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MAKING SENSE OF STRUCTURAL ASPECTS OF EQUATIONS BY USING ALGEBRAIC-LIKE FORMALISM

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This paper reports on a design experiment conducted to explore the construction of meanings by 17-year-old students, emerging from their interpretations and uses of algebraic-like formalism. The students worked collaboratively in groups of two or three, using MoPiX, a constructionist computational environment with which they could create concrete entities in the form of Newtonian models by using equations and animate them to link the equations' formalism to its visual representation. Some illustrative examples of two groups of students' work indicate the potential of the activities and tools for expressing and reflecting on the mathematical nature of the available formalism. We particularly focused on the students' engagement in reification processes, i.e. making sense of structural aspects of equations, involved in conceptualising them as objects that underlie the behaviour of the respective models.

INTRODUCTION

In this paper we report on a classroom research [1] aiming to explore 17-year-old students' construction of meanings, emerging from the use of algebraic-like formalism in equations used as means to create and animate concrete entities in the form of Newtonian models. The students worked collaboratively in groups of two or three using a constructionist computational environment called "MoPiX" [2], developed at the London Knowledge Lab (http://www.lkl.ac.uk/mopix/) (Winters et al., 2006). MoPiX allows students to construct virtual models consisting of objects whose properties and behaviours are defined and controlled by the equations assigned to them. We primarily focused on how students interpreted and used the available formalism while engaged in *reification* processes (Sfard, 1991), i.e. making sense of structural aspects of equations, involved in conceptualising them as objects that underlie the behaviour of the respective models.

THEORETICAL BACKGROUND

Recognising the meaning of symbols in equations, the ways in which they are related to generalisations integrated within specific equations and also the ways in which a particular arrangement of symbols in an equation expresses a particular meaning, are all fundamental elements to the mathematical and scientific thinking. Research has been showing rather conclusively that the use of symbolic formalisms constitutes an obstacle for many students beginning to study more advanced mathematics (Dubinsky, 2000). Traditional approaches to teaching equations as part of the mathematics of motion or mechanics seem to fail to challenge the students' intuitions since they usually encompass static representations such as tables and graphs which are subsequently converted into equations. Lacking any chance of interacting with the respective representations, students fail to identify meaningful links between the components and relationships in such systems and the extensive use of mathematical expressions (diSessa, 1993). Indeed, students tend to use and manipulate physics equations in a rote manner, without understanding the concepts they covey (Larkin et al., 1980). Sherin (2001) argued that, in order to overcome this obstacle, students need to acquire knowledge elements that he termed *symbolic forms*. The acquisition of *symbolic forms* would help students make connections between an algebraic expression's conceptual content and its structure, which is considered to be crucial for the understanding, meaningful use and construction of physics equations.

In the mathematics education field, the relevant research is mainly based on the distinction between the two major stances that students adopt towards equations: the process stance and the object stance (Kieran, 1992; Sfard, 1991). The process stance is mainly related with a surface "reading" of an equation, concentrated into the performance of computational actions following a sequence of operations (i.e. computing values). In contrast, according to the object stance, an equation can be treated as an object on its own right, which is crucial to the students' development of the so-called *algebraic structure sense* (Hoch and Dreyfus, 2004), i.e. the act of being able to see an algebraic expression as an entity, recognise structures, sub-structures and connections between them, as well as to recognise possible manipulations and choose which of them are useful to perform. This development, linking procedural and structural aspects of equations, has been termed *reification* (Sfard, 1991) and has been considered to underlie the learning of algebra in general.

Recently, students' uses and interpretations of symbolic formalism in understanding mathematical and scientific ideas have been studied in relation to the representational infrastructure of new computational environments designed to make the symbolic aspect of equations more accessible and meaningful to children, especially through the use of multiple linked representations (Kaput and Rochelle, 1997). Adopting a broadly constructionist framework (Harel and Papert, 1991), we used a computer environment that is designed to enhance the link between formalism and concrete models, allowing us to study the ways in which the use of formalism, when put in the role of an expression of an action or a construct (a model), can operate as a mathematical representation for constructionist meaning-making. Our central research aim was to study students' construction of meanings emerging from the use of mathematical formalism when engaged in reification processes. We mainly focused on the development of their understanding on the structure of an equation based primarily on the conception of it as a system of connections and relationships between its component parts.

THE COMPUTATIONAL ENVIRONMENT

MoPiX (Winters et al. 2006) constitutes a programmable environment that provides the user the opportunity to construct and animate in a 2d space, models representing phenomena such as collisions and motions. In order to attribute behaviours and properties to the objects taking part in the animations generated, the user assigns to the objects equations that may already exist in the computational environment's Equations Library or equations that she constructs by herself.

Figure 1 shows a red ball performing in the MoPiX environment a combined motion both in the vertical and horizontal axis, leaving a green trace behind. As one may observe, the equations attributed to the object incorporate formal notation symbols (Vx, x, t) as well as programming–natural language utterances (ME, appearance, Circle). However, their main characteristic is that they constitute functions of time, as it is stated by the second argument on the parentheses on their left side. For example, the horizontal motion equations attributed to the ball define the object's: horizontal position at the 0 time instance (1), horizontal position at any time instance (2), the horizontal velocity at the 0 time instance (3), the horizontal velocity at any time instance (4) and the horizontal acceleration at any time instance (5). The MoPiX environment constantly computes the attributes given to the objects in the form of equations and updates the display, generating on the screen the visual effect of an animation.

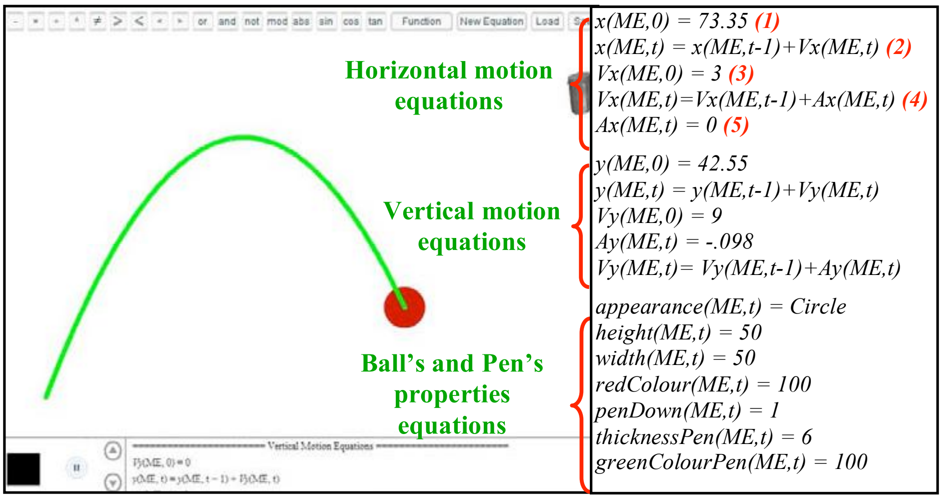

Figure 0. The MoPiX environment

Some specific features of MoPiX, underlying the novel character of the representations provided, may offer students opportunities to further appreciate utilities of the algebraic activity around the use of equations. The first of these features is that MoPiX offers a strong visual image of equations as containers into which numbers, variables and relations can be placed. The meaningful use of the environment may allow students to easily make connections between the structure of an equation and the quantities represented in it. The second feature of MoPiX is that it allows the user to have deep structure access (diSessa, 2000) to the models animated. The equations attributed to the objects and underpin the models' behaviour do not constitute "black boxes", unavailable for inspection or modifications by the user (for a discussion on black and white box approaches see Kynigos 2004). The third feature of MoPiX is that the manipulations performed to a model's symbolic facet (e.g. changing a value or removing an equation from the model) produce a visual result on the Stage, from which students can get meaningful feedback. "Debugging" a flawed animation demands students' engagement in a back and forth process of constructing a model predicting its behaviour, observing the animation generated, identifying the equations that are responsible for the "buggy" behaviour and specifying which and how particular parts need to be fixed.

TASKS

For the first phase of the activities we developed, using exclusively "Library" equations, the "One Red Ball" microworld which consisted of a single red ball performing a combined motion in the vertical and the horizontal axis. The students were asked to execute the model, observe the animation generated, discuss with their teammates and other workgroups the behaviours animated and write down their remarks and observations on a worksheet. In order to provoke discussions regarding the equations' role and stimulate students to start using the equations themselves, we asked them to try to reproduce the red ball's motion. In this process, we encouraged them to interpret and use equations from the "Library", add and remove equations from their objects so as to observe any changes of behaviour and link the equations they used to the behaviours they had previously identified. As we deliberately made the original red ball move rather slowly, near the end of this phase, we expected students to start expressing their personal ideas about their own object's motion (e.g. make it move faster) and thus start editing the model's equations, using the "Equations Editor", so as to describe the new behaviours they might have in mind.

For the second phase of the activities we designed a half–baked microworld (Kynigos 2007), i.e. a microworld that incorporates an interesting idea but it is incomplete by design so as to invite students to deconstruct it, build on its parts, customize and change it. In this case we built a game–like microworld –called "Juggler" (Kynigos 2007)– consisting of three interrelated objects: a red ball and two rackets with which the ball interacted. The ball's behaviour was partially the same as the "One Red Ball's". However, certain equations underpinning its behaviour, did not derive from the environment's "Library" but were created by us. Using the mouse the rackets could be moved around and make the ball bounce on them, forcing it to move away in specific ways.

We asked the students to execute the Juggler's model, observe the animation generated and identify the conditions under which each object interacted with each other. The students were encouraged to discuss with their teammates on how they would change the "Juggler" microworld and embed in it their own ideas regarding its behaviour. In the process of changing the half–baked microworld, students were expected to deconstruct the existing model so as to link the behaviours generated on the screen to its equations' formalism and reconstruct the microworld, employing strategies that would depict their ideas about the new model's animated behaviours.

METHOD

The experiment took place in a Secondary Vocational Education school in Athens with one class of eight 12^{th} grade students (17 years old) studying mechanical engineering and two researchers -the one acting also as a teacher- for 25 school hours. Students were divided in groups of two or three. The groups had at their disposal a PC connected to the Internet, the MoPiX manual, translations in Greek of selected equations' symbols and a notebook for expressing their ideas. The adopted methodological approach was based on participant observation of human activities, taking place in real time. The researchers circulated among the teams posing questions, encouraging students to explain their ideas and strategies, asking for refinements and revisions when appropriate and challenging them to express and implement their own ideas. A screen capture software was used so as to record the students' voices and at the same time capture their interactions with the MoPiX environment. Apart from the audio/video recordings, the data corpus involved also the students' MoPiX models as well as the researchers' field notes. For the analysis we transcribed verbatim the audio recordings of two groups of students for which we had collected detailed data throughout the teaching sequence and also several significant learning incidents from other workgroups. The unit of analysis was the episode, defined as an extract of actions and interactions performed in a continuous period of time around a particular issue. The episodes which are the main means of presenting and discussing the data were selected (a) to involve interactions with the available tool during which the MoPiX equations were used to construct mathematical meaning and (b) to represent clearly aspects of the reification processes emerging from this use.

ANALYSIS AND INTERPRETATIONS

Interpreting existing equations' symbols

In the first phase of the experimentation, the students in their attempt to reproduce the red ball' motion, started interpreting and using equations that already existed in the environment's "Equation Library". The natural language aspect incorporated in the MoPiX formalism was the element that guided their actions. The equations that they chose to assign first to their object were those whose symbols (at least some of them) were close to everyday language utterances and provided them some indication on the kind of the behaviour they described (e.g. the "amIHittingtheGround" symbol). Equations that contained symbols that didn't satisfy the "natural language" criterion (e.g. the "Ax") were simply disregarded.

As they continued their experimentations with MoPiX, the students seemed to gradually abandon the "natural language" criterion and shifted their attention into identifying the meaning of the symbols. The students of Group B for instance came across two "Library" equations that seemed to describe the velocity in the x axis, the "Vx(ME,0)=3" and the "Vx(ME,t)=Vx(ME,t-1) + Ax(ME,t)". Their decision to attribute the second one to their object, so as to define its velocity at any time instance, came as a result of a comparison between the two equations' left parts. Yet again, the students seemed to interpret specific symbols of the equations and completely disregard others (e.g. the "Ax" on the second equation's right part).

In a number of subsequent episodes, the same students seem to articulate their understanding not just about particular symbols but also about the whole string of the equation's symbols and the relations among them. In the following excerpt the students of Group B talk about the "x(ME,t)=x(ME,t-1)+Vx(ME,t)" equation.

S1 It *[i.e. x(ME,t)]* is the object *[i.e. "ME"]* in function with time *[i.e. "t"]*.

R2 What does this mean?

S1 *[goes on disregarding the question and points at the x(ME,t-1)]* It's your object *[i.e. "ME"]* in function with time minus 1 *[i.e."t-1"]*.

R2 What does "in function with time" mean? Can you explain it to me?

S1 How much... In every second, for example, how much it moves.

R2 Meaning?

S2 Wait a minute! *[Showing both parts of the equation]* The equation is this one. All of this. It's not just these two *[i.e. the x(ME,t) and the x (ME,t-1)]*.

S1 Minus 1, which means that in every second of your time it subtracts always 1, resulting to something less than the current time. Plus your velocity.

Drawing on his previous experience with the MoPiX equations, S1 starts to independently interpret the equation's symbols moving from left to right. Having interpreted the first two of them, he attempts to also interpret the relationship between them and defines it as the distance that the object has covered in a second of time. S2, who understands the kind of correlation S1 has made, intervenes and stresses the fact that he hasn't taken into account all the symbols in the equation. S1, who up to that point disregarded the "Vx(ME,t)" on the right part, takes an overall view of the equation and interprets it not by merely referring to the comprising symbols but by also referring to the connection between them. It is noticeable that at this point the students' actions demonstrate an emerging awareness of the equation's structure as a system of connections and relationships between the component parts.

Variables and numerical values to control motion animations

As students gained familiarity with the MoPiX formalism, they started expressing their own personal ideas about the ways their objects should move. In order to put into effect those ideas, the students initially modified the existing equations' symbols and left the structure intact. One of the main elements that they often altered was the equations' arithmetic values. The students of Group B, for instance, attributed to their object the "Vy(ME,0)=0" equation which prescribed the object's y axis initial velocity to be 0. The observation of the animation triggered the implementation of a series of changes to the equation's arithmetic values starting with the conversion of the "0" on the right part into "3". The successive changes of the arithmetic value on the equation's right part didn't cause the object to constantly move since the equation referred just to the initial velocity. To make the velocity for "all the next time instances to come" to be "3", the students replaced the "0" on the left part (i.e. an arithmetic value) with "t" (i.e. a variable).

S2 Do we need a symbol for this?

R2	Do we need a symbol? It's a good question. How do you plan to express it?
S2	With symbols we usually express something that we can't describe accurately.
S1	Plus… t. *[He writes down Vy(ME,t)=3]. [Showing the "t"]* So, when I look at this symbol
S2	I'll know it represents the infinity.

We suggest that the students relocated their focus from just attributing arithmetic values, which indicates a process stance to equations, into forming functional relationships. The fact that they were involved in a process of recognizing which manipulations were possible and at the same time useful to perform so as to express their idea, indicates a implicit focus on the structure of the equations. Furthermore, the statements concerning the use of symbols to express "something that we can't describe accurately" seems to constitute an indication of a progressive acquisition of algebraic structure sense through "mixed cues" (Arcavi, 1994) (i.e. interpreting symbols as invitations for some kind of action while working with them).

Relating different objects' behaviours by constructing new equations

The next episode describes how the Group A students, in the course of changing the "Juggler" microworld, didn't just use or edit existing equations but constructed from scratch two new ones. The idea they wanted to bring into effect was to "make a ball on the Stage change its colour according to an ellipse's position". Knowing that there was no such equation in the "Library", they started talking about how they would correlate those two objects using the Y coordinates.

S1	When it *[i.e. the ball]* is situated in a Y below the Y of this one *[i.e. the ellipse]* for example.
R1	I'm thinking… Will the ball know when it is below or above the ellipse?
S2	That's what we will define. We will define the Ys.
S1	This. The: "I am below now". How will we write this?
S2	Using the Ys. Using the Ys. The Ys. That is: when its Y is 401, it is red. When the Y is something less than 400, it's green!

Having conceptualized the effect they would like their new equation to have, the students in the above excerpt decide about two distinct elements regarding the equation under construction: its content (i.e. the symbols) and its structure (i.e. the signs between the symbols). Subsequently, encountering the fact that there was no in-built MoPiX symbol to express the idea of an object becoming green under certain conditions, the students came to invent one. The "gineprasino" (i.e. "become green" in Greek) symbol was decided to represent a varying quantity taking two distinct values (1 and 0, according to if the ball was below the ellipse or not). To represent the ball's position they chose to use its Y coordinate in terms of a quantity varying over time (i.e. "y(ME,t)") while for the ellipse's position they chose to use its Y coordinate in terms of the constant arithmetic value corresponding to the object's position on the Stage at that time (i.e. "274"). Adding a "less than" sign in between, the equation eventually developed was the "gineprasino(ME,t)=y(ME,t)≤274".

Unexpectedly, this equation didn't cause the ball to become green since it described solely the event to which the ball would respond (being below the ellipse) and not the ball's exact behaviour after the event would have occurred (change its colour). To overcome this obstacle, the students decided to construct another equation in which they tried to find out ways to integrate the "gineprasino" variable. A "Library" equation which explains what happens to a ball's velocity when it hits on one of the Stage's sides and the way in which a variable similar to the "gineprasino" was incorporated in it, led students to duplicate this equation's structure, eliminate any content and use it as a template to designate what happens to the ball's colour when it is below the ellipse. The second equation encompassed in-built MoPiX symbols (the "greenColour"), the "gineprasino" variable in two different forms (not(gineprasino) and gineprasino) and numerical values (0 and 100) to express the percentage of the green colour the ball would contain in each case (i.e. the ball being above and below the ellipse). Thus, the second equation developed was the: "greenColour(ME,t) = not(gineprasino(ME,t))×0 + gineprasino(ME,t)×100".

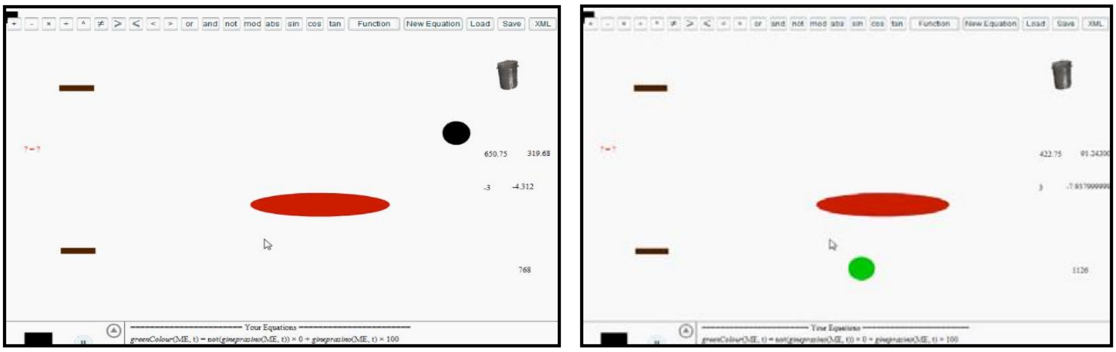

Figure 2: The ball's different percentage of green colour according to its Y position

The above episode contains many interesting events that indicate the existence of a qualitative transformation of the students' mathematical experience in reifying equations that emerged through their interaction with the available tools.

While building the first equation the students got engaged in processes such as inventing and naming variables, relating symbols with mathematical systems (i.e. the XY coordinate system) and manipulating inequality symbols to relate arithmetic values to variables. However, in building the second equation, the meaning generation evolved to include the students' view of equations as objects. The students extracted mathematical meaning from an equation that seemed to describe a behaviour similar to the one they intended to attribute to their ball. Conceptualizing a mapping between the ideas behind the two equations, the students duplicated the similar equation's structure and inserted new terms so as to define a completely novel behaviour for their object. This is a clear indication that they recognised the existence of structures external to the symbols themselves and used them as landmarks to navigate the second equation's construction process.

The manipulation of the second equation's new terms reveals further their developing structural approach to equations. By inserting in the second equation the the "gineprasino" variable which was introduced in the first one and providing it new forms (i.e. not(gineprasino)), the students seem to have conceptualised the first equation as a mathematical object which it could be used means to encode structure and meaning in the second equation. We think that this reflects a kind of mathematical thinking that has a great deal to do with developing a good algebraic structural sense accompanied with the acquisition of a functional outlook to equations as objects which is a warranty of relational understanding.

CONCLUDING REMARKS

Our purpose in this paper was to illustrate a particular approach to studying the student's construction of meanings for structural aspects of equations, emerging from the use of novel algebraic-like formalism. In the first part of the results, an initial icon-driven conceptualisation of the MoPiX equations seemed to have been leading students towards the development of criteria for an isolated interpretation of the MoPiX equations' symbols. As soon as the students became familiar with testing their models and observing the animations generated on the "Stage", their interactions with the computer environment became strongly associated with the editing of the existing equations' content. As expressed in the second part of the results, the editing of equations revealed a subtle shift from a process-oriented view to equations into an object- oriented one as well as a progressive development of algebraic structure sense. In the last part of the results, students' previous experience with the MoPiX tools seemed to become part of their repertoire, allowing them to construct new equations following specific structural rules, invent variables and specify their values, and use the equations as objects to represent variables in other equations. Concluding, we suggest that in the present study reifying an equation was not a one–way process of understanding hierarchically–structured mathematical concepts but a dynamic process of meaning–making, webbed by the available representational infrastructure (Noss and Hoyles, 1996) and the ways by which students drew upon and reconstructed it to make mathematical sense.

NOTES

1. The research took place in the frame of the project "ReMath" (Representing Mathematics with Digital Media), European Community, 6th Framework Programme, Information Society Technologies, IST-4-26751-STP, 2005-2008 (http://remath.cti.gr)

2. "MoPiX" was developed at London Knowledge Lab (LKL) by K. Kahn, N. Winters, D. Nikolic, C. Morgan and J. Alshwaikh.

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