Numerical solution of the Benjamin equation

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Abstract

In this paper we consider the Benjamin equation, a partial differential equation that models one-way propagation of long internal waves of small amplitude along the interface of two fluid layers under the effects of gravity and surface tension. We solve the periodic initial-value problem for the Benjamin equation numerically by a new fully discrete hybrid finite-element / spectral scheme, which we first validate by pinning down its accuracy and stability properties. After testing the evolution properties of the scheme in a study of propagation of single - and multi-pulse solitary waves of the Benjamin equation, we use it in an exploratory mode to illuminate phenomena such as overtaking collisions of solitary waves, and the stability of single-pulse, multi-pulse and ‘depression’ solitary waves.

Key words: Benjamin equation, Solitary waves, Hybrid Finite Element-Spectral method

1991 MSC: 76B15 (primary), 65M60, 65M70 (secondary)

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1 Introduction

In this paper we will consider the *Benjamin equation*

\[ u_t + \alpha u_x + \beta uu_x - \gamma \mathcal{H}u_{xx} - \delta u_{xxx} = 0, \quad (1.1) \]

where \( u = u(x,t), x \in \mathbb{R}, t \geq 0, \alpha, \beta, \gamma, \delta \) are positive constants, and \( \mathcal{H} \) denotes the Hilbert transform defined on the real line as

\[ \mathcal{H}f(x) := \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} \, dy \]

or through its Fourier transform as

\[ \widehat{\mathcal{H}f}(k) = -i \text{sign}(k) \hat{f}(k), k \in \mathbb{R}. \]

The Benjamin equation, cf. [1,2,3], is a model for *internal* waves propagating under the effect of gravity and surface tension in the positive \( x \)-direction along the interface of a two-dimensional system of two homogeneous layers of incompressible, inviscid fluids consisting at rest of a thin layer of fluid 1 of depth \( d_1 \) and density \( \rho_1 \) lying above a layer of fluid 2 of very large depth \( d_2 \gg d_1 \) and density \( \rho_2 > \rho_1 \). The upper layer is bounded above by a horizontal ‘rigid lid’ and the lower layer is bounded below by an impermeable horizontal bottom.

It is further assumed that the following physical regime of interest is to be modelled: Let \( a \) be a typical amplitude and \( \lambda \) a typical wavelength of the interfacial wave. The parameters \( \epsilon = a/d_1 \) and \( \mu = d_1^2/\lambda^2 \) are assumed to be small and satisfy \( \mu \sim \epsilon^2 \ll 1 \); it is also assumed that capillarity effects along the interface are not negligible. Under these assumptions \((1.1)\) was derived in [1] from the two-dimensional, two-layer Euler equations in the presence of interface surface tension by dispersion relation arguments. The variables in \((1.1)\) are nondimensional and scaled, and the coefficients are given by

\[ \alpha = \sqrt{\frac{\rho_2 - \rho_1}{\rho_1}}, \quad \beta = \frac{3}{2} \alpha \epsilon, \quad \gamma = \frac{1}{2} \alpha \sqrt{\mu} \frac{\rho_2}{\rho_1}, \quad \delta = \frac{\alpha T}{2g\lambda^2(\rho_2 - \rho_1)}, \]

where \( T \) is the interfacial surface tension and \( g \) the acceleration of gravity. The variables \( x \) and \( t \) are proportional to distance along the channel and time, respectively, and \( u(x,t) \) denotes the downward vertical displacement of the interface from its level of rest at \((x,t)\). The interfacial surface tension \( T \) is assumed to be much larger than \( g(\rho_2 - \rho_1)d_1^2 \). (For a further discussion of the physical regime of validity of \((1.1)\) cf. [3].) Note that if the parameter \( \delta \) is taken equal to zero, \((1.1)\) reduces to the Benjamin-Ono (BO) equation, [4,5], while, if we put \( \gamma = 0 \) we obtain the KdV equation with negative dispersion coefficient.
It is well known, cf. [1], that sufficiently smooth solutions of (1.1) that vanish suitably at infinity preserve the functionals

\[ m(u) = \int_{-\infty}^{\infty} u \, dx, \quad (1.2) \]
\[ I(u) = \frac{1}{2} \int_{-\infty}^{\infty} u^2 \, dx, \quad (1.3) \]
\[ E(u) = \int_{-\infty}^{\infty} \left( \frac{6}{5} u^3 - \frac{1}{2} \gamma u H u_x + \frac{1}{2} \delta u_x \right) \, dx. \quad (1.4) \]

Global well-posedness in \( L^2 \) for the Cauchy problem and also for the periodic initial-value problem for (1.1) was established in [6].

In this paper we will study (1.1) numerically, paying particular attention to properties of its solitary-wave solutions. These are travelling-wave solutions of the form \( u(x,t) = \varphi(x - c_s t), c_s > 0 \), such that \( \varphi \) and its derivatives tend to zero as \( \xi = x - c_s t \) approaches \( \pm \infty \). Substituting this expression in (1.1) and integrating once we obtain

\[ (\alpha - c_s) \varphi + \frac{\beta}{2} \varphi^2 - \gamma H \varphi - \delta \varphi'' = 0, \quad (1.5) \]

where \( \prime = d/d\xi \), and the operator \( H \) is defined by \( H := H \partial_x \), i. e. by \( \hat{H} \hat{f}(k) = |k|\hat{f}(k), k \in \mathbb{R} \). We will assume that \( \alpha - c_s > 0 \).

If we perform the change of variables

\[ \varphi(\xi) = -\frac{2(\alpha - c_s)}{\beta} \psi(z), \quad z = \sqrt{\frac{\alpha - c_s}{\delta}} \xi, \]

in (1.5), we see that the solitary-wave profile \( \psi(z) \) satisfies the ordinary differential equation (o.d.e.)

\[ \psi - 2\tilde{\gamma} H \psi - \psi_{zz} - \psi^2 = 0, \quad z \in \mathbb{R}, \quad (1.6) \]

where

\[ \tilde{\gamma} = \frac{\gamma}{2\sqrt{\delta(\alpha - c_s)}}. \quad (1.7) \]

This change of variables and the resulting equation (1.6) was used in [1,2], and [3]. (In these references \( \tilde{\gamma} \) is denoted by \( \gamma \).) In his papers Benjamin showed that for each \( \tilde{\gamma} \in [0, 1] \), there exists a solution \( \psi \) of (1.6) which is an even function
of \( z \) with \( \psi(0) = \max_{z \in \mathbb{R}} \psi(z) > 0 \). He also argued by formal asymptotics that for each \( \tilde{\gamma} \in [0, 1) \) there is a bounded interval centered at \( z = 0 \), in which \( \psi \) oscillates (with the number of oscillations increasing as \( \tilde{\gamma} \) approaches 1), while outside this interval he concluded in [2] that \( |\psi| \) decays like \( 1/z^2 \). In addition, in the same paper he outlined an orbital stability theory for these solitary waves for small \( \tilde{\gamma} \). In [3] a complete theory of existence and orbital stability of the solitary waves for small \( \tilde{\gamma} \) was presented. Further issues of existence and rigorous asymptotics of the solitary waves of (1.1) and related equations were explored in [7]. In [8] concentration compactness arguments were used to establish existence and a weaker version of stability of the solitary waves of (1.1) for \( 0 < \tilde{\gamma} < 1 \).

In this paper we will employ the solitary-wave equation in the form (1.5). As a result, normally the solitary waves will have negative maximum excursions from their level of rest.

Since explicit formulas for the solitary waves of the Benjamin equation are not known (except when one of \( \gamma \) or \( \delta \) is set equal to zero), one must resort to approximate techniques for their construction. The presence of the nonlocal terms in (1.1) and (1.5), which have a handy Fourier representation in the periodic case as well, naturally suggests using spectral-type methods for approximating their solutions. The preceding discussion of the Benjamin equation applies to its associated Cauchy problem on \( \mathbb{R} \). Solving it numerically requires posing it on a finite \( x \)-interval \([-L, L]\) with, say, periodic boundary conditions, assuming \( 2L \)-periodic initial data. In case solitary waves, their generation and interactions, are the focus of interest, one should take into account that they decay quadratically and, consequently, the interval \([-L, L]\) should be taken sufficiently large.

In Albert et al., [3], the equation (1.6) was discretized in space by a pseudospectral technique and the resulting nonlinear system of equations for the Fourier coefficients of \( \psi = \psi_\gamma \) for a desired value of \( \tilde{\gamma} \in (0, 1) \) was solved by incremental continuation coupled with Newton’s method. With this technique the authors of [3] were able to construct accurate approximate solutions of (1.6) that were even functions with a positive absolute maximum at \( z = 0 \).

In [9], Bona and Kalisch solved numerically the periodic initial-value problem for the Benjamin equation using a pseudospectral (collocation) method in space coupled with a second-order time-stepping procedure. They confirmed that resolution of suitable general initial profiles into a number of solitary waves plus a dispersive tail (a phenomenon that has been observed in other nonlinear dispersive wave equations) also occurs in the case of the Benjamin equation. In some cases they observed, in addition to detached solitary waves, the emergence of clusters (pairs, triplets, etc.) of ‘orbiting’ solitary waves that interacted among themselves. They also constructed approximate solitary
waves, using the resolution property, by truncating and iteratively ‘cleaning’ a separated solitary wave. They used two such approximate solitary waves of different speeds to study their overtaking collision and observed that the interaction was not elastic, a fact indicating that the Benjamin equation is not integrable.

In [10], Calvo and Akylas considered solitary waves of the Benjamin equation and compared them to solitary waves of the full Euler equations for interfacial flows in the presence of surface tension when the parameters of the problem are close to the Benjamin equation regime of validity and also farther from it. The numerical scheme they used for approximating the solitary waves was based on a hybrid spatial discretization that employed fourth-order finite differences on a uniform grid for the derivatives, and the discrete Fourier transform for the nonlocal term. The resulting nonlinear system of equations was solved again by a continuation-Newton technique. The temporal discretization of the periodic initial-value problem was effected by an explicit predictor-corrector scheme. They identified another branch of solitary wave solutions of the Benjamin equation, the ‘depression’ solitary waves (resembling analogous solutions of the Euler equations), and tested their stability numerically. They observed that such a wave becomes unstable and resolves into pulses resembling usual (‘elevation’) solitary waves of the Benjamin equation plus small-amplitude dispersive oscillations. This was also supported by a linearized stability of the depression solitary waves carried out in [10].

In a recent paper [11], we made a study of several incremental continuation techniques for approximating solitary waves of the Benjamin equation that satisfy (1.5). (The values of $\alpha$, $\beta$, $\delta$ and $c_s$ were fixed, and $\gamma$ was used as continuation parameter.) A standard pseudospectral (collocation) method yielded the underlying discrete nonlinear system. We found that Newton’s method, combined with a suitably preconditioned conjugate gradient technique for solving the attendant linear system at each Newton iteration, was the generally most efficient technique of implementing the incremental step and produced very accurate approximations of the solitary waves for $0 \leq \gamma < 1$. With this method we also computed other branches of solutions of (1.5), namely multipulse solitary waves, by starting the homotopy path from linear combinations of solitary waves of the KdV equation. We verified the accuracy of these profiles as travelling waves of the Benjamin equation by using them as initial values in a full discretization of the periodic initial-value problem for (1.1) with a modified third-order accurate two-stage DIRK time-stepping technique.

Much of numerical work with spectral-type methods for one-dimensional, nonlocal, nonlinear dispersive wave equations has been centered around the Benjamin-Ono (BO), [4,5], and the Intermediate Long Wave (ILW) equation, [12,13]. Early computational work for these equations was reviewed in [14]; rigorous error estimates for semidiscrete and fully discrete schemes of second-
order temporal accuracy for spectral and pseudospectral (collocation) spatial discretizations were derived e. g. in [15], [16], and [17]. In [18], a Crank-Nicolson time-stepping scheme was coupled with a spatial difference discretization while the nonlocal term was discretized in physical space by the midpoint quadrature formula, interpreted as a discrete convolution and computed by the discrete Fourier transform. This hybrid method was shown to be of second-order accuracy in $L^2$ in space and time.

In the paper at hand we continue our numerical study of the Benjamin equation with a new scheme based on a hybrid finite element-spectral method. We consider the periodic initial-value problem for (1.1) and discretize it in space by the Galerkin method using smooth periodic splines of order $r \geq 3$ on a uniform mesh with meshlength $h$. (Cubic splines, i. e. $r = 4$, are mainly used in the computations.) The nonlocal term is computed using a spectral approximation as described in Section 2. Then, the system of o.d.e.’s representing the semidiscrete problem is discretized in time; we use as a base time-stepping scheme the two-stage, fourth-order accurate, Gauss-Legendre implicit Runge-Kutta method. This scheme has high accuracy and good stability properties and has previously been extensively used for the temporal discretization of stiff partial differential equations with a KdV term, cf. e. g. [20] and its references. We then make a computational study of the accuracy and stability properties of this hybrid scheme.

In Section 3 we review the continuation-conjugate gradient-Newton technique of [11] for generating single and multi-pulse solitary-wave solutions of the Benjamin equation for various values of $\gamma$ with particular attention to values close to 1. We use these numerical profiles as initial conditions in numerical evolution experiments with the hybrid scheme and investigate with various metrics their accuracy as travelling wave solutions of the Benjamin equation. Our conclusion from the numerical experiments of Sections 2 and 3 is that the hybrid scheme yields very accurate and stable approximations of solutions of the Benjamin equation, and in particular of the solitary waves for values of $\gamma \in (0, 1)$ that can be taken quite close to 1.

In Section 4 we make a detailed computational study of overtaking (‘one-way’) collisions of solitary waves of the Benjamin equation and compare the inelastic character of these interactions with the analogous, ‘clean’ interactions in the case of the integrable BO equation. Finally, in Section 5 we explore issues of stability and instability of single-pulse and multi-pulse solitary waves of the Benjamin equation under small and larger perturbations. Our computational study confirms the stability of the single-pulse solitary waves for small and moderate values of $\gamma$ but is inconclusive for cases of $\gamma$ very close to 1. The multi-pulse waves appear to be unstable. Finally, we examine the stability of the ‘depression’ solitary waves and confirm the results of [10] regarding their instability.
In the paper we denote, for integer \( r \geq 0 \), by \( C^r_p \) the periodic functions, on \([-L, L]\) or \([0, 2\pi]\) as the case may be, that belong to \( C^r \), and by \( P_r \) the polynomials of degree at most \( r \). The inner product for real or complex-valued functions in \( L^2 \) is denoted by \((\cdot, \cdot)\) and the associated norm by \(|| \cdot ||\).

Interested readers may find an extended version of the present paper, including additional numerical results, in [22].

2 The hybrid spectral-finite element scheme

We consider the periodic initial-value problem for the Benjamin equation, i.e. for \( t \geq 0 \) we seek a \( 2L \)–periodic real function \( u = u(x, t) \) such that

\[
\begin{align*}
    u_t + \alpha u_x + \beta uu_x - \gamma \mathcal{G} u_{xx} - \delta u_{xxx} &= 0, \quad x \in [-L, L], \quad t > 0, \\
    u(x, 0) &= u_0(x), \quad x \in [-L, L],
\end{align*}
\]

where \( u_0 \) is a given smooth \( 2L \)–periodic function and \( \alpha, \beta, \gamma, \delta \) positive constants. The operator \( \mathcal{G} \) is the Hilbert transform acting on \( 2L \)–periodic functions; for the purposes of this section it will be represented by its principal-value integral form [13]

\[
\mathcal{G} f(x) := \frac{1}{2L} \text{p.v.} \int_{-L}^{L} \cot \left( \frac{\pi(x-y)}{2L} \right) f(y) dy,
\]

where \( f \) is \( 2L \)–periodic. In the sequel we will assume that the solution of (2.1) is sufficiently smooth.

2.1 The semidiscrete hybrid scheme

In this section we assume for simplicity that the problem (2.1) has been transformed onto the spatial interval \( I = [0, 2\pi] \). For integer \( r \geq 3 \) and an even integer \( N \), let \( h = 2\pi/N \), \( x_j = jh, j = 0, \ldots, N \), and consider the finite dimensional spaces

\[
S_N = \text{span} \left\{ e^{ikx} : k \in \mathbb{Z}, -N/2 \leq k \leq N/2 - 1 \right\},
\]

and

\[
S_h = \left\{ \phi \in C^{r-2}_p : \phi|_{[x_j, x_{j+1}]} \in P_{r-1}, 0 \leq j \leq N - 1 \right\}.
\]

The hybrid spectral-finite element approximation \( u_h \) of the solution \( u \) of (2.1) is a real \( S_h \)-valued function \( u_h(t) \) of \( t \geq 0 \) defined by the o.d.e. initial-value
problem
\[(u_{ht}, \chi) + (\alpha u_{hx} + \beta u_h u_{hx}, \chi) + \gamma (P_N G u_{hx}, \chi) + \delta (u_{hxx}, \chi) = 0, \forall \chi \in S_h, t \geq 0, \]
u_h(0) = P_h u_0, \tag{2.3}

where \(P_h, P_N\) are the \(L^2\) projections onto \(S_h\) and \(S_N\), respectively, given for \(w \in L^2\) as
\[(P_h w, \chi) = (w, \chi), \quad \forall \chi \in S_h\]
and
\[(P_N w, \phi) = (w, \phi), \quad \forall \phi \in S_N,\]
where \((\cdot, \cdot)\) is the \(L^2(0, 2\pi)\) inner product. For \(f \in L^2\), \(P_N f\) is represented by
\[P_N f(x) = \sum_{k=-N/2}^{N/2-1} \hat{f}_k e^{ikx},\]
where \(\hat{f}_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx, k \in \mathbb{Z}\) are the Fourier coefficients of \(f\). Note that \((G f)_k = -i \text{sign}(k) \hat{f}_k\) and that \(G\) is antisymmetric in \(L^2\).

2.2 The fully discrete hybrid scheme

We discretize the o.d.e. initial-value problem (2.3) in the temporal variable using the analogous scheme derived in [20] in the case of the generalized KdV equation. (This scheme was also used in [21].) Denoting again by \((\cdot, \cdot)\) the \(L^2(0, 2\pi)\) inner product, we define, for each \(t \in [0, T]\), the map \(F : S_h \rightarrow S_h\) by the equation
\[(F(u_h), \chi) = -[(\alpha u_{hx} + \beta u_h u_{hx}, \chi) + \gamma (P_N G u_{hx}, \chi) + \delta (u_{hxx}, \chi)], \quad \forall \chi \in S_h.\]

Then, (2.3) may be written as
\[u_{ht} = F(u_h), \quad 0 \leq t \leq T, \quad u_h(0) = P_h u_0. \tag{2.4}\]

The initial-value problem (2.4) is stiff. We discretized it in time by the 2-stage Gauss-Legendre implicit Runge-Kutta method, which is fourth-order accurate and has good nonlinear stability properties. It corresponds to the Butcher table
\[
\begin{array}{cccc}
 a_{11} & a_{12} & \tau_1 \\
 a_{21} & a_{22} & \tau_2 \\
b_1 & b_2 & \\
\end{array}
\begin{array}{c}
\frac{1}{4} \\
\frac{1}{4} + \frac{1}{2\sqrt{3}} \\
\frac{1}{2}
\end{array}
\begin{array}{c}
\frac{1}{4} - \frac{1}{2\sqrt{3}} \\
\frac{1}{4} + \frac{1}{2\sqrt{3}} \\
\frac{1}{2}
\end{array}
\begin{array}{c}
1 - \frac{1}{2\sqrt{3}} \\
\frac{1}{2} + \frac{1}{2\sqrt{3}} \\
\frac{1}{2}
\end{array}
\begin{array}{c}
\frac{1}{4} - \frac{1}{2\sqrt{3}} \\
\frac{1}{4} + \frac{1}{2\sqrt{3}} \\
\frac{1}{2}
\end{array}
\end{array}
\]
The resulting fully discrete scheme is the following. Let \( t^n = nk, n = 0, 1, \ldots, M \), where \( T = Mk \). We seek \( U^n \) approximating \( u_h(t^n) \), and \( U^{n,i} \) in \( S_h, i = 1, 2 \), as solutions of the system of nonlinear equations

\[
U^{n,i} = U^n + k \sum_{j=1}^{2} a_{ij} F(U^{n,j}), \quad i = 1, 2, \quad 0 \leq n \leq M - 1, \quad (2.5)
\]

and set

\[
U^{n+1} = U^n + k \sum_{j=1}^{2} b_j F(U^{n,j}), \quad 0 \leq n \leq M - 1, \quad (2.6)
\]

where \( U^0 = u_h(0) \).

The numerical solution of the nonlinear system of equations represented by (2.5) is effected, at each time \( n; 0 \leq n \leq M - 1 \), by a Newton-type method, cf. [20]. The reader is referred to [20] (and of course to [22]) for full details of its implementation. Here it suffices to note that the usual Newton’s method generates approximations \( U^{n,i}_0 \) of \( U^{n,i} \) in \( S_h \) for \( j = 1, \ldots, J_{out} \), called ‘outer iterates’; the required starting values \( U^{n,i}_0 \) are computed by extrapolation of the appropriate order from previous values \( U^m, m \leq n \). In order to avoid forming the Jacobian in Newton’s method for each \( j \) we replace it for each \( n \) by a fixed matrix \( J(U^*) \) whose elements are evaluated at \( U^* = \frac{1}{2}(U^{n,1}_0 + U^{n,2}_0) \), and compute approximations \( U^{n,i, \ell}_j, \ell = 0, 1, \ldots, J_{inn}, \) of \( U^{n,i} \), called ‘inner iterates’, by an iterative scheme that requires solving linear systems with \( J(U^*) \). In the approximate Jacobian \( J(U^*) \) we did not include its part corresponding to the nonlocal term but transferred it to the right-hand side of the systems in order to have a sparse \( J(U^*) \) when a basis of small support is chosen for \( S_h \). (This is the main difference between the present algorithm and that of [20] which corresponds to the KdV case.).

For each \( j, 0 \leq j \leq J_{out} - 1, U^{n,i}_{j+1} \) is approximated by the last inner iterate \( U^{n,i}_{j+1,J_{inn}} \) of the sequence of inner iterates \( U^{n,i, \ell}_j, 0 \leq \ell \leq J_{inn}, J_{inn} \) and \( J_{out} \) are such that

\[
\left( \sum_{k=1}^{2} \| U^{n,k, \ell+1}_j - U^{n,k, \ell}_j \|_{\ell_2}^2 \right)^{1/2} \leq \varepsilon,
\]

and

\[
\left( \sum_{k=1}^{2} \| U^{n,k}_j - U^{n,k}_j \|_{\ell_2}^2 \right)^{1/2} \leq \varepsilon,
\]

where \( \| v \|_{\ell_2} \) denotes the Euclidean norm of the coefficients of \( v \in S_h \) with respect to its basis, and \( \varepsilon \) is usually taken to be \( 10^{-10} \). Typically \( J_{out} \) is equal to one and \( J_{inn} \) is less or equal to four.

The integrals involving the local terms in (2.3) are computed in general using the 5-point Gauss-Legendre quadrature rule in each spatial interval. The inner product \( \langle P_N G u_{hx}, \chi_x \rangle \) involving the nonlocal term is approximated by
\((I_N \mathcal{G} u_{h,x}, \chi_x)\) where the Fourier interpolant \(I_N\) is defined as

\[
I_N v(x) = \sum_{k=-N/2}^{N/2-1} \hat{v}_k e^{ikx},
\]

and \(\hat{v}_k\) denote the discrete Fourier coefficients of \(v\), computed by the Fast Fourier Transform. The inner product involving \(I_N\) is approximated by the trapezoidal quadrature rule, which is very accurate for periodic functions.

In the sequel, we shall use the fully discrete scheme described above with the \(C^2\) cubic splines \((r = 4)\) as the finite element subspace \(S_h\), equipped with the usual periodic B-spline basis. We shall refer to this method as the hybrid scheme/method.

We checked numerically the orders of convergence of the hybrid scheme as follows. Due to lack of analytical formulas for solutions of the Benjamin equation we considered the nonhomogeneous equation

\[
u_t + u u_x + \mathcal{G} u_{xx} + \frac{1}{2} u_{xxx} = f(x,t), \quad (x,t) \in [-1,1] \times [0, T], \quad (2.7)
\]

with periodic boundary conditions and

\[
f(x,t) = e^t \left( \sin(\pi x) + \frac{\pi}{2} e^t \sin(2\pi x) + \left( \pi^2 - \frac{\pi^4}{2} \right) \cos(\pi x) \right),
\]

so that the specific equation has a solution \(u(x,t) = e^t \sin(\pi x)\). We solved it numerically up to \(T = 1\) and we computed the discrete maximum error on the quadrature nodes and the normalized \(L^2\) error defined as \(\|e_h(\cdot, t^n)\| / \|e_h(\cdot, 0)\|\), where \(e_h = u - U\).

<table>
<thead>
<tr>
<th>(N)</th>
<th>(M)</th>
<th>(L^\infty) Error</th>
<th>Rate</th>
<th>(L^2) Error</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1000</td>
<td>0.2630 × 10^{-1}</td>
<td>–</td>
<td>0.4263 × 10^{-1}</td>
<td>–</td>
</tr>
<tr>
<td>8</td>
<td>1000</td>
<td>0.2654 × 10^{-2}</td>
<td>3.309</td>
<td>0.4125 × 10^{-2}</td>
<td>3.370</td>
</tr>
<tr>
<td>16</td>
<td>1000</td>
<td>0.1916 × 10^{-3}</td>
<td>3.793</td>
<td>0.2686 × 10^{-3}</td>
<td>3.941</td>
</tr>
<tr>
<td>32</td>
<td>1000</td>
<td>0.1243 × 10^{-4}</td>
<td>3.945</td>
<td>0.1693 × 10^{-4}</td>
<td>3.988</td>
</tr>
<tr>
<td>64</td>
<td>1000</td>
<td>0.7863 × 10^{-6}</td>
<td>3.983</td>
<td>0.1060 × 10^{-5}</td>
<td>3.997</td>
</tr>
<tr>
<td>128</td>
<td>1000</td>
<td>0.5068 × 10^{-7}</td>
<td>3.956</td>
<td>0.6636 × 10^{-7}</td>
<td>3.998</td>
</tr>
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</table>

Table 1
Spatial rates of convergence

Tables 1 and 2 show the numerical spatial and temporal rates of convergence of
Table 2
Temporal rates of convergence

<table>
<thead>
<tr>
<th>N</th>
<th>M</th>
<th>$L^\infty$ Error</th>
<th>Rate</th>
<th>$L^2$ Error</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>20</td>
<td>$0.1301 \times 10^{-3}$</td>
<td>–</td>
<td>$0.1249 \times 10^{-3}$</td>
<td>–</td>
</tr>
<tr>
<td>40</td>
<td>40</td>
<td>$0.1866 \times 10^{-4}$</td>
<td>2.802</td>
<td>$0.1678 \times 10^{-4}$</td>
<td>2.896</td>
</tr>
<tr>
<td>80</td>
<td>80</td>
<td>$0.3888 \times 10^{-5}$</td>
<td>2.262</td>
<td>$0.3733 \times 10^{-5}$</td>
<td>2.169</td>
</tr>
<tr>
<td>160</td>
<td>160</td>
<td>$0.5566 \times 10^{-6}$</td>
<td>2.804</td>
<td>$0.5465 \times 10^{-6}$</td>
<td>2.772</td>
</tr>
<tr>
<td>320</td>
<td>320</td>
<td>$0.7289 \times 10^{-7}$</td>
<td>2.933</td>
<td>$0.7101 \times 10^{-7}$</td>
<td>2.944</td>
</tr>
<tr>
<td>640</td>
<td>640</td>
<td>$0.9443 \times 10^{-8}$</td>
<td>2.948</td>
<td>$0.8994 \times 10^{-8}$</td>
<td>2.981</td>
</tr>
</tbody>
</table>

The error for this experiment and norms at $t = T = 1$. Here $N$ is the number of spatial intervals and $M = T/k$. We observe that the spatial rate is practically optimal (four) and that the temporal rate approximates the value $p = 3$ as $N, M$ increase. (For this experiment, with the tolerance set at $\epsilon = 10^{-10}$, the number of Newton iterations $J_{\text{out}}$ came out to be always one and $J_{\text{inn}}$ varied in general between one and four provided $k$ and $h$ were sufficiently small.) The theoretical order of accuracy of the two-stage Gauss-Legendre RK method is of course equal to four. In our case, the loss of one order of temporal accuracy is apparently caused by the presence of the nonlocal term: Recall that in the approximate Jacobian $J(U^*)$ we did not include its part corresponding to the nonlocal term, but transferred it to the right-hand side in order to retain sparsity in the operators on the left when a basis of small support is chosen for $S_h$. This efficiency consideration renders the scheme explicit with respect to the nonlocal term and linearly implicit with respect to the rest of the terms in the equation, and causes the loss of temporal accuracy by one order.

Hence, the major difficulty in the numerical solution of the Benjamin equation, as compared e. g. to that of the KdV equation by a similar scheme, is the presence of the nonlocal term. Both equations lead to semidiscretizations which are highly stiff o.d.e. systems. Consequently, their temporal discretizations should be effected by implicit, high order, nonlinearly stable schemes, such as the 2-stage Gauss-Legendre RK scheme, for example: When a finite element spatial discretization is used, preservation of sparsity leads to an iterative implementation of Newton’s method. The temporal accuracy of the resulting linearized scheme is equal to four in the case of the KdV equation (see [20], Table 3), but is reduced to three in the case of the Benjamin equation as seen above, not as a result of order reduction due to stiffness, but for reasons of numerical efficiency, i. e. for solving sparse systems.

We should mention that we did not detect any need for a stability bound on $k/h$ for the hybrid scheme. (Values as high as $k/h = 8$ were tried.) Of course accuracy is reduced as $k$ increases and so in the numerical experiments of
sections 3-5 \( k/h \) was taken much smaller.

In [22] the interested reader may find an experimental error analysis of the hybrid scheme applied to the BO equation (i.e. in the case \( \delta = 0 \).) For this problem the hybrid scheme converges under a stability condition of the form \( k = O(h^2) \).

2.3 Validation of the hybrid method

We now present the results of some numerical tests that we performed in order to validate further the hybrid method.

In our first experiment we simulate the propagation of a periodic travelling-wave solution of the Benjamin-Ono equation that was used in [18]. This solution resembles a solitary wave and is given by the formula

\[
    u(x, t) = \frac{2c_s A^2}{1 - \sqrt{1 - A^2 \cos(c_s A(x - c_s t))}},
\]

where \( A = \frac{x}{c_s L} \). This is a 2\( L \)-periodic solution of the BO with coefficients \( \alpha = \delta = 0, \beta = \gamma = 1 \) in (1.1). We approximated it by the hybrid method in two runs with \( N = 256 \) and \( k = 0.01 \) and with \( N = 1024 \) and \( k = 5 \times 10^{-4}, \) respectively, on the interval \([-L, L]\) with \( L = 15 \) and \( c_s = 0.25 \) for \( 0 \leq t \leq 100 \), using (2.8) at \( t = 0 \) as initial condition. The numerical solution is shown in Figure 1 at \( t = 0, 10 \) and 100. (The two numerical profiles coincided within graph thickness.)

![Figure 1. Numerical evolution of the periodic-travelling wave solution (2.8) of the Benjamin-Ono equation.](image)

In this example, the normalized \( L^2 \) error, defined as \( \max_n \frac{\|u^n - U^n\|}{\|U^0\|} \), was of \( O(10^{-7}) \) for \( N = 256 \) and of \( O(10^{-11}) \) for \( N = 1024 \). In both cases, the \( L^2 \) norm of the numerical solution was equal to 2.50662827463 while the Hamiltonian
(invariant $E(u)$ given by (1.4)) was equal to $-0.47344593881$. (Both were preserved for $0 \leq t \leq 100$ up to the twelve significant digits shown.) In addition, we computed for each $t^n$ several other types of errors that are relevant in assessing the accuracy of approximation of solitary-type waves, cf. [20], [21]. These were: (i) The (normalized) amplitude error $AE(t^n) = \frac{u_{\text{max}} - U^n(x^*)}{u_{\text{max}}}$, where $u_{\text{max}}$ is the maximum value of the exact solution and $x^*$ is the point where the approximate solution $U^n$ achieves its maximum, found by applying Newton’s method to compute the root of the equation $\frac{d}{dx}U^n(x) = 0$ that corresponds to the maximum of $U^n$. (ii) The $L^2$ (normalized) shape error defined as $SE(t^n) = \inf_{\tau} ||U^n - u(\cdot, \tau)||/||u_0||$, computed as $SE(t^n) = \xi(\tau^*)$, where $\tau^*$ is the point near $t^n$ (found by Newton’s method) where $\frac{d}{dx}(\xi^2) = 0$, with $\xi(\tau) = ||U^n - u(\cdot, \tau)||/||u_0||$. (iii) The associated phase error $PE(t^n) = \tau^* - t^n$. Figure 2 shows these errors as functions of $t^n$ up to $T = 100$, for $N = 256$ and $N = 1024$. The speed $c_s = 0.25$ of the travelling wave was preserved for $N = 256$ to 6 digits up to $t = 50$ and to 5 digits up to $t = 100$, while for $N = 1024$ up to at least 7 digits up to $t = 100$.

In a second validation experiment we computed the evolution of a solitary wave for the Benjamin equation (2.1) with $\gamma = 0.5$ (all other coefficients being
equal to one) with $L = 128$ up to $T = 100$. The initial solitary-wave profile was generated with high accuracy by numerical continuation with the CGN method as explained in [11] and in Section 3 of the present paper. Table 3 presents the parameters and results of this run.

Table 3
Errors at $T = 100$ and parameters for the hybrid method. Solitary wave, Benjamin equation, $\gamma = 0.5$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$k$</th>
<th>$L^2$ error</th>
<th>$H^1$ error</th>
</tr>
</thead>
<tbody>
<tr>
<td>2048</td>
<td>$1 \times 10^{-2}$</td>
<td>$0.4398 \times 10^{-6}$</td>
<td>$0.3664 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

In the table the $L^2$ and shape errors are normalized as explained earlier. The (normalized) $H^1$ error, defined analogously, is a useful error metric for oscillatory profiles such as the solitary waves of the Benjamin equation.

In our third experiment we solved the Benjamin equation in the form $u_t + uu_x + G u_{xx} + u_{xxx} = 0$ for $x \in [-300, 300]$ up to $T = 100$ using as initial condition the Gaussian $u(x, 0) = 2e^{-(x/4)^2}$. As expected, [9], the initial profile resolves itself into a series of solitary waves. As Figure 3 shows, by $T = 100$ three solitary waves have appeared, followed by a dispersive tail.

We computed the solution with the hybrid scheme for various values of the discretization parameters $h$ and $k$ starting from $h = 0.1, k = 0.1$ and reducing $h$ and/or $k$. Some of the profiles produced by the hybrid runs are shown in Figure 3; they all coincide within graph thickness.

A pseudospectral spatial discretization for all terms of the p.d.e., coupled with a suitable time-stepping procedure, can also of course be used for the numerical solution of the periodic initial-value problem for the Benjamin equation. In [11], for example, we used a pseudospectral scheme coupled with a third-order accurate SDIRK procedure for the temporal discretization. For the purposes of the paper at hand we compared the performance of the hybrid scheme with that of a fully discrete method obtained in the standard way, cf. [19], by first discretizing in space by Fourier collocation, using the linear part of the resulting system of o.d.e.’s for the Fourier coefficients in an integrating factor, and discretizing the new system in time by the explicit, fourth-order accurate, four-stage classical RK scheme. Full details about this scheme and its comparison with the hybrid one may be found in [22]. Our general conclusions from that comparison were that for comparable $L^2$ accuracy the spectral method was faster, but that the hybrid scheme conserved the invariant $E$ to more digits and required less stringent stability conditions. (The choice of a different time integrator might remedy these observed drawbacks of the spectral scheme.)
Our main motivation for constructing a hybrid spectral-finite element scheme for the Benjamin equation was to use it as a first step towards developing similar hybrid schemes for other nonlinear dispersive wave equations and systems with nonlocal terms and discretizing on different meshes the local and nonlocal terms of the p.d.e.’s. This could possibly yield efficient schemes for studying problems that develop singularities in one spatial dimension, and nonlocal problems in 2D, where a general triangulation could be used for the local terms and a coarser uniform mesh for the nonlocal ones.

3 Generation and propagation of solitary waves

In this section we first review the numerical technique that we used to generate solitary-wave solutions of the Benjamin equation. These solitary-wave profiles were taken as initial values for the hybrid time-stepping method and integrated forward in time. We present in some detail the temporal evolution of various error metrics suitable for assessing the accuracy of these numerically generated travelling waves.
As was already mentioned in the Introduction, the solitary waves of the Benjamin equation are travelling-wave solutions of (1.1) of the form \( u(x, t) = \varphi(x - c_s t) \), \( c_s > 0 \), such that \( \varphi \) and its derivatives tend to zero as \( \xi = x - c_s t \) approaches \( \pm \infty \). Consequently, \( \varphi \) satisfies the equation (1.5), from which, taking Fourier transforms, we obtain

\[
(-c_s + \alpha - \gamma |k| + \delta k^2)\hat{\varphi} + \frac{\beta}{2} \hat{\varphi}^2 = 0, \quad k \in \mathbb{R},
\]

where \( \hat{\varphi}(k) \) is the Fourier transform of \( \varphi \). If we discretize this equation assuming periodic boundary conditions on \( [-L, L] \) and using the formula

\[
\mathcal{F}\left(\mathcal{F}^{-1}\left(\hat{\varphi}^2\right)\right),
\]

where \( \mathcal{F} \) is the discrete Fourier transform to compute the convolution, we obtain the \( N \times N \) nonlinear system of equations

\[
(-c_s + \alpha - \gamma |k| + \delta k^2)\varphi^N_k + \frac{\beta}{2} \left(\varphi^N \ast \varphi^N\right)_k = 0,
\]

for \( k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1 \), where \( \varphi^N \) is the approximation of \( \varphi \) in \( S_N \) and \( \varphi^N_k \) denotes its \( k \)th Fourier coefficient.

To solve (3.1) we use an incremental continuation technique with respect to the parameter \( \gamma \), following e. g. [3]. For a fixed set of constants \( \alpha, \beta, \delta, c_s \) in (3.1) we consider a homotopic path \( \gamma_0 = 0 < \gamma_1 < \ldots < \gamma_M = \gamma \) and solve (3.1) successively for \( \gamma_0, \gamma_1, \ldots, \gamma_M \) with an iterative nonlinear solver, using for each \( j \) the numerical solution for \( \gamma = \gamma_{j-1} \) as an initial guess in solving for \( \gamma = \gamma_j \). (The starting value \( \gamma_0 = 0 \) of the path corresponds to the KdV equation for which exact solitary-wave solutions are available.) The incremental continuation technique has the added advantage that it produces a series of solitary waves for varying values of \( \gamma \) with a fixed speed \( c_s \).

The nonlinear system solver that we used to generate the solution of (3.1) for each \( \gamma_j \) was Newton’s method, wherein the attendant linear systems were solved by an inner iteration performed by the preconditioned conjugate gradient technique. The resulting iterative scheme, called CGN in the sequel, was described in detail in [11], where it was also compared with several other nonlinear solvers and found to be more efficient, with respect to a variety of metrics, for approximating solutions of (3.1). We refer the reader to [11] for the implementation of CGN; let us just mention that for the computations in the present paper the Newton iteration was terminated when the quantity \( \frac{\|\varphi^N_{[\nu]} - \varphi^N_{[\nu-1]}\|}{\|\varphi^N_{[\nu]}\|} \) became less than \( 10^{-15} \). (Here \( \varphi^N_{[\nu]} \) is the \( \nu \)-th Newton iterate approximating \( \varphi^N \). The preconditioned conjugate-gradient inner iter-
ation was terminated when \( ||R^{(i)}||_M/||R^{(0)}||_M \) became less than \( 10^{-2} \). Here \( R^{(i)} \) is the residual defined in the standard way in the conjugate-gradient algorithm, and the norm \( || \cdot ||_M \) is the weighted \( L^2 \) norm \((\cdot, M^{-1}\cdot)^{1/2}\), where \( M = cI - \partial_{xx} \) is the preconditioning operator that we used; its action in Fourier variables is \( c + k^2 \) and the value \( c = 0.275 \) was found to be optimal in computations. The number of CG inner iterations needed to reach the threshold defined above varied between 3 and 10 typically.

![Figure 4](image1.png)

**Figure 4.** Solitary waves of the Benjamin equation for various values of \( \gamma, c_s = 0.45 \).

![Figure 5](image2.png)

**Figure 5.** Solitary waves of the Benjamin equation for various values of \( \gamma, c_s = 0.75 \).

Using this algorithm we produced solitary waves of the Benjamin equation in \([-256, 256]\) with \( N = 4096 \) using \( \gamma_j = j\Delta\gamma, j = 1, \ldots, 99 \), with \( \Delta\gamma = 0.01 \) and an exact solitary wave of the KdV equation at \( \gamma_0 = 0 \). In all computations
we took $\alpha = \beta = \delta = 1$. Figure 4 shows the computed profiles of the solitary waves for $c_s = 0.45$ and $\gamma = 0, 0.5, 0.9, 0.99$, while Figure 5 shows the solitary waves corresponding to $c_s = 0.75$ for the same values of $\gamma$. As is well-known, the number of oscillations increases with $c_s$ and $\gamma$.

We also constructed with the same technique *multi-pulse* solitary waves by starting at $\gamma_0 = 0$ with a superposition of translated KdV solitary waves as explained in [11]. Two-pulse and three-pulse such solitary waves are shown for $\gamma = 0.1, 0.5$, and $0.9$ and $c_s = 0.75$ in Figure 6.

![Figure 6. Two-pulse (a,b,c) and three-pulse (b,d,f) solitary waves of the Benjamin equation for $\gamma = 0.1, 0.5, 0.9, c_s = 0.75$](image)

As a measure of the accuracy of the CGN method for approximating the solution of (3.1) for each value of $\gamma$ we computed the $L^2$ norm of the residual $r$, whose $k$-th Fourier component is defined as the left-hand side of (3.1) with $\phi^N$ replaced by its numerical approximation. The value of $||r||$ for single-pulse and two-pulse and three-pulse solitary waves as a function of $\gamma$ remained smaller than $5 \times 10^{-13}$ but in general the residual increases as $\gamma$ approaches one, a fact that reflects the difficulty in solving the nonlinear systems with $\gamma$ close to one.
The above-described technique for generating solitary waves of the Benjamin equation was found to be more accurate, compared to iterative ‘cleaning’, cf. e. g. [9], wherein one isolates and ‘cleans’ iteratively solitary waves that are produced by resolution of suitable initial data, and which works well in case the solitary waves decay exponentially. In the case of the Benjamin equation, for which the solitary waves are known to decay quadratically, [2,7], we found that even for large spatial computational intervals it was very hard to make the values at the boundaries of the solitary waves produced by iterative cleaning less than $O(10^{-5})$. This small truncation error produced dispersive oscillations of the same order of magnitude that very fast polluted the ensuing solution when such solitary-wave profiles were used as initial values in evolution studies. Of course, for solitary waves produced by iterative cleaning one does not have a priori knowledge of their speed, so it is not easy to design systematic experiments with families of solitary waves of varying speed.

We used the numerical solitary waves that we constructed as initial values $u_0$ and integrated in time the Benjamin equation using the fully discrete hybrid scheme described in Section 2. As a further test of the accuracy of the numerical solitary waves and the time-stepping technique we computed several invariants of the evolution and various pertinent error measures. In all cases we used the spatial interval $[-256, 256]$ and $N = 4096$ and we integrated the equation up to $T = 300$.

Table 4 shows the values of the $L^2$ norm, of the invariant $K = I + E$, where $I$ and $E$ are discrete versions of the quantities defined in (1.3) and (1.4), respectively, and of the amplitude of the numerically propagated single-pulse solitary waves with $c_s = 0.75$ for various values of $\gamma$. The digits shown for each quantity were conserved up to $T = 300$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$L^2$-norm</th>
<th>$K$</th>
<th>amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.6096361661</td>
<td>1.09624383030</td>
<td>-0.7183404</td>
</tr>
<tr>
<td>0.5</td>
<td>1.08290587306</td>
<td>0.48258984490</td>
<td>-0.541174</td>
</tr>
<tr>
<td>0.9</td>
<td>0.44162186544</td>
<td>0.07565402212</td>
<td>-0.2280941</td>
</tr>
<tr>
<td>0.95</td>
<td>0.33588124247</td>
<td>0.04319622837</td>
<td>-0.165667</td>
</tr>
<tr>
<td>0.99</td>
<td>0.2429264136</td>
<td>0.022247817281</td>
<td>-0.090357</td>
</tr>
</tbody>
</table>

Table 4
Conserved quantities for numerical evolution up to $T = 300$ of single-pulse solitary waves of speed $c_s = 0.75$ for various values of $\gamma$.

Table 5 shows the conserved digits of the same quantities for the analogous propagation experiment with two-pulse and three-pulse solitary waves with $\gamma = 0.5$. 

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Table 5
Conserved quantities for numerical evolution up to $T = 300$ of multi-pulse solitary waves of speed $c_s = 0.75$ for $\gamma = 0.5$

<table>
<thead>
<tr>
<th>Number of pulses</th>
<th>$\gamma$</th>
<th>$L^2$-norm</th>
<th>$K$</th>
<th>amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.5</td>
<td>1.6419433913</td>
<td>1.1164182800</td>
<td>−0.582995</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>2.0816580537</td>
<td>1.800497679</td>
<td>−0.618111</td>
</tr>
</tbody>
</table>

In these computations the quantity $K$ was defined at $t^n$ as

$$K := \frac{1}{2} \int_{-L}^{L} \left( U^2 + \frac{\beta}{3} U^3 - \gamma U U_N U_x \right) dx,$$

where $U = U^n$, the integrals being evaluated by numerical quadrature as described in Section 2.

In Figure 7 we show the $L^2$ (normalized) shape error of the propagating numerical single-pulse solitary wave for $c_s = 0.75$ and $\gamma = 0.95$ and $\gamma = 0.99$, as function of $t^n$. This quantity is defined as

$$SE(t^n) = \inf_{\tau} \| U^n - \varphi_h(\cdot - c_s \tau) \| / \| \varphi_h \|,$$

where $\varphi_h = P_h \varphi^N = U^0$ is the $L^2$-projection on $S_h$ of the numerically generated initial solitary wave $\varphi^N$. As in section 2, $SE(t^n)$ is again computed as $\xi(\tau^*)$, where $\tau^*$ is the point near $t^n$ (found by Newton’s method) where $\frac{d}{d\tau} \xi^2(\tau^*) = 0$, with $\xi(\tau) := \| U^n - \varphi_h(\cdot - c_s \tau) \| / \| \varphi_h \|$. We observed that for $\gamma \leq 0.95$ the shape errors increased with $\gamma$ and eventually stabilized as $t$ increased. However, for $\gamma = 0.99$ a linear temporal growth was observed. (The magnitudes of the shape error ranged from $O(10^{-8})$ for $\gamma = 0.1$ to $O(10^{-6})$ for $\gamma = 0.99$.) Figure 8 shows the analogous graphs for the phase error, defined...
as \( PE(t^n) = \tau^* - t^n \). The phase errors increase linearly with \( t \) and with \( \gamma \) for fixed \( t \), ranging from \( O(10^{-7}) \) when \( \gamma = 0.1 \) to \( O(10^{-5}) \) when \( \gamma = 0.99 \) at \( t = 300 \).

![Figure 8. Phase error of the numerical propagation of single-pulse solitary waves with \( c_s = 0.75 \) and \( \gamma = 0.95, 0.99 \).](image)

Finally, we computed the relative speed error of the simulations, defined as \((C^n - c_s)/c_s\), where \( C^n = (x^*(t^n + \delta t) - x^*(t^n))/\delta t \) and \( x^* \) an approximation of the center of the pulse, i.e., the position of its most negative excursion. When we chose \( \delta t = 1 \) the absolute values of the specific error never exceeded \( 5 \times 10^{-7} \) for all \( \gamma \); the mean value of the speed remained constant during the computations.

We observed a similar qualitative behaviour of all these error indicators in simulations of numerically generated travelling two-pulse and three-pulse solitary waves. (For \( c_s = 0.75 \) and \( \gamma = 0.5 \) the shape errors were of \( O(10^{-7}) \) while the phase errors of \( O(10^{-5}) \) at \( t = 300 \).)

In conclusion, the outcome of the numerous tests (performed in this and the preceding section) of the validity and accuracy of the numerical technique for generating initial solitary-wave profiles and of the fully discrete hybrid scheme that was used for their numerical evolution, give us enough confidence to use these schemes in the study of interactions and stability of solitary waves of the Benjamin equation to be undertaken presently.

4 Overtaking collisions of solitary waves

In this section we study in some detail, by computational means using the hybrid method, overtaking collisions of solitary waves of the Benjamin equation. For a given value of \( \gamma \in (0, 1) \) solitary waves with smaller (absolute) amplitude (i.e., a smaller in absolute value maximum negative excursion) have
larger speed and will consequently overtake solitary waves with larger (absolute) amplitude, which are slower. The solitary waves interact nonlinearly and emerge largely unchanged; their interaction is inelastic, i.e. it is accompanied by the production of a small amplitude dispersive tail since the Benjamin equation does not appear to be completely integrable, as already noted in \([9]\) where results of a simulation of an overtaking collision for solitary waves of the Benjamin equation have been shown.

To set the stage we first present, as a benchmark, the results of a simulation with the hybrid method of an overtaking collision of two solitary waves of the BO equation. The initial solitary waves (cf. (2.8)) had amplitudes \(A_1 = 4, A_2 = 1\) and corresponding speeds \(c_{s,1} = 2\) and \(c_{s,2} = 1.25\), and were centered at \(x_{0,1} = -100\) and \(x_{0,2} = 100\), respectively. The computation was effected with \(N = 4096\) and \(k = h/20\) on \([-256, 256]\), and produced the evolution depicted in Figure 9 at selected instances of \(t \in [0, 400]\). The two solitary waves interact elastically around \(t = 265\). During the interaction there always are two distinct peaks present. No artificial oscillations accompany the numerical solution after the interaction.

![Figure 9. Overtaking collision of two solitary waves of the Benjamin-Ono equation.](image-url)

We now turn to the simulations of overtaking collisions of pairs of solitary waves of the Benjamin equation. We studied such collisions for various values
of $\gamma$; we present here the results for $\gamma = 0.1$ and $\gamma = 0.99$. For all cases we used the hybrid method on the spatial interval $[-512, 512]$ with $h = 0.125$ and $k = 0.02$ and constructed initial solitary-wave profiles of various speeds (centered at $x_1 = 256$ and $x_2 = -256$) by the procedure described in Section 3.

Figure 10 shows several temporal instances of the overtaking collision of two solitary waves of speeds $c_{s,1} = 0.45$ and $c_{s,2} = 0.75$ in the case $\gamma = 0.1$. (During this simulation the $L^2$ norm of the solution was $\|u\| = 3.387194802$, and the value of the quantity (3.2) was $K = 4.04751039$ up to $T = 3000$.) The faster solitary wave overtakes the slower and they interact nonlinearly with two peaks always present during the interaction. The collision produces a dispersive tail (see Figure 10(g)), a fact suggesting that the Benjamin equation is not integrable. Note that the dispersive tail precedes the solitary waves being of smaller amplitude and hence faster in our framework. Figure 11 shows some details of the interaction: In (a) the maximum negative excursion of the solution is plotted versus time. In (b)--a magnification of (a)--one may observe how the maximum negative excursion of the faster wave approaches asymptotically its initial value. The paths of the solitary waves are plotted in

![Figure 10](image_url)
Figure 11. Overtaking collision of solitary waves of the Benjamin equation for \( \gamma = 0.1, c_{s,1} = 0.45, c_{s,2} = 0.75 \). Evolution of Figure 10. (a): Temporal evolution of the maximum negative excursion of the solution. (b) Magnification of (a). (c): Paths of solitary waves. The dotted lines would be the paths if no interactions occurred.

(c): The faster wave is shifted slightly forward and the slower backward after the interaction.

(In an analogous simulation of the overtaking collision of two solitary waves of the Benjamin equation of initial speeds \( c_{s,1} = 0.25 \) and \( c_{s,2} = 0.85 \), again for \( \gamma = 0.1 \), we observed that the larger difference of the speeds of the solitary waves apparently causes the formation of a single peak momentarily during the interaction. Otherwise the details of the overtaking collision are qualitatively the same with those in Figures 10-11.)

We noticed that the collisions became harder to simulate for \( \gamma > 0.9 \). Figure 12 shows the interaction of two solitary waves of speeds \( c_{s,1} = 0.45 \) and \( c_{s,2} = 0.75 \) in the case \( \gamma = 0.99 \). The \( L^2 \) norm was preserved to ten digits (it was equal to 1.532051456) up to \( t = 3000 \), but \( K = 6.821038 \) was preserved to 7 digits, reflecting the increased difficulty of the computation. It is not clear whether the small oscillations in front of the smaller, highly oscillatory solitary wave in Figure 12(g) at \( t = 2900 \) belong to a dispersive tail or are numerical artifacts or somehow indicate that the smaller wave has not stabilized after the interaction. We observe that after about \( t = 2500 \) as shown in Figure 13 in which the maximum negative excursion of the solution is plotted versus time, after achieving again its pre-interaction value, the maximum negative excursion of the slower wave starts oscillating as it interacts with the dispersive tail or, possibly, as numerical or real instability of the solitary waves sets in.

We also performed numerical experiments simulating overtaking collisions in-
Figure 12. Overtaking collision of solitary waves of the Benjamin equation for \( \gamma = 0.99, c_{s,1} = 0.45, c_{s,2} = 0.75 \).

Figure 13. Overtaking collision of solitary waves of the Benjamin equation for \( \gamma = 0.99, c_{s,1} = 0.45, c_{s,2} = 0.75 \). Graphs analogous to (a) and (b) of Figure 11.

Involving multi-pulse solitary waves of the Benjamin equation. Figures 14 and 15 show such an interaction of a fast two-pulse solitary wave of speed \( c_{s,2} = 0.75 \) with a slower single-pulse wave with \( c_{s,1} = 0.45 \) for \( \gamma = 0.5 \). During this simulation we observed that \( \|u\| = 2.873492446, K = 2.8836586 \) up to \( t = 3000 \). After the interaction the waves separate and there is evidence of a dispersive tail, but the two-pulse wave has not apparently recovered its shape and initial amplitude by \( t = 3000 \). The same is true for the single-pulse wave whose maximum negative excursion has not returned to its initial value by \( t = 3000 \).
Figure 14. Overtaking collision of a two-pulse and an ordinary solitary wave of the Benjamin equation for $\gamma = 0.5, c_{s,1} = 0.45, c_{s,2} = 0.75$.

Figure 15. Overtaking collision of solitary waves for the Benjamin equation for $\gamma = 0.5, c_{s,1} = 0.45, c_{s,2} = 0.75$, evolution of Figure 14. Maximum negative excursion of the solution versus time.

as Figure 15 indicates.

5 Stability of solitary waves

In this section we first study by computational means the stability of single-pulse and multi-pulse solitary waves of the Benjamin equation under small
As was mentioned in the Introduction, a theory of stability of single-pulse waves was outlined in [2] and a complete proof for small $\gamma$ was given in [3]. Another proof, valid for all $\gamma \in [0,1)$, of stability in a weaker sense was given in [8].

We start with the single-pulse case. Figure 16(a)–(d) shows the evolution, on the spatial interval $[-2048, 2048]$ with $h = 0.0625$ and $k = 0.02$, ensuing from a single-pulse solitary wave with $\gamma = 0.5$ and $c_s = 0.75$, centered at $x_0 = 0$, when it is perturbed by a multiplicative factor $r = 1.1$. As expected, the perturbed solitary wave evolves into a new one of slightly larger maximum negative excursion plus a preceding dispersive tail. Figure 16(e) shows the evolution of the maximum negative excursion of the solitary wave from its initial value $-0.59526$ to its eventual value which is equal to $-0.60523$. We also simulated the evolution of a perturbed solitary wave corresponding to $\gamma = 0.99$. Figure 17(a)–(d) shows this evolution. The initial solitary wave had $c_s = 0.75$ and was perturbed by a multiplicative factor of $r = 1.2$. (The computation was effected on $[-1024, 1024]$ with $h = 0.0625, k = 0.02$ up to $T = 1000$.) The wave radiates forward a small-amplitude oscillatory wavetrain.

![Figure 16](image)

**Figure 16.** Evolution of a perturbed single-pulse solitary wave of the Benjamin equation ($\gamma = 0.5$). (b) and (d) are magnifications of (a) and (c), respectively. (e): Evolution of the maximum negative excursion of the solution.
which has not separated from the main wave up to $T = 1000$. This fact, and also the temporal variation of the maximum negative excursion of the wave (Figure 17(e)) which has not achieved an asymptotic state by $t = 1000$, does not allow us to reach a conclusion about the stability of solitary waves for $\gamma = 0.99$. The wave may be unstable and keep radiating small-amplitude oscillations for all $t$ or may stabilize into a nearby solitary wave after very long time.

![Figure 17. (a)-(c) Evolution of a perturbed solitary wave of the Benjamin equation ($\gamma = 0.99$). ((b) and (d) are magnifications of (a) and (c), respectively.) (e): Evolution of the maximum negative excursion of the solution.](image)

We turn now to a computational stability study of a two-pulse solitary wave. We took as initial condition a two-pulse solitary wave in the case $\gamma = 0.5$ and perturbed it asymmetrically multiplying it by a factor $r(\tanh x + 1) + 1$ with $r = 0.05$. Figure 18 shows the evolution that ensues. (The computation was done on $[-1024, 1024]$ up to $T = 1000$ using $h = 0.0625, k = 0.02$.) The perturbed two-pulse wave radiates forward the usual small-amplitude oscillatory wavetrain. We observe that its two negative peaks oscillate exchanging heights in a periodic-like manner (Figure 19(a)), while their distance is also oscillating apparently periodically (Figure 19(b)). This ‘dance’ of the twin peaks went on up to the end of our computation at $t = 1000$, but it is unlikely to continue
unaltered for ever due to the apparent constant shedding of radiation.

In a related numerical experiment, whose outcome is shown in Figure 20, we perturbed the same initial two-pulse solitary wave with a larger asymmetric factor ($r$ was taken now to be 0.4) of the same form as above. (All computational parameters remained the same.) After a brief initial dancing phase (up

Figure 18. Evolution of a perturbed two-pulse solitary wave of the Benjamin equation. ($\gamma = 0.5$. ((b) and (f) are magnifications of (a) and (e), respectively.))

Figure 19. (a): Amplitudes (maximum negative excursions) of the two negative peaks of the perturbed two-pulse solitary wave of Figure 18, and (b): Distance between the two peaks, as functions of $t$.

to about $t = 40$) accompanied by radiation, we observed that two single-pulse
solitary waves were generated. Figure 21 shows the evolution of the maximum negative excursions of the two negative peaks up to $T = 1000$.

Figure 20. (a)–(e): Evolution of a more perturbed two-pulse solitary wave of the Benjamin equation ($\gamma = 0.5$).

Figure 21. Maximum negative excursions of the two negative peaks as functions of $t$.

We conclude then that the effect of the larger perturbation is apparently to accelerate the end of the dance and initiate resolution into solitary waves.
In [9], Kalisch and Bona describe numerical experiments in which they observed resolution into solitary waves for the Benjamin equation with initial Gaussian profiles of the form $Ae^{-(x/\lambda)^2}$. As $\lambda$ was increased the emergence of a pair of ‘orbiting’ solitary waves was observed which danced in the way previously described. For larger values of $\lambda$, the authors of [9] report that ‘triplets’ and ‘quadruplets’ of such orbiting solitary waves appeared. It was further conjectured in [9] (on the basis of the observed increase of the distance between the peaks of the orbiting pairs of solitary waves) that the system ‘may eventually transform into two separately propagating solitary waves’.

In the light of the numerical experiments of the present paper one could interpret the orbiting solitary waves of [9] as perturbed multi-pulse solitary waves, which, after an intermediate dancing stage, resolve themselves into separate single-pulse solitary waves.

We also computed the evolution of ‘depression’ solitary waves of the Benjamin equation considered by Calvo and Akylas in [10] with the aim of studying their stability properties. In order to facilitate comparisons with the results of [10], we computed the initial ‘depression’ wave profile by solving the solitary-wave equation in the form given by equation (44) of [10], i.e., as solution $\phi = \phi(x)$ of

$$\nu \phi - \phi^2 - 2\gamma H \phi_x - \phi_{xx} = 0,$$

with $\nu = 1, \gamma = 0.94$. For this purpose we used the CGN algorithm (without continuation) taking as initial guess the usual (‘elevation’) solitary wave of the Benjamin equation corresponding to $\gamma = 0.94, c_s = 0.9$, reflected about the $x-$axis and multiplied by a factor of two. (We performed 175 iterations with a final residual error of the order of $10^{-13}$.) The profile $\phi(x) = u_0(x)$ that was obtained is shown in Figure 22; it corresponds to the profile of the uppermost snapshot of Figure 6 of [10].

![Figure 22. Initial ‘depression’ solitary wave $\phi(x) = u_0(x), \gamma = 0.94, c_s = 0.9$.](image)

We then integrated forward in time with our hybrid scheme using the appro-
appropriate transformed version of the p.d.e. (43) of [10]. Specifically, if \( \eta = \eta(X, \tau) \) is the solution of that equation, our change of variables was defined by

\[
\eta(X, \tau) = u(x, t), \quad x = X + 2.8\tau, \quad t = 2\tau. \tag{5.1}
\]

This gave for the variable \( u(x, t) \) the Benjamin equation of the form

\[
u_t + 1.4u_x - uu_x - 0.94\mathcal{H}u_{xx} - 0.5u_{xxx} = 0, \tag{5.2}
\]
i.e. of the form (1.1) with \( \beta = -1, \alpha, \gamma, \delta \) positive, which we integrated with the hybrid method on \([-1024, 1024]\) using \( h = 0.125(N = 16384), k = 0.02 \) up to \( t = 1120 \). We observed that the initial profile moves to the right with speed \( c_s = 0.9 \), apparently unchanged until about \( t = 250 \). (Note that the analogous wave in Figure 6 of [10] moves to the left because its speed is equal to \( -1 \).)

This follows from our change of variables (5.1) which implies that \( u(x, t) = \phi(x - 0.9t) \) if and only if \( \eta(X, \tau) = \phi(X + \tau) \).) After that time, perturbed by the errors inherent in the numerical scheme, the ‘depression’ wave starts losing its shape and eventually develops into one main pulse, apparently a solitary wave of ‘elevation’, which continues travelling to the right, preceded by a dispersive oscillatory wavetrain. This instability confirms the results of [10] and may be seen more clearly in another numerical experiment in which we explicitly perturbed the initial profile, taking as initial value the function \( ru_0(x) \) with \( r = 1.1 \). The evolution that resulted was simulated again up to \( t = 2200 \) with the hybrid scheme for (5.2) with the same discretization parameters as before and is depicted in Figure 23. The perturbed initial ‘depression’ solitary wave loses its shape fast and apparently evolves in two usual (‘elevation’) solitary waves of different heights that travel to the right preceded by a dispersive tail. (Note that in this figure the solitary waves have positive peaks, while in previous sections of the paper at hand they had negative. This is due to the negative sign of the \( uu_x \) term in (5.2): If we make the change of variable \( v = -u, v \) satisfies the Benjamin equation \( v_t + 1.4v_x + vv_x - 0.94\mathcal{H}v_{xx} - 0.5v_{xxx} = 0 \), which is our usual form. For the latter equation the solitary waves of ‘elevation’ type have negative maximum excursions from zero and waves of smaller absolute amplitude are faster than those of larger absolute amplitude, cf. e.g. Figure 10. Hence in the \( u \)-equation (5.2) the solitary waves have positive maximum excursions and still move to the right with the waves of smaller amplitude being faster than those of larger amplitude and with the tiny dispersive oscillatory wavetrain being even faster as observed in Figure 23.)
6 Conclusions

In Section 2.1 of this paper a fully discrete hybrid numerical scheme involving in space a spectral approximation of the nonlocal term and a smooth spline approximation of the other terms of the p.d.e., and in time an implicit Runge-Kutta discretization, was proposed for the periodic initial-value problem for the Benjamin equation (1.1). The implementation of the scheme was described in Section 2.2. The rates of convergence of the method in $L^2$ and $L^\infty$ were experimentally found to be equal to four in the spatial and three in the temporal variable; in addition, the scheme appears to be unconditionally stable, as no relation between the spatial and temporal discretization parameters was found to be necessary for stability. The accuracy and stability of the scheme was further validated in Section 2.2 by means of numerical experiments that verified the suitability of the method for approximating well solitary-wave solutions of the Benjamin and the related Benjamin-Ono equation (i.e. the case $\delta = 0$ in (1.1)).

In Section 3 we reviewed a continuation conjugate gradient-Newton solver of the nonlinear system (3.1), which represents a Fourier spatial discretization
of the o.d.e. (1.5) satisfied by the solitary waves of the Benjamin equation. This technique was proposed by the authors of the present paper in [11] and produces very accurate approximations of single-pulse and multi-pulse solitary waves for any value of the speed \( c_s > \alpha \) and \( \gamma \in [0, 1) \), where \( \alpha, \gamma \) are the parameters occurring in (1.1). Solving directly the nonlinear solitary wave o.d.e. avoids the drawbacks of generating solitary waves by iterative ‘cleaning’ of pulses produced as time increases by resolution of initial data in sequences of solitary waves. This property is shared by the Benjamin equation with other nonlinear dispersive wave propagation models. (Iterative cleaning is not easily accomplished in the case of the Benjamin equation for the additional reason that the tails of its solitary waves decay only algebraically.) These numerically generated solitary waves were used as initial data in the evolution equation that was subsequently integrated forward in time by the hybrid scheme.

A variety of error metrics pertinent to the propagation of solitary waves and the conservation of the discrete analogs of the p.d.e. invariants by the scheme were used to assess and confirm the accuracy of the ensuing numerical solutions as approximations of travelling wave solutions of the Benjamin equation. However, for values of the parameter \( \gamma \) very close to one the accuracy of the computations is somewhat reduced.

In the last two sections of the paper the hybrid scheme was used as a computational tool for studying interaction and stability properties of the solitary waves of the Benjamin equation. In Section 4 one-way interactions (overtaking collisions) of solitary waves were considered. It was pointed out in [9] that these interactions are not exact in the sense that a small amplitude dispersive ‘tail’ appears after the collisions. (This ‘tail’ is faster than the solitary waves in our formulation and hence precedes the main pulses.) This is an indication that the Benjamin equation is not integrable (as opposed to the BO equation.) We made a detailed study of the interaction of two single-pulse solitary waves of different speeds for several values of \( \gamma \) and found that in general the waves emerging from the interaction were single-pulse solitary waves whose amplitudes as time increases approach their pre-interaction levels but suffer slight phase changes. In addition, a small, faster dispersive tail is generated. The collisions became harder to simulate for \( \gamma > 0.9 \). For \( \gamma = 0.99 \) we observed small oscillations of the amplitude of the emerging slower wave. While numerical loss of accuracy cannot be ruled out, this phenomenon may be related to a possible instability of solitary waves for \( \gamma \) very close to one. (Recall that the orbital stability of solitary waves of the Benjamin equation has been established for small enough \( \gamma \in [0, 1) \), [2], [3], while the theory of [8], which is valid for all \( 0 < \gamma < 1 \), concerns a weaker type of stability.) When a two-pulse solitary wave was let to collide with a single-pulse one, we observed, even for \( \gamma = 0.5 \), that the amplitudes of the emerging single-pulse and two-pulse waves had not returned to their initial values even after a long time had elapsed. This phenomenon may be related to the apparent instability of the multi-pulse solitary wave...
waves that was also discussed in the following section.

In Section 5 we studied computationally issues of stability of the solitary waves under small perturbations. For moderate values of $\gamma$ we observed that, when perturbed by a small multiplicative factor, single-pulse solitary waves of the Benjamin equation evolve into a nearby solitary wave plus the usual dispersive ‘tail’. The experiment for $\gamma = 0.99$ was inconclusive as it generated an oscillatory pulse similar to a solitary wave, which did not stabilize but kept radiating small-amplitude oscillations, even after a very long computational time had elapsed. When we perturbed a two-pulse solitary wave we observed that it evolved into a structure with two peaks whose height and distance oscillated apparently periodically in time up to the end of our computation. When the amount of the perturbation was increased, this ‘dancing’ phase ended and the twin-peak structure evolved into two separate single-pulse solitary waves. We believe that this experiment clarifies the nature of the ‘orbiting’ solitary waves observed in [9] and verifies the conjecture put forward in [9] about their eventual break-up into separate solitary waves. This section of the paper ends with a computational study of the stability of the ‘depression’ solitary waves identified in [10]. We were able to construct such ‘depression’ waves in our framework and check that they are indeed unstable under small perturbations, thus verifying the results of [10].

Our aim in the future is to develop similar hybrid schemes for other nonlinear dispersive wave equations and systems and use them to discretize on different meshes the local and nonlocal terms of the p.d.e.’s; this would yield efficient schemes, especially in two spatial dimensions.

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References


