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Exploring the role of context in students' meaning making for algebraic generalization

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In this paper we present results of a study aiming to investigate 7th grade students' construction of meanings for algebraic generalization. The students worked in groups using a specially designed microworld to explore tasks aiming to link figural patterns to realistic situations. Our focus is on the role of specific contextual elements (i.e. realistic task, digital tools, means of symbolization) on students' meaning making. The combined use of Abstraction in Context and microgenetic analysis indicates the critical role of the available software structures in mediating students' generalizations from the real context of the task to the algebraic context of school mathematics.

Keywords: Generalization, Algebraic thinking, Objectification, Microgenetic analysis.

Introduction

In this study we investigate 7th grade students' construction of meanings for algebraic generalization. The students collaborated in groups of three using the exploratory microworld eXpresser (Noss et al., 2009) to create figural patterns by expressing their structure through repeated building blocks of square tiles and developing the rules underpinning the calculation of the number of tiles in the patterns. The microworld allows students to use (iconic) variables to reproduce their constructions for different number of repetitions, to express generalization and to check their correctness through appropriate feedback. In this paper, we focus on the role of specific elements of context (i.e. realistic task context, digital tools, various means of symbolization inside and outside the digital environment) on students' meaning making for algebraic generalization.

Theoretical framework

A way to introduce students to algebra is to be aware of a pattern or regularity and then try to express it through a relationship (Mason, Graham, & Johnston-Wilder, 2005). There is something inherently arithmetic in algebra and something inherently algebraic in arithmetic, and pattern activity brings these two aspects together (Radford, 2014). Algebraic thinking is characterized by its analytical nature and it is related to the semiotic system used by students to work with symbolic expressions and relations including not necessarily alphanumeric symbolism but also non-symbolic and embodied forms of expression (ibid). Radford (2010) developed the theory of objectification with which he describes the semiotic transition of students from the distinction of a similarity in its expression as a generalization in more mathematical ways through the use of signs (gestures, words, symbols). In order to investigate the process of generalization by students as an algebraic activity, Radford (2014) suggested three features of algebraic thinking: (1) indeterminacy: the existence of unknown quantities (e.g., variables, parameters), (2) denotation: the need to name and symbolize these indeterminate quantities in different ways (not only with algebraic symbolism, but also with alphanumeric signs, natural language, gestures, or a mixture of these); (3) analyticity: the manipulation of indeterminate quantities (through operations such as addition, multiplication) as if they were known. According to the theory of objectification, knowledge moves through specific problem-solving/posing activities from an abstract indeterminate form of possibilities (what students can potentially do and think and in what ways) to a concretized form of reasoning and action ("bringing-forth" something to the realm of attention and understanding). Recognizing the centrality of the abstract-concrete duality in the process of objectification, in this study we consider objectification as an abstraction process taking also into account the link between knowledge construction and context (Cole, 1996). Context includes dynamic factors such as student interactions with peers, teachers, tools, realistic scenarios and algebraic expressions, that may affect an abstraction process (e.g., for algebraic generalization). The role of context is crucial to learning processes and the complexity of learning processes is due, at least in part, to the context's influences on the student's constructions of knowledge (Dreyfus, 2010).

We combine Abstraction in Context (AiC) (Hershkowitz, Schwarz, & Dreyfus, 2001) and microgenetic analysis (Siegler, 2006) in order to address the role of specific contextual elements on the development of students' algebraic thinking. AiC offers a way to describe at the micro-level how meanings are constructed by shedding light on their connections to the existing mathematical knowledge through three epistemic actions: recognizing (R), building-with (B) and constructing (C). Recognizing an already known mathematical concept, process or idea occurs when a student recognizes it as inherent in a given mathematical situation. Building-with involves combining existing knowledge elements (i.e. recognized constructs) to achieve a goal, such as solving a problem or justifying a solution. Constructing is carried out by assembling or integrating previous knowledge elements by vertical mathematization to produce a new structure. The microgenetic analysis, originated in the Vygotskian psychology, provides tools and techniques to analyze discourse data and take a deeper look at the genesis of knowledge construction and the role of contextual factors including the social interactions and the use of tools in the learning environment. In this study, we investigate 7th grade students' objectification of algebraic generalization. AiC offers a framework to describe and analyze objectification as an abstraction process while microgenetic analysis allows us to take a deeper look at how the realistic task context, the available tools and the various means of symbolization inside and outside the digital environment influence students' construction of meanings. We aim to contribute in the research literature that explores the role of contextual factors in the development of students' algebraic thinking by taking a deeper look at the genesis of knowledge construction at the micro-level.

The microworld

EXpresser is a mathematical microworld designed to support 11-14 year-old students in their reasoning and problem-solving of generalization tasks (Noss et al., 2009). It supports students to perceive structure and find ways to express structural relationships, to identify variants and invariants in patterns and to recognize and articulate generalizations. In Figure 1 there is a model of a snake where the body combined by red tiles while the head and tail by blue. Students are asked to construct a model that works for any number of red tiles according to how old is the snake and find a rule for the total number of tiles that compose snake in any month of its life. EXpresser consists of two main areas: (a) My Model (a work area, Figure 1 on the right of the screen); and (b) Computer's Model

(Figure 1 on the left). In My Model, students can use building blocks of square tiles to make patterns that can be combined to models.

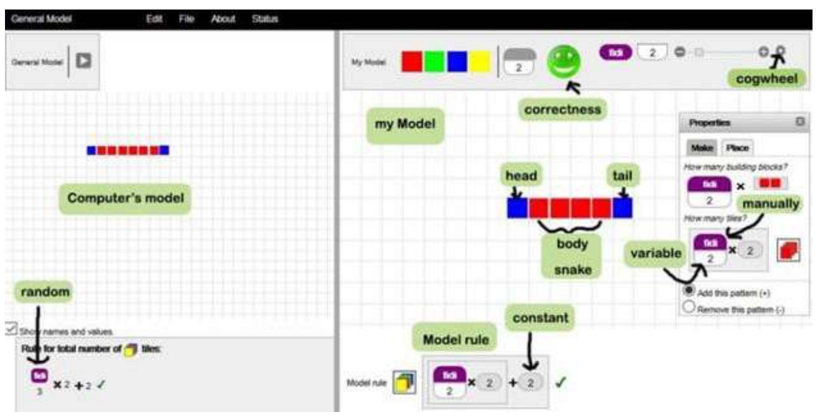

Figure 1: Pattern on eXpresser

EXpresser allows students to work with (icon) variables so as to reproduce dynamically (i.e. to 'animate') their patterns for different numbers of repetitions, to express generality through semi-algebraic relations and to test the validity of these relations for random values of repetition through appropriate feedback. By default, all numbers in eXpresser are constants and it is possible to change its value by "unlocking" them to become variables that can be handled by a slider (Figure 1). From the "cogwheel" icon (next to the slider) students can configure the variable's "domain" and specify its interval of values and the step of the variation (Figure 2). Students can convert a constant number to a variable in My Model while in Computer's Model the variables take random values.

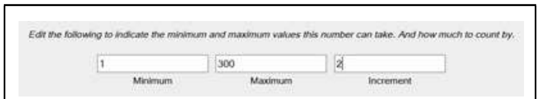

Figure 2: Properties of slider

Students can construct a model rule for the total number of tiles and only if it is correct pattern will be coloured (Figure 1). Otherwise, as an indication of error, the pattern appears colorless. Finally, when students find the general rule in Model Rule the face icon on the central toolbox becomes green and smiling (Figure 1). As variables take random values, they provide a rational for generality (Mavrikis et al., 2013). Thus, the environment incorporates an 'algebra' and a language aiming to make generalization more concrete for the students by facilitating expression of structure.

Methodology

Our research approach is informed by the influential idea of "design" in learning (Cobb et al., 2003) aiming to explore the role of alternative representations and means of expression on students' meaning making for algebraic generalization. The study took place in a lower secondary experimental school in Athens with one class of 7th grade (13-year-old) with 18 students and one experimenting teacher. The students worked in groups of three for 12 teaching hours (6 two-hour sessions, one session per week) over two months. In the end of the implementation, 1-hour interviews with each group of students were conducted intended to capture details about students' thinking and approaches over the whole implementation. At the beginning of the study we knew that students had not worked with patterns before (patterns are not included in the Greek mathematics curriculum) and they had minimum use of algebraic symbols. Thus, we expected to see if and how their interaction with the available tools and resources would influence the meanings that they would create for algebraic generalization.

Task design aimed to link patterns to problems associated with realistic workplace contexts, thus the need for algebraic generalization was expected to arise as part of problem solving. The activity sequence was divided in three phases and for each one of them we designed a series of tasks. In the first phase students assumed the role of a herpetologist by studying the development of snakes through a simple linear pattern (*Snakes*). The second phase referred to the organization of a wedding party by professionals (*Table Arranging*). Initially students investigated the problem for 38 guests with concrete manipulatives and, next, a demanding version of the problem (132 guests) leading to a more complex linear pattern of unified tables was explored through eXpresser. In the third phase, students acted as pool designers to explore high complexity second degree patterns (*Pool designer*). In this paper, we analyze data from the first phase. *Snakes* engaged students in exploring the growth rate of a snake (i.e. grows by 2 cm every month, lives 25 years, its length reaches 6 meters) in order to identify the appropriate size of its cage. Assuming that one square tile in eXpresser corresponds to 1 cm, students were asked to construct a simple linear pattern that would grow by 2 squares tiles each time and thus would depict the snake in any month of its life. The questions involved in the worksheet were: "1. How many squares will the pattern have when the snake is 5 months old? (Respectively for 10, 25,100). 2. Describe with words how the pattern works. 3. Describe with an algebraic formula how many red square tiles appear for each month". Our general goal was to engage students in investigating the relationship between the age of the snake and the total square tiles of the pattern.

The collected data consists of video and audio recordings (four groups). The data were fully transcribed for the analysis. The unit of analysis was the thematic episode defined as an extract of actions and interactions around students' conceptualization and expression of generalizations. The analysis was carried out in two levels. First, the episodes were analyzed through AiC to highlight the evolution of students' epistemic actions while constructing generalizations. Next, the same episodes were analyzed through microgenetic analysis (Siegler, 2006) that involved: (a) coding of students' and teacher's utterances in relation to contextual elements (i.e. task, tools, symbolization) that appeared to be crucial in students' conceptualization and expression of generalizations, namely context snake, context snake in eXpresser, context algebra in eXpresser and context algebra; (b) categorizing the utterances in clusters of meanings emerging through constant comparison. In this paper we analyze an episode from the interview of one group of three students (Group 1).

Results

In this episode, students had already constructed the snake pattern in eXpresser and they had responded to question 1 through a numerical generalization. As they wrote on the worksheet: "each month the snake grows 2 cm, so for 5 months the number of square tiles would be 2x5, for 10 months 2x10, for 25 months 2x25, and for 100 months 2x100". They explained that for providing the answer for 5 months they counted one by one the red square tiles on the screen, they did not do the same for the rest of the numbers. In question 2 the students answered that "the pattern is increased by 2" and in question 3 they provided the algebraic formula: "x+2". This answer, that appears to be wrong (the correct one is 2x+2), challenged the researcher to engage students in a discussion in order to justify their choice of symbols in relation to their designed pattern (Figure 3, 4). We note that the designed pattern appeared to work correctly in eXpresser. The students had unlocked a number called "fidi" (snake in Greek) to become a variable and used the corresponding slider to choose values for the variable (in Figure 3 the value ‘2’ is chosen). The students recognized that the pattern consisted of 2 constant blue square tiles (for the snake’s head and tail) and one building block of 2 red square tiles that had to be repeated for designing the body. In order to create the variable the students pulled a red square tile to My Model and unlocked it (Figure 4). Then, they opened the cogwheel in the corresponding slider (Figure 2) and set the domain of the variable, by changing the minimum value from 1 to 2, the maximum from 1 to 300 and the increment from 1 to 2. As they explained, they put 2 to the minimum because the snake at birth is 2cm, they put 300 to maximum because it grows for 25 years until reaching 6 meters and they put 2 in the increment because it increases 2cm each month.

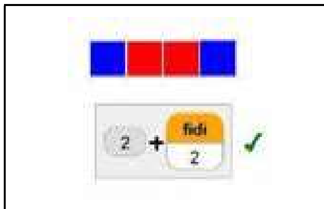	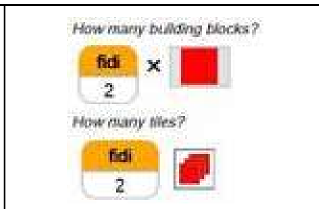	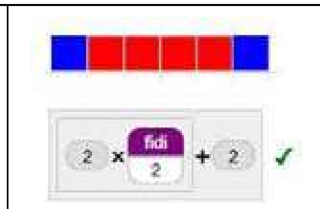	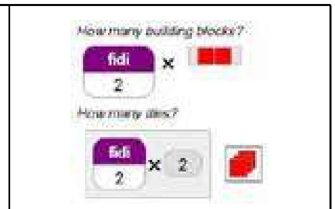
Figure 3:Pattern Group 1	**Figure 4: Pattern properties Group 1**	**Figure 5: Pattern Group 2**	**Figure 6: Pattern properties Group 2**

For correct coloring the red squares, the students used the variable to complete the Model Rule (Figure 3) by adding to “fidi” the constant number 2 representing the constant (blue) squares tiles. This way the values of the variable in the slider represent the length of the snake in cm. These values actually result by a ‘hidden’ multiplication of the number of months by 2. For instance, value 6 in the slider means 6cm length indicating that 3 months are multiplied by 2. By exploiting the structures provided by the software, the students achieved to construct a pattern working correctly and thus they concluded that the required symbolic generalization was “x+2”. However, this formula cannot be used for calculating the total length of the snake in relation to its month of life. We note that students had already answered questions 1 and 2 by multiplying specific number of months by 2. In the following excerpt, the researcher discusses with group 1 students about their pattern and formula. She attempts to understand their formula and brings to the fore the fact that although the pattern works in eXpresser the formula cannot be used for answering the questions.

11	Researcher:	What does x+2 means to you?
12	Student 2:	We put x for the body and 2 for the head and tail. So 2 would be constant independently of the length of the snake. (**Recognizing**)
13	Researcher:	So x refers to red squares and 2 to the blue ones.
14	Student 1:	Yes for the tail and the head. (**Recognizing**)
15	Researcher:	Can this formula help us answer the question 1? What about the 5 months?
16	Student 1:	5 times 2.
17	Researcher:	But here you write plus 2.
18	Student 2:	[*She seems to be confused*] Is it the squares? ... 7 [*adding 2 to 5*] times 2 gives 14? ... (**Recognizing**)
	Student 1:	Our x here [*showing the icon of “fidi”*] will not help us because it shows how many red tiles appear in the pattern. If we don’t know the number of red tiles we can’t find it very easily [*refers to number of months*]. (**Recognizing**)
19	Researcher:	So what does your x show?

20 Student 1: Our x is for the red tiles. It basically expresses what is already on the screen.

21 Researcher: What do you mean exactly? Can you change something to help us more?

22 Student 2: For the months…?

Student 1: Probably… Yes, 2 times the months. Actually x should have to express the number of months. (**Building-with**)

23 Researcher: In your pattern x doesn't refer to months. It refers only to even numbers, 2 months, 4, or 6. For 5 months, what would you do?

24 Student 1: Our x refers to the whole body of snake…it refers to something different…

In the above dialogue the students recognized that the variation of months is embedded implicitly in their formula in eXpresser without being symbolized in some way. In the next part of the episode, the researcher aims to challenge students' views on symbolization by presenting them a pattern created by another group (Group 2) in a different way (Figure 6). In particular, group 2 constructed a pattern based on the formula '2x+2' where x takes all natural numbers as values in the slider. The researcher invites the students to reflect on the two formulas.

25 Researcher: In your pattern, x does not refer to months. In order to find out how old the snake is you told me that you have to multiply the variable by 2… Here is a pattern of another group. [*The Researcher shows the pattern of Figure 5*]

26 Student 1: Yes actually it [*the slider*] gives values expressing months ... These students (Group 2) multiplied by 2 (Figure 5) so as to refer to the whole month otherwise this [*she showing the iconic variable in properties, Figure 5*] would refer only to a half month. Therefore "fidi" [*iconic variable*] refers to the half body of snake, so they multiplied by 2 in order to find the whole. (**Recognizing, Building-with**)

27 Researcher: So when the slider shows 2, we see how long the snake would be in 2 months. Which is the formula of this pattern. (Group 2)?

28 Student 2: [*calculates total number of tiles for value 2 in the slider*] 2 times 2 add 2, its 6. (**Building-with**)

29 Researcher: Can you express it by symbols?

30 Student 1: 2 times x add 2, because x here refers to something different from our x. Here it refers to months. (**Construction**)

The above episode represents an instance of objectification since the students appeared to conceptualize in more sophisticated ways different symbolic forms. In terms of AiC, the episode represents a construction process leading to students' construction of meanings for their own formula as well as the formula '2x+2'. The main challenge faced by the students was to conceptualize the role of iconic variables and linking them both to the construction of different patterns and to the realistic context of the task. We observe that initially the students recognized that the environment's iconic structures are associated directly by their previous mathematical constructs and they linked the variable x to eXpresser's unlocked variable (line 12). Then they recognized that the x of their formula doesn't work for answering to questions 1 and 2 in the worksheet (line 18) since it refers implicitly to numbers of months (line 22). Thus, they were able to build with it an explanation of their own strategy. In line 26, the students recognized that the value of x in the new pattern represents number of months and the graphical outcome "refers to the half body of snake". Building-with these two recognizing actions they provided a justification for group's 2 solution: the multiplication with 2 is necessary in order to calculate the total length of the snake (line 26). This is a sophisticated justification since the students conceptualized the variation of months inside the variation of snake's length. Next, the students constructed the new formula initially by the use of specific numbers (line 28) and subsequently with the use of variables (line 30) following an analytic algebraic approach. At the same time they developed a new meaning for the variable by objectificating x as a sign representing number of months.

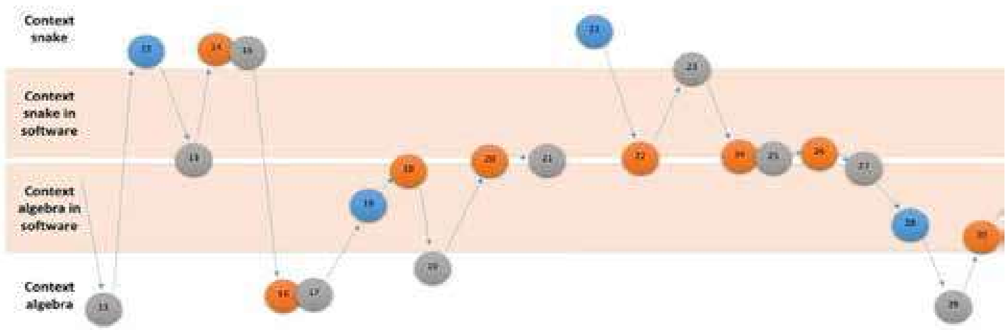

Figure 7: Microgenetic analysis: Gray is for Researcher, Orange for Student 1 and Blue for Student 2

The microgenetic analysis of the above episode shows a rich meaning generation where each symbol is conceptualized in different ways in relation to the different elements of context and utterances cannot be confined to a single context (Figure 7). The analysis revealed four clusters of meanings corresponding to specific contextual elements: real world snake (context snake); snake represented in eXpresser (context snake in software); quasi-algebraic symbolic expression in eXpresser (context algebra in software); algebraic symbolism (context algebra). In the beginning of the episode we see that students refer to symbols in relation to the real context or in relation to school mathematics. In the progression of the episode we note that students' utterances belong to different levels simultaneously. For example, when students use x they refer to snake body while with number 2 on head and queue respectively (line 12). Then the same x in the software is a set of red squares. Indeed, when the researcher asks whether "x+2" can help us to calculate the red squares in relation to the snake's life (line 15), the students realize that the x of their formula doesn't work for answering to questions referring to the snake's length in different months (line 18). Looking at Figure 7, the students' answer is based on the properties of iconic variable in eXpresser as well as on the representation of the snake in the environment for different values in the slider. This way student's utterance is placed at the intersection of two contextual elements: context snake in software and context algebra in software. Finally, while observing the pattern created by another group, the students identify that symbol x refers at the same time to red squares and months of snake's life. This is why the constructed meaning is placed again at the intersection of two contextual elements (Figure 7, line 26). Taking a global view of the microgenetic diagram, we see that students' interaction that leads to the construction of algebraic generalization in the algebraic context (line 30) takes place mainly within the two contextual elements in the middle. This finding highlights the critical role of the software structures in mediating students' generalization from the real context of the task to the algebraic context of school mathematics.

Conclusion

The analysis revealed an objectification process accompanied by meaning generation for algebraic generalization. In terms of AiC, the students conceptualized different symbolic forms of expression related to different patterns. This was carried out through a sequence of epistemic actions including: linking previous mathematical constructs to eXpresser's variable (unlocked numbers); recognizing the role of variable in the constructed patterns; relating variable values to the graphical outcome of patterns; conceptualizing covariation of different variables for the construction of patterns (e.g., months and snake's length); and constructing formulas for patterns not through a trial-and-error arithmetic method but in algebraic analytic way. The microgenetic interpretive analysis allowed us to identify, analyze and discuss the role of the different contextual elements to students' construction of meanings. Four clusters of meanings appeared interrelated to the context of real snake, the representation of snake in eXpresser, the quasi-algebraic context of eXpresser and the context of school algebra. As regards the role of contextual elements in students' construction of algebraic generalizations, the analysis revealed the critical role of software representations/structures in mediating the making of links between realistic tasks and algebra.

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