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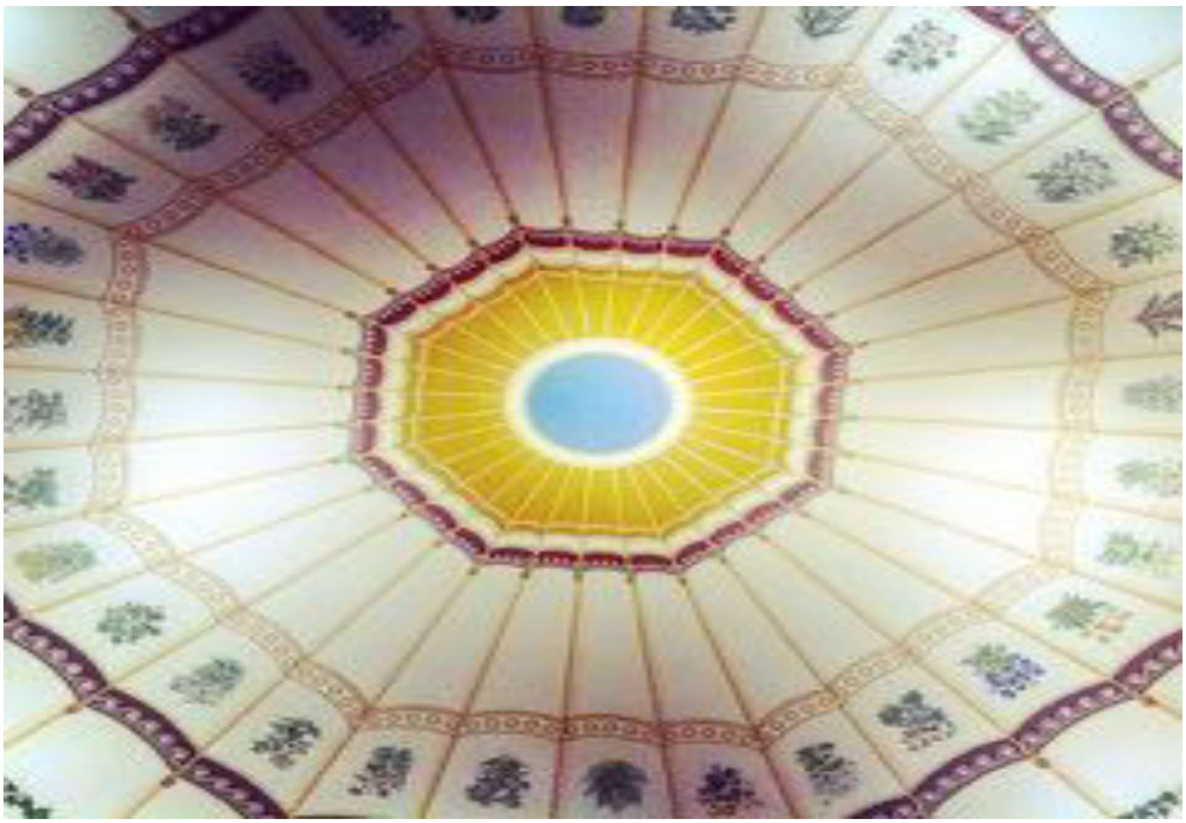

Edited by: Hans-Georg Weigand, Alison Clark-Wilson, Ana Donevska-Todorova, Eleonora Faggiano, Niels Grønbæk and Jana Trgalova

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Editors:	Hans-Georg Weigand, Alison Clark-Wilson, Ana Donevska-Todorova, Eleonora Faggiano, Niels Grønbæk and Jana Trgalova
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Conceptualization of function as covariation through the use of learning trajectories

Georgios - Ignatios Kafetzopoulos and Giorgos Psycharis

National and Kapodistrian University of Athens, Greece
gkafetzo@math.uoa.gr, gpsych@math.uoa.gr

This study aims to identify levels of 11th grade students' conceptualization of function as covariation. The students worked with modelling tasks involving the use of the digital environment Casyopée which combines algebraic and geometrical representations of functions. The results indicated six hierarchical levels of thinking about function as covariation though the use of learning trajectories.

Keywords: function, covariation, learning trajectories, modelling tasks, Casyopée.

THEORETICAL FRAMEWORK

This paper reports classroom based research aiming to identify levels of 11th grade students' conceptualization of function as covariation. The students were engaged in modelling realistic problems through the use of the digital environment Casyopée, which involves interconnected representations and allows the manipulation of covarying quantities and the treatment of the corresponding functions.

The notion of function plays a predominant role in secondary education and can be conceived in two ways: (a) as a correspondence of two variables and (b) as a covariation, which is related to the understanding of the way in which dependent and independent variables change as well as to the coordination between these changes (Carlson et al., 2002). Recent studies connect directly the definition of function with the idea of covariation:

> A function, covariationally, is the conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other (Thompson & Carlson, 2017, p. 444).

This definition is based on the person's conceptualization of function as covariation emphasizing the ways by which two quantities (corresponding to dependent and independent variables) covary in relation to each other. Carlson et al. (2002) studied the development of undergraduate student's thinking about the covariation of quantities in dynamic situations such as filling a bottle with water. Their research results led to a framework of five levels of covariation which were described through corresponding mental processes: *Dependence* (observation of changes in the two variables); *Directed change* (increase or decrease - with changes of the other); *Quantitative correlation* (coordination of the amount change of a variable with changes of the other); *Average rate* (correlation of the mean rate of change with uniform increases of the independent variable); and *Instantaneous rate of change* (correlation of the instantaneous rate with continuous increases of the independent variable). Therefore, the concept of covariation links functional relationships to the rate of change and is essential for understanding the fundamental concepts of advanced mathematics. The present study aims to contribute to the research literature related to students' understanding about function as covariation at the upper secondary education level since research at this level is rather limited.

Modelling tasks involving the use of digital tools have been indicated as providing rich opportunities for students to engage in functional thinking and therefore to interpret function as covariation (Lagrange & Psycharis, 2014; Psycharis, 2015). Lagrange (2014) describes the process of modelling a problem in Casyopée through a "modelling cycle" that includes four settings: (a) a physical object (e.g., paper), allowing students to experiment; (b) the dynamic figure resulting by modelling the dependencies in a digital tool (e.g., a dynamic rectangle in a Dynamic Geometry window); (c) the covarying magnitudes (e.g., the side length and the area of the rectangle); (d) the algebraic functions that model problem. In this approach, students' transition from experimentation with the physical object (quantities) to working with functions (variables) is mediated by working with covarying magnitudes and measurements, through the use of multiple representations such as algebraic notation, graphs and tables (Lagrange, 2014). In this study, we use realistic problems and specially designed digital tools for designing modelling activities to encourage students' transitions in the different settings of the modelling cycle.

Another strand of research that influenced this study concerns *learning trajectories* (Clements & Sarama, 2009), which include three essential elements: (a) a mathematical goal, (b) educational activities to achieve the goal, and (c) a description of the development of students' thinking as they are engaged with the activities. In this paper, we use the idea of trajectories to describe the progression of students' thinking about function as covariation. The trajectories define different layers of thinking from simple to more complex understandings. However, these levels do not indicate a unique sequence of stages from which all students pass in the same way. Students can move to different levels in both directions as their learning progresses depending on the difficulties they face (Clements & Sarama, 2014). Furthermore, in order to study the role of context and available resources in the learning process, we consider construction of knowledge about function as covariation as an abstraction process and we use the *Abstraction in Context* theory (AiC, Hershkowitz et al., 2001). According to AiC, the construction of mathematical knowledge in a specific context takes place through three epistemic actions: (a) *recognizing* a previous construction as relevant to the situation; (b) *building-with*: rebuilding existing knowledge to achieve a localized goal (e.g., the solution of the problem); and (c) *constructing* a new construct through the integration and consolidation of previous constructions. In this study, we use learning trajectories to identify levels of students' conceptualization of function as covariation and AiC to highlight the development of students' thinking within the learning trajectories.

THE DIGITAL ENVIRONMENT CASYOPÉE

Casyopée combines an Algebra window and a Dynamic Geometry window, which are interconnected. Students can create in Casyopée free or fixed geometric objects (e.g., points), define independent and dependent quantities as magnitudes (e.g., lengths and areas symbolized as c0, c1, c2, etc.) in "geometric calculations" tab and investigate their covariation.

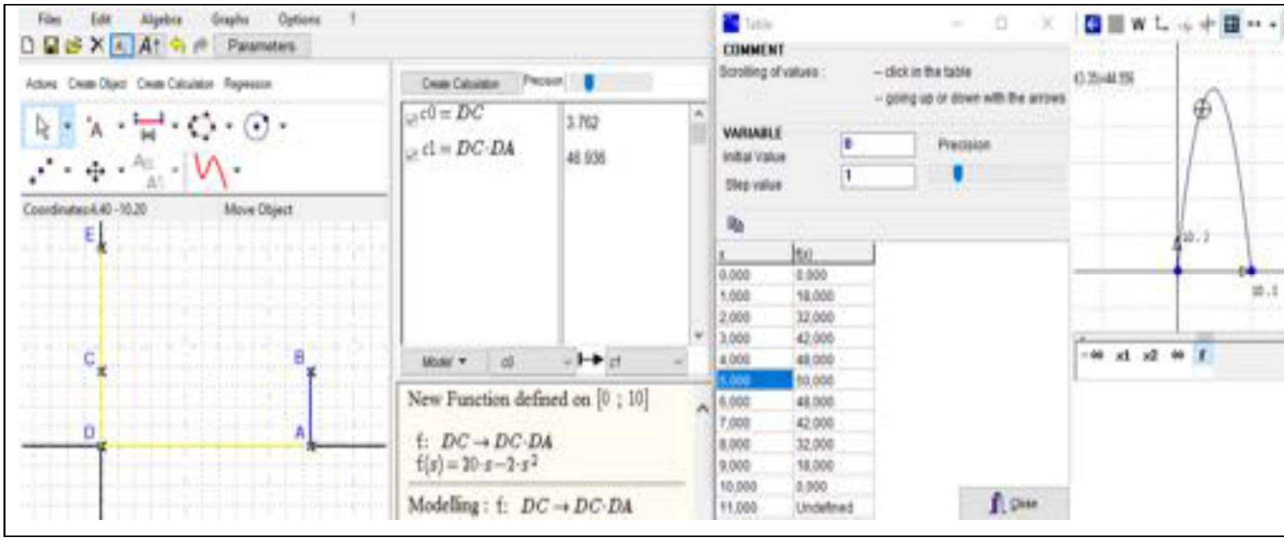

Figure 1: Dynamic Geometry, geometric calculations, table of values and graph.

In addition, through the "automatic modelling" functionality students have the opportunity to check whether a function can be defined using two covarying magnitudes (e.g., c0=DC and c1=DC*DA, Fig. 1). If a function can be defined, its algebraic formula is automatically extracted in the Algebra window, otherwise appropriate information is provided. Finally, a function can be interpreted using different representations, such as the table of values and the graph.

METHODOLOGY

Method, framework, tasks and data collection

This research is characterized as a design research (Cobb et al., 2003) since there were two cycles of implementation, in two secondary schools in Athens. Before the study the students had been introduced to functions according to the mathematics curriculum, including the definition of function, monotonicity and extreme points. A series of three modelling tasks was implemented through the close collaboration between two mathematics teachers (one in each school) and one researcher who acted as participant observer in the classroom.

The three tasks were related to realistic optimization problems and their sequence was such that the covariation appeared from simple to more complex situations according to the expected learning trajectories. A priori we anticipated students' transition from the intuitive approach of covariation by experimenting with manipulatives to deeper conceptualization of covariation between magnitudes and further between variables. The first task (*Gutter Design*) required optimal gutter design to maximize water flow.

The design followed the modelling cycle including students' engagement in: (1) experimenting with the folding of a paper (10 cm X 20 cm), observing the covariations and expressing the algebraic relationship using a variable, (2) designing and exploring a dynamic model that models the problem in Casyopée, (3) experimenting with covarying magnitudes and (4) creating the function that models the problem and resolving it through the available representations. In the first school, 23 students worked in eight groups of three for 14 hours over four months. In the second school, 25 students worked in eight groups of three for six hours over three months. The data collected consists of video recordings and audio recordings (four groups). The data were fully transcribed for the analysis. In this article we analyze the data from the two focus groups from two schools (first school: group 1 – S1, S2, S3 and second school: group 2 – S4, S5) during the implementation of the *Gutter Design* (three hours at one occasion in each school).

Method of analysis

In the first phase of the analysis we coded categories of episodes (open coding, Strauss & Corbin, 1998) based on students' references to covarying magnitudes in the different settings of the modelling cycle. Then we analyzed qualitative elements (e.g., use of symbolism) in students' thinking about function as covariation, taking into account existing classifications (e.g., Carlson et al., 2002) and the expected learning trajectories from the *Gutter Design*. Using continuous comparisons, we traced the initial categorization of the episodes, taking into account students' conceptualizations of function as covariation from simple to complex ones. This resulted in the identification of six levels of students' conceptualization. In the second phase, we analyzed line by line the transcripts in every category of episodes with the help of AiC (recognizing, building-with, constructing) in order to describe students' thinking about function as covariation as an abstraction process. In this analysis we put our notes in brackets within the transcripts.

RESULTS

Level 1: Identifying dependencies

In this level, the students recognized the dependencies of the covarying quantities needed in order to model the problem (e.g., the side length and the cross-sectional area). In the beginning, the students were able to experiment with a paper model and later to model the problem in the software by creating a dynamic rectangle referring to the cross section of the gutter. Level 1 appeared at the first hour of experimentation of the students with (a) the paper model (physical object) and (b) the dynamic rectangle constructed in Casyopée, where they recognized the interdependence between the one side (e.g., DC) and the area of the rectangle ABCD in the Dynamic Geometry window (Fig. 1). For example, in group 1 (school 1) students were experimenting with the paper model with appropriate folds and observed the interdependence of the sides in order to maximize the amount of water passing through the cross-sectional area of the gutter.

1 S1: In order to maximize the water flow, we need to maximize the base, but up to a point. The bigger she gets the more water will pass, but the side walls will become smaller. We need both of them to find the...

2 S2: Aha! We need to maximize the area of this rectangle! [*the cross-section area*]

In this episode, students' experimentation with the paper model helped them to understand the situation. Using the model, S1 recognized the dependence of the two sides in order to maximize the amount of water and observes that both sides are needed to identify a new quantity (building-with) to work with for solving the problem. Finally, S2 constructs the identification of dependencies pointing that the required quantity is the rectangular area.

Level 2: Conceptualizing covarying quantities as magnitudes

The transition from quantities to magnitudes is a critical step in covariation towards more abstract conceptualizations. In this level, the students recognized that changing a magnitude causes a corresponding change to another magnitude. This level also includes cases where students linked the two magnitudes by recognizing that they are proportional. Level 2 episodes appeared during the first and the second teaching hours during an introductory whole class discussion as well as while the students were working with Casyopée and linked the changes between quantities (Dynamic Geometry window) to those between magnitudes (Geometric Calculation window, Fig. 1) (*"As long as one magnitude changes, the other changes too"*).

Level 3: Conceptualizing the direction of change

In this level, the students were able to describe the direction of changes between two magnitudes. Level 3 episodes appeared during the second hour while the students (a) were experimenting with specific values while folding the paper to determine the cross-section area, and (b) were observing the changing values of magnitudes in geometric calculations tab. Level 3 is more sophisticated than the previous level as the students emphasized the direction of change (*"As long as one magnitude decreases, the other decreases too"*). The selected episode (school 1) refers to the experimentation in the geometric calculations tab in Casyopée.

1 R: How did you construct the rectangle?

2 S1: Look at the area here. We see that the maximum area is 50 and as we change this value... [*the value of DC*].

3 S2: Ok. We selected point C, we constructed a parallel line and then we created it. We inserted the coordinates, y-coordinate is equal to zero, but x-coordinate is equal to AD.

4 S1: Ok. Here we cannot say that it is the maximum. We can see that if we change point C in this straight line [*on the segment DC*] the area continuously decreases and maximizes when it [*DC*] gets its maximum value.

5 S2: Look here [*in the geometric calculations tab*] it says 50 and we have the maximum value of segment DC. While we move down point C, we see that the area is decreasing too.

By moving the point C, the students correlated the changes in the length of the segment DC with the changes in the area of the rectangle ABCD in order to maximize the cross-section area of the gutter. As we see in the excerpt, the interconnected representations helped S2 to observe the covariation of measurements. S1 observed that changes on the length DC change also the area values (line 2, recognizing). Then, he was able to link specifically the two covarying magnitudes (line 4, building-with) to answer the problem. Finally, S2 conceptualized the direction of the change of the covarying magnitudes as an abstraction (line 5, constructing) stating that the area decreases as the length of DC decreases.

Level 4: Conceptualizing covarying magnitudes as variables

In the beginning of the third hour, students were able to observe that the pair of the covarying magnitudes can be considered as a pair of covarying variables. In this level, while the students were creating the function to model the problem (automatic modelling) they were able to observe that one variable can be considered as an independent variable and a second one as a dependent. This level is more sophisticated than the previous one as it emphasizes the functional relationship between the two variables (*"If we say DA*DC as dependent (the area) then the independent must be DC, because through its movement the area changes"*).

Level 5: Formalizing the covariation of variables

In this level, the students described the covariation of variables more formal through the use of algebraic symbolism. Conceptualizations of that kind emerged gradually during the third hour. In the beginning, the students conceived the changes of each variable separately and later they connected these variations formally. For example, group 1 students modelled the dynamic rectangle ABCD in Casyopée, defined the independent and dependent variables in automatic modelling and opened the table to examine the values. In the next excerpt, we see how they conceptualized function as covariation formally by observing the changes in each column of the table of values.

1 S1: From the table we see the maximum value [*of DC*], that at 5… [*the area is 50*]
2 S2: It shows the area for each value that x takes with the restrictions we set.
3 S1: If we change the step it shows us the area in relation to the side DC that changes by 0.5. We see that 5 remains the value of the side DC so as to have the maximum area. We notice that for the different values of x the area changes and reaches its maximum in DC [*equal to 5*]
4 S3: Wait. For the various values of x, the area changes and finds a maximum for x = 5 with the area equal to f (5) = 50.

The three students observe the variation of the side DC and its value that maximizes the area ABCD. By linking DC with column x of the table values (line 2), S2 helps S1 to conceptualize the variation of DC as the variation of the independent variable x (line 3, recognizing level 4). Then, S1 experiments with different values in the step of the table so as to determine the maximum area (line 3, building-with). Finally, S3 conceptualizes function as covariation by relating the changes in the two columns of the table as dependent and independent variables (line 3, constructing level 5).

Level 6: Formalizing covarying variables and connecting representations

In this level, the students described the function as covariation using algebraic terms as well as different representations of function. Level 6 appeared during the last experimental hour in both schools when the students had already created the required function in Casyopée using the automatic modelling functionality and answered the problem. For example, using the graph and the algebraic formula $f(x) = 20x - 2x^2$ provided in the Algebra window the students of group 2 (school 2) observed the changes of the two variables and identified that the maximum point is (5, 50). Challenged by the researcher, they were engaged in linking the table of values and the graph in order to explain how they got to their final answer. In doing so, they used primarily algebraic terms.

2 R: Using the graphs can you answer the problem? Which is the better folding?

3 S4: We will find out which point on the x axis has the maximum point and we will identify the coordinates of the peak. Look here, it changes! Look, I move the point and the area changes.

4 S5: We can get the same result, I mean we can find which x corresponds, which y corresponds here and we say that in the top, where the highest point lies, it is the largest and we have found the coordinates from both the graph and the table of values.

CONCLUSION

We used modelling activities to identify levels of students' conceptualization of function as covariation in their transitions in the different settings of the modelling cycle (Lagrange, 2014). The analysis revealed six original hierarchical levels, which inform existing research about the evolution of students' functional thinking in secondary education while working with digital tools combining algebraic and geometrical representations. Our study enriches existing levels of covariation by providing a more subtle categorization of students' thinking taking into account the specificities of the learning context and the rich available repertoire of tools and representations. The crucial role of tools can be highlighted as follows: (a) at level 1 and 2 the transition to the identification of dependencies and the conceptualization of covariation perceptually at the level of quantities was supported by the manipulation of the dynamic rectangle ABCD in the Dynamic Geometry window, (b) transition to levels 3 and 4 where the students moved to describing covariation as the direction of change and working with covarying magnitudes was facilitated by the use of geometric calculations functionality, and (c) further move to levels 5 and 6 indicated by the formal use of algebraic terms and the extended use of multiple representations of functions was promoted through the automatic modelling functionality (i.e. favoring definition of independent and dependent variables) and the availability of multiple and interconnected representations of Casyopée.

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