the ideal that does the job is

$$J = \{u \in A(G) : N(u)\chi_{L\times L} \in \mathcal{U}_\perp \text{ for every compact set } L \subseteq G\}.$$

# Harmonic functionals, harmonic operators

Given $\sigma : G \to \mathbb{C}$ a completely bounded multiplier of $A(G)$, Neufang and Runde define

$$H_\sigma = \{T \in \mathrm{VN}(G) : \sigma \cdot T = T\}$$
$$\text{and} \quad \tilde{H}_\sigma = \{T \in \mathcal{B}(L^2(G) : S_{N(\sigma)}(T) = T\}.$$

We show that:

**Theorem**

*If* $\Sigma \subseteq M^{cb}A(G)$*, then* $\tilde{H}_\Sigma = \mathrm{Bim}\, H_\Sigma$*.*

As a corollary we obtain their result that if $\sigma \in P^1(G) \cap A(G)$ then $\tilde{H}_\sigma$ is a von Neumann algebra, namely

$$\tilde{H}_\sigma = (H_\sigma \cup L^\infty(G))''.$$

This is still in progress...

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