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Using critical incidents as a tool for promoting prospective teachers' noticing during reflective discussions in a fieldwork-based university course

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Abstract

Preparing prospective mathematics teachers to become teachers who recognize and respond to students' mathematical needs is challenging. In this study, we use the construct of critical incident as a tool to support prospective mathematics teachers' reflection on their authentic fieldwork activities, notice students' thinking, and link it to the complexity of mathematics teaching. Particularly, we aim to explore the characteristics and evolution of prospective mathematics teachers' noticing of students' mathematical thinking when critical incidents trigger reflective discussions. Critical incidents are moments in which students' mathematical thinking becomes apparent and can provide teachers with opportunities to delve more deeply into the mathematics discussed in the lesson. In the study, twenty-two prospective mathematics teachers participated in fieldwork activities that included observing and teaching secondary school classrooms. The prospective teachers identified critical incidents from their observations and teaching, which were the foci for reflective discussion in university sessions. By characterizing the prospective teachers' reflective talk in these discussions, we demonstrate the discussion's evolution. In it, participants questioned learning and teaching mathematics and suggested alternate explanations. This characterization also shows that using critical incidents in the university discussions enabled the prospective teachers to link students' thinking with the teacher's teaching practices while supporting their reflection using classroom evidence. We emphasize the importance of descriptive talk in the discussion, which allows for deepening the prospective teachers' reflections. Further, we explore the teacher educator's contributions in those discussions, showing that the teacher educator mainly maintained the reflective talk by contextualizing the critical incidents and pressing the participants to explain further issues they raised in the discussions. Implications for mathematics teacher education are discussed.

Keywords Critical events · Teacher noticing · Field-based teacher preparation program · Secondary school mathematics

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1 Introduction

Teaching mathematics is a complex task affected by many factors found inside and outside the classroom setting (e.g., Ball & Forzani, 2009). It involves responding to students' thinking to deepen their mathematical knowledge (e.g., Dyer & Sherin, 2016; Thanheiser et al., 2021), forming appropriate socio-mathematical norms that encourage students' engagement (e.g., Yackel & Cobb, 1996), considering students' affects as they engage in the mathematics presented in the classroom (e.g., Op't Eynde et al., 2006), and seeing the social factors that exist inside and outside the classroom and influence students' learning (e.g., Jaworski & Potari, 2009; Rezat & Sträßer, 2012). Seeing the complexity of mathematics teaching is even more demanding for prospective teachers (PTs), as they also need to connect teaching practices and students' learning (e.g., Warshauer et al., 2021) without having enough classroom experience that allows them to make these connections (e.g., Leikin, 2008). Thus, teacher education programs try to bring the complexity of being attuned to the students' learning into the university using fieldwork activities, such as classroom observations and teaching, together with reflection on these activities (e.g., Zeichner, 2012). Nevertheless, when reflecting on these authentic fieldwork activities, PTs have difficulties in developing an interpretive stance of classroom events (e.g., Barnhart & van Es, 2015; Phelps-Gregory & Spitzer, 2021; Rotem & Ayalon, 2023).

Research suggests that developing structures can foster teachers' systematic reflection on teaching practice to make noticing classroom complexity more concrete (Mason, 2002; Santagata et al., 2007). Using teacher noticing in teaching preparation programs was found to be a productive way to structure PTs' reflection on teaching (e.g., Santagata et al., 2007) and focus on students' mathematical thinking (Jacobs et al., 2010; Schack et al., 2013). Noticing students' thinking comprises two interconnected skills: attending to students' mathematical thinking in classroom interactions and interpreting that thinking (van Es, 2011). Here, we use *critical incident* (CI) as a tool to support PTs' reflection on their authentic fieldwork activities, notice students' mathematical thinking, and link it to the complexity of mathematics teaching. Furthermore, adopting a community of inquiry perspective (Jaworski, 2006), where collaborative discussions are a way to reflect on and inquire into teaching, we focus on the PTs' discussions in which they reflect on CIs they identify from their fieldwork. This perspective also emphasizes the role of the teacher educator (TE).

This study is part of a larger research project conducted in the context of a fieldwork-based university course that involves the development of a community of inquiry (Potari & Psycharis, 2018). In it, PTs identified CIs during fieldwork—lesson observation and teaching. These CIs were then used to prompt reflective discussions in the university. The study's originality lies in its structure, where the CIs that PTs pre-identified from their field work trigger discussions in the university course. The few studies that have used this type of structure (Rotem & Ayalon, 2022), typically focused on characterizing CIs. In contrast, our study focuses on the reflective discussions at university with the TE to explore the characteristics and evolution of PTs' noticing of students' mathematical thinking as expressed in these discussions. This focus makes theoretical, methodological, and practical contributions to the underrepresented issue of how individual teacher noticing is shaped by the community and vice versa (König et al., 2022; Scheiner & Kaiser, 2023). Thus, to address the aforementioned research challenges our research questions are as follows:

RQ1: What characterizes PTs' noticing of students' mathematical thinking during reflective discussions triggered by their chosen CIs?
RQ2: How does the PTs' noticing of students' mathematical thinking evolve during these reflective discussions?
RQ3: What is the role of the TE in these discussions?

2 Theoretical background and literature review

Following our main constructs, CI and teacher noticing, and this study context, we first present the theoretical perspective guiding our study (Section 2.1), then the current use of CIs and teacher noticing in fieldwork-based teacher education (Section 2.2). Lastly, we discuss the role of the TE in promoting teacher noticing during reflective discussions triggered by the CIs (Section 2.3).

2.1 Theoretical perspective

This study's unique structure *coordinates* (Prediger et al., 2008) cognitive-psychological and socio-cultural perspectives on teacher noticing. König et al. (2022) distinguish between these two perspectives. The cognitive-psychological perspective "characterizes noticing as a set of cognitive processes or processes that take place in the minds of individual teachers, such as perceiving salient incidents in a classroom and interpreting and making sense of those incidents" (p. 2). The socio-cultural perspective "draws on the socio-cultural and practice-based construct of professional vision developed by Goodwin (1994)" (p. 3). It emphasizes the societal and contextual aspects of teacher noticing, such as participating in the university discussions where discourse and community practices of inquiry shape the PTs' interpretations together with the diverse sources upon which the PTs base their interpretations of CIs (e.g., prior educational experience as a student; prior experience as a private tutor and so forth).

Indeed, PTs' noticing is a dynamic process, which "is not solely a psychological process, but also a socially situated activity shaped by discursive practices and sociopolitical contexts" (Scheiner & Kaiser, 2023; p. 1-105). Building on this view that the two perspectives of noticing are interwoven, in this study, we coordinate them to better understand the characteristics and evolution of PTs' noticing of students' mathematical thinking when CIs trigger reflective discussions. When coordinating the perspectives, we take complementary components from the cognitive-psychological and socio-cultural perspectives of teacher noticing to build a comprehensive framework (see Section 3) (Prediger et al., 2008).

In this study, the PTs initially identify CIs from the fieldwork activities, where they are placed in pairs and experience the classroom individually. This reflects their individual noticing of student's mathematical thinking. Then, they share the CIs in discussions in the university, where the PTs' cohort form a community and share a "collective experience… [of] construct[ing] meaning over a period of time from their lived interactions with each other" (Amador, Wallin, et al. 2021; p. 559). In the discussion around a CI, the PTs and the TE may raise questions and interpretations about the student's mathematical thinking. The questions and interpretations one participant raises may influence the interpretations and recollections of the CI by other participants, making inquiry *a tool* that supports *being* a participant in the university discussions (Jaworski, 2006) while also introducing the need to use the socio-cultural perspective in this study.

Indeed, in the discussion around a CI, which is a socially situated activity, the roles and goals of the PTs and the TE "are both distinct and deeply intertwined" (Jaworski, 2006; p.187). Usually, when reflecting on practice, mathematics teachers align with current teaching norms (Wenger, 1998). This in turn requires critical alignment where the teachers are "critically questioning roles and purposes as a part of their participation for the ongoing regeneration of the practice" (Jaworski, 2006; p.190). Thus, part of the TE's role is prompting and facilitating the community of inquiry so that participants reflect on the CI from different perspectives: that of the students, and that of the teacher who was teaching during the identified CI.

Consequently, we use frameworks guided by both perspectives: (1) frameworks that align with the cognitive-psychological perspective of teacher noticing (König et al., 2022; van Es, 2011) and (2) frameworks that show how individual noticing is shaped by community participation (Bragelman et al., 2021; König et al., 2022) and vice versa, enacting the socio-cultural perspective of teacher noticing. More details of the particular frameworks we use in this study are provided next.

2.2 CIs and teacher noticing in fieldwork-based teacher education

Preparing PTs to become teachers who recognize and respond to students' mathematical needs (e.g., Hiebert et al., 2019; Jacobs et al., 2010) is challenging. PTs are expected to recognize the complexity of mathematics learning and teaching without having enough practical experience (Leikin, 2008). Therefore, most teacher preparation programs involve fieldwork to bring theory with practice closer and serve as a primary source for PTs' learning (e.g., Ball & Forzani, 2009; Grossman et al., 2009). In recent years, creating a framework for reflection for PTs is an issue that has triggered researchers' attention in mathematics teacher education (e.g., Karsenty & Arcavi, 2017; Santagata et al., 2007). In this study, we facilitate the PTs' reflection using CIs and teacher noticing.

We address a CI as a construct that focuses on students' learning to bring forward the complexity of mathematics learning and teaching. Rotem and Ayalon (2022) defined a CI as a "moment in which students' mathematical thinking becomes apparent and thus can provide teachers opportunities to delve more deeply into the mathematics discussed in the lesson" (p. 1). They suggest characterizing the CIs that PTs identify during classroom observation and teaching through a three-axis model containing the event's focus (i.e., mathematics, pedagogy, combined mathematics and pedagogy), the event's participants (i.e., the students, the teacher, and the students and the teacher combined), and dimensions of learning and teaching (i.e., cognitive, social, and affective). Consequently, a CI can be characterized using several combinations of the model's elements. In this study, we follow Rotem and Ayalon's (2022) definition and use CIs to promote PTs' noticing. Furthermore, when using the construct of a CI we draw on scholars who argue for CI's subjective nature (e.g., Rotem & Ayalon, 2022; Yang & Ricks, 2012). According to this perspective whatever the PTs identify as a CI is crucial for their learning and therefore it is considered a CI.

Teacher noticing originated as a practice that involved identifying what is significant in a classroom interaction, interpreting this noteworthy incident based on knowledge and experiences, and linking it with broader principles of learning and teaching (e.g., van Es, 2011; van Es & Sherin, 2002). Teacher noticing is the most prominent construct in the field of mathematics teacher education (Amador, Bragelman, et al. 2021) and is considered to be an appropriate framework for reflection (e.g., Santagata et al., 2007) on a CI. A predominant framework used to promote teacher noticing (Amador, Bragelman, et al. 2021) is van Es's (2011) *learning-to-notice* framework. It consists of four levels of noticing according to "what teachers notice" and "how teachers notice." This framework is detailed in Table 1.

When the learning-to-notice framework is used in a methods course, it can support PTs' in shifting from attending to and describing teaching to interpreting and linking it with students' learning (e.g., Roth McDuffie et al., 2014). Most research that has addressed teacher noticing as a way to promote PTs learning uses the PTs' written work (e.g., Amador, Bragelman, et al. 2021).

However, many teacher preparation programs include both written assignments and discussions. These emphasize the importance of learning about the PTs' noticing as it is expressed during discussions. In fact, Amador, Wallin, et al. (2021) suggested that when participating in teacher professional community discussions over time, the participants' noticing converges as the result of the "social construction of meaning based on collective experiences" (p. 1). Moreover, two decades ago, teacher noticing was commonly viewed as comprising of two linearly enacted skills—first attending to the students' mathematical thinking and then interpreting it. Recent theoretical and empirical works, though, have suggested that this might not be the case (Scheiner, 2021). Particularly in professional community discussions, when a teacher attends to a CI, they might ask further questions to support their interpretation of the students' mathematical thinking. This in turn can influence what the teacher is attending to in the CI (van Es & Sherin, 2021).

Bragelman et al. (2021) investigated PTs' noticing of students' thinking during collaborative discussions around videotaped lessons. They used a modification of learning to notice framework to show PTs' noticing trajectories within a whole-class discussion and how individual PTs' noticing shifts alongside the TE's contributions. The two studies above (Amador, Wallin, et al. 2021; Bragelman et al., 2021) are the only ones we found that investigated how one PT's noticing, as expressed when they talk, might influence another PT's noticing in the collaborative discussions. Hence, our study aims to contribute to this underrepresented, yet important issue of the way individual teacher noticing is shaped by the community and vice versa. However, we use a different structure from the usual setting documented in most studies. In this study, the discussions were triggered by authentic real-life CIs that PTs identified during their fieldwork. Further, here we consider

Table 1 The van Es learning to notice framework (Van Es, 2011; p.139)

Level	What teachers notice	How teachers notice
1 – Baseline	Making general observations about the whole class environment	Providing general impressions and descriptive comments
2 – Mixed noticing	Focusing on teacher pedagogy and beginning to attend to students' thinking	Providing primarily evaluative with some interpretative comments and beginning to refer to specific events and interactions as evidence
3 – Focused noticing	Attending to particular students' mathematical thinking	Providing interpretative comments, referring to specific events and interactions as evidence, and elaborating on events and interactions
4 – Extended noticing	Interrelating particular students' mathematical thinking and teachers' teaching strategies	Making connections between events and principles of learning and teaching and suggesting alternative pedagogical actions

the TE as part of the community as they also shape and influence the social construction of meaning in the discussion.

2.3 The teacher educator's contributions when facilitating reflective discussions

Facilitating the discussions around a CI is crucial to successfully supporting teachers' learning (Gibbons et al., 2021). Much of the recent research addresses the TEs' challenges (e.g., Borko et al., 2014), their own professional learning (e.g., Jacobs et al., 2017; Schwarts et al., 2021), and their own noticing of students' thinking (e.g., Amador, 2022). Here, we are interested in the TE's role as part of the community of inquiry (Jaworski, 2006) when facilitating collaborative discussions.

Few studies address TEs' facilitation moves of collaborative discussions around videotaped cases and CIs. This issue started to receive attention about two decades ago (e.g., Nemirovsky & Galvis, 2004). Coles (2013) built on Mason's work (Mason, 2002) and suggested that part of the TE's role is to give the participants time to reconstruct what they observed in a lesson while avoiding judgment and interpretation, and deciding when to move into interpretation. van Es et al. (2014) examined facilitators' in-the-moment contributions in video-based professional development programs and offered a framework for facilitation. In this study, we follow researchers that build on van Es et al.'s (2014) framework (e.g., Tekkumru-Kisa & Stein, 2017) and explore the TE's role when facilitating discussions to characterize how one PT's contribution might influence another PT's interpretation.

3 Methodology

3.1 Research context and participants

The study took place in the context of a 14-week mathematics education undergraduate course (taught by the second author) at a leading university in Greece. The course is a part of the bachelor's degree in the university's mathematics department, and it targets students who intend to become secondary school mathematics teachers (i.e., PTs). It is the main route through which the PTs engage in fieldwork, and its goal is to allow them to reflect critically on aspects of mathematics learning and teaching that emerge from the complex classroom environments.

Our study participants were 22 PTs (9 males, 13 females) who attended the course, eight postgraduate students who served as PTs' mentors and accompanied them to schools, and the second and third authors. The PTs' mentors were informed about the aims of the course and the idea of CIs and had access to the course materials. The mentors' goal was to support the PTs by providing feedback on their lesson designs and chosen CIs. The second author served as the TE and as a mentor for one pair of PTs. The third author participated in the university sessions as a participant-observer.

3.2 Research design and data collection

The mathematics education undergraduate course alternated fieldwork in schools (4 h per visit) and 3-h university sessions (details in Table 2).

As CIs are the main tool for promoting teachers' noticing during reflective discussions in this study, we introduced the participants to the idea of a CI in the first two university sessions prior to the fieldwork. In session one, the TE presented the meaning of a CI through an existing literature definition, which appears in the beginning of this paper (taken from Rotem & Ayalon, 2022; p. 1). Then, participants watched some clips showing classroom interactions taken from a videotaped lesson from Japan about inequalities (TIMSSVIDEO website: link). Afterwards, they worked in groups to identify a CI by reading the videoclips' transcript. The participants could identify numerous CIs in these clips. Emphasizing the perspective that whatever they chose is crucial for their learning (e.g., Rotem & Ayalon, 2022; Yang & Ricks, 2012), the TE did not limit them to selecting a particular CI, but rather choosing CIs that focus on students' thinking according to our definition. Finally, the participants answered a *Critical Incident (CI) Analysis Worksheet* containing five questions about their chosen CI (Fig. 1). The questions aim to support the PTs CIs' interpretation, using prompts to facilitate varying attention and interpretation levels, ranging from baseline to extended noticing (van Es, 2011). For example, question 2 prompts PTs to think on teacher pedagogy

Table 2 PTs' fieldwork and university activities in the course

Week	Fieldwork activity	University session activity
1, 2		Presenting the idea of CIs using existing literature definitions.
3	Describing students' engagement in the lesson briefly and how the teacher managed students' queries and/or wrong responses.	
4		Collaborative, reflective discussions around CIs
5	Analyzing teachers' and students' questions.	
6		Collaborative, reflective discussions around CIs
7	Identifying an unexpected contribution by a student and discussing it with the student and the teacher after the lesson.	
8		Collaborative, reflective discussions around CIs
9	Designing and teaching a lesson to one group of students outside the classroom.	
10		Collaborative, reflective discussions around CIs
11	Designing a part of a lesson around one task and implementing it in the classroom.	
12		Collaborative, reflective discussions around CIs
13	Designing and teaching a whole lesson in the classroom	
14		Collaborative, reflective discussions around CIs

Critical Incident Analysis Worksheet

Please answer all the following five questions:

1). Why is the CI significant to mathematics learning and teaching?

2). How do you interpret what is happening in this CI? You can use the course readings.

3). How can the CI be linked to broader mathematics learning and teaching principles?

4). How do you evaluate the way the teacher managed this CI?

5). What would you do if you were the teacher?

Fig. 1 A critical incident analysis worksheet

(learning-to-notice level 1), while question 5 prompts them to connect principles of learning and teaching and suggest alternative pedagogical actions (learning-to-notice level 4).

In the second university session, participants presented their answers to the worksheet questions and were further challenged by the TE to justify why they considered the selected incident as critical. Participants argued their interpretations, commented on/questioned existing teaching practices, and suggested alternative ones.

After the first two university sessions, the PTs were paired and placed in schools for the fieldwork activities (classroom observations and teaching) under the supervision of their mentors (see below). In the fieldwork activity, PTs were asked to (1) keep systematic notes and/or recordings (audio/video) of the lessons, (2) select a CI (from their notes/recording), (3) justify its selection, and (4) analyze it according to the CI analysis worksheet. As seen in the fieldwork activities column (Table 2), the fieldwork activities included a theme for the fieldwork, which was the main task for the PTs during that week. The assumption was that in the following week's university session, the PTs' identified CIs would align with this theme. Nevertheless, the TE was open to the emergence of other CIs, even those not concerned with these themes.

After a fieldwork week, the PTs would each present their selected CIs for reflective discussion in a 3-h session at the university (about three PTs each session). Usually, the discussion would begin with the PT presenting their identified CI while the TE facilitated the discussion using the facilitation moves described in van Es et al. (2014). These moves were modified for the purpose of facilitating discussions around authentic CIs the PTs had identified from their own fieldwork in schools. Table 6 details the TE contributions to the discussions. The structure for each discussion followed the CI analysis worksheet (Fig. 1). Yet, it was unique due to the different CIs presented by various PTs while bringing forward different issues concerning learning and teaching mathematics.

For the large research project of which this study is a part of (Potari & Psycharis, 2018), we collected all data related to the discussions around the CIs. This data consisted of (a) PTs' CI analysis worksheets, lesson plans, presentation files and materials used in their own teaching; (b) video recordings/transcripts of all university sessions (8 in total) that captured the whole-class discussions, and (c) researchers' field notes. This study focuses on the reflective discussions around a CI, so here we analyze the video recordings of the university sessions.

3.2.1 Ethical considerations

All rights of confidentiality and withdrawal from the study were confirmed via written consent from the participants. To avoid biases related to the dual role of the TE as teacher and researcher, the third author was responsible for the data collection process.

3.3 Data analysis

We carried out a three-phase data analysis of the transcribed university discussions. Building on our study goal of exploring the characteristics and evolution of PTs' noticing of students' mathematical thinking when CIs trigger reflective discussions, we first reduced the data of the large research project. We focused here on extracts from the reflective discussions in which the CIs focus on students' mathematical thinking. Other issues that were explored using the full data set collected in the large research project can be found in Potari and Psycharis (2018). In the second phase, we considered each of the participants' and the TE's talk-turns (turn at speaking, see Bragelman et al., 2021) in the transcribed discussion as a unit of analysis. In line with our theoretical perspective, we coded (1) the participants' talk-turns and (2) the TE's facilitation moves. For the participants' talk-turns we built on the learning-to-notice framework (van Es, 2011), which is often viewed as the cognitive-psychological perspective, together with Bragelman et al.'s (2021) framework, and a bottom-up analysis. For the TE's facilitation moves, we used van Es et al.'s modified framework (Van Es et al., 2014). Third, we used talk-turns graphs (Bragelman et al., 2021) to track patterns across the discussions. The following details each data analysis phase.

3.3.1 Phase 1

The first phase goal was to reduce the data and focus on extracts in which the CIs bring students' mathematical thinking forward. Overall, the university reflective discussions comprised 14 extracts, of which six met our goal in the sense that each one consisted of a discussion around one CI dealing with students' thinking in the lesson. These six extracts consisted of 207 talk-turns (from participants and TE). Table 3 details them.

Table 3 Details of the six extracts analyzed in this study

Extract	Course week	University session	Participants' talk-turns	TE's talk-turns	Total no. of talk-turns
1	8	5	11	9	20
2	10	6	35	35	70
3	10	6	14	8	22
4	12	7	33	17	50
5	14	8	10	8	18
6	14	8	19	8	27
Total			122	85	207

3.3.2 Phase 2

The Phase 2 goal was to code all analysis units, both the participants' and the TE's talk-turns, in each of the six extracts. We could then use this coding for answering our RQ1 (characterizing PTs' noticing of students mathematical thinking during the reflective discussions) and RQ3 (the role of the TE). Of the 207 talk-turns, 14 turns were not related to noticing as they were too short to code (i.e., when participants agreed or disagreed with other participants or the TE: "yes," "true," and so forth). This resulted in 193 talk-turns, representing 93% of the extracts. One hundred eight talk-turns were the participants' turns, and 85 turns were of the TE.

To characterize the participants' talk-turns, we started with top-down coding according to van Es's (2011) levels of noticing. As discussed in the literature review, van Es's (2011) framework suggests that an interpretation contains three aspects: quality, some degree of evidence (from none to evidence-based), and links to broader learning and teaching issues. While coding, more categories emerged, so we fine-grained the characterization in a bottom-up coding as follows. We separated van Es' (2011) highest level into its three main sub-characterizations—quality, sources, and links—and considered each aspect a talk-turn's characterization. We define *sources* as the evidence brought in the talk-turn and *links* as connections to broader learning and teaching issues. Then, we further elaborated and developed the sub-characterization using other frameworks that built on van Es' (2011) framework: Bragelman et al. (2021) to refine the sub-characterization of quality types (Table 4) and Rotem and Ayalon (2022) to refine the sub-characterization of the links. Sub-characterization for sources was drawn solely on the bottom-up characterization (Table 5).

To characterize the TE role as revealed during her talk-turns, we used the van Es et al. (2014) framework to first carry out top-down coding. However, van Es et al.'s (2014) framework was designed to describe the facilitators' moves during video-based professional development programs. We therefore changed the wording of the framework's definitions to fit our study's structure in which the CIs were identified in fieldwork activities. We also refined and contextualized the van Es et al. (2014) framework so it would encompass all the TE's talk-turns in the university sessions. Table 6 depicts the van Es et al. (2014) framework that we modified to fit our context.

Table 4 Quality types for each of the participants' talk-turns

Quality type	Definition
Descriptive	Participants' talk-turns that describe the CI. They give details about the task, textbook, number of students in class, grade, school, etc. The participants could also describe the behavior of the students or the teacher.
Evaluative	Participants' talk-turns that evaluate some occurrences in the CI. This type of quality includes expressions like "Nice," "I liked that…", or "I don't think that this is good for…"
Interpretive	Participants' talk-turns that account for some occurrences in the CI. They try to explain why something happened in the CI.
Alternate explanation	Participants' talk-turns that account for some occurrences in the CI while adding a different explanation to the already discussed.
Questioning	Participants talk-turns that raise questions regarding some occurrences in the CI. These questions address essential issues indicating an inquiry stance to mathematics learning and teaching.

Table 5 Types of possible links and sources characterizing the participants' talk-turns

Types of links	Affective issues; cognitive issues; socio-cultural aspects inside the classroom; socio-cultural aspects outside the classroom; mathematical content; teaching practices; and research findings.
Types of sources	Prior educational experience as a student; prior experience as a private tutor; textbook and other curriculum resources; classroom evidence; teachers/mentors explanations; mathematics education principles/papers/courses.

Appendix depicts an example of the coding process using the coding schema for a reflective talk-turn (Table 4), source, and links (Table 5) together with the TE talk-turn (Table 6). The second and third authors transcribed and translated the data set from Greek to English. They carried out the first phase of the data analysis, the initial top-down coding, according to the van Es (2011) framework. The first author fine-grained the characterization in

Table 6 The van Es et al. (2014) framework for TE's contributions as modified for this study

TE's contributions		Definition
Orienting group to the CI	Launching	Pose general prompts to elicit participants' ideas.
	Contextualizing	Provide or ask the PTs to provide additional information about the classroom context, the lesson, the mathematical content, or other relevant information related to the incident.
Sustaining an inquiry stance	Highlighting	Direct attention to noteworthy student ideas in the CI.
	Lifting up	Identify an important idea that a participant raised in the discussion for further discussion.
	Pressing	Prompt participants to explain their reasoning and/or elaborate on their ideas.
	Offering an explanation	Provide an interpretation of an event, interaction, or mathematical idea, from a stance of inquiry or while connecting to research in math education.
	Countering	After the participant's explanation, the TE offers an alternative perspective.
	Clarifying	Restate and revoice to ensure a common understanding of an idea.
Maintaining a focus on the CI and the mathematics	Redirecting	Shift the discussion to maintain focus on the CI analysis.
	Pointing to evidence	Contribute substantively to the conversation, using evidence for reasoning about learning and teaching with CIs (describing the classroom's observation).
	Connecting ideas	Make connections between ideas raised in the discussion.
Supporting group collaboration	Standing back	Allow the group members time to discuss an issue.
	Distributing participation	Invite participants to share different ideas based on who is (and is not) participating. This is a verbal invitation and also through gestures where the TE approaches other participants in the discussion.
	Validating participant ideas	Confirm and support participant contributions.

a bottom-up data analysis and completed the coding process for the participants and TE's talk-turns.

To ensure the validity and reliability of the analysis, each author analyzed individually 1/3 of the data using the coding scheme. Then, we compared our coding, disagreements were discussed until a consensus was reached, and some categories were refined (Lincoln, 1995). Finally, the first author completed the data analysis, while the second and third authors cross checked the entire analysis while also referring to the original discussions' scripts.

3.3.3 Phase 3

The goal of this phase was to answer our research questions. First, to answer RQ1 (characterizing the PTs' noticing) and RQ3 (examining the role of the TE), we used the codes from the previous phase and counted their frequencies in relation to qualities, links, and sources. Second, to answer RQ2 and explore how the discussion evolved and note the TE's contributions, we drew on a micro-analysis conducted by Bragelman et al. (2021) and graphed the participants and the TE's talk-turns in each of the reflective discussions around the CIs (for extracts 1–6). Some of the graphs are presented in Section 4 (Figs. 3, 6, and 7).

We used three graphs to show the evolution of the discussion in terms of the quality and the contributor of the talk-turn (Fig. 3); the evolution of the discussion in terms of the quality, links, and sources (Fig. 6); and the relationship between the participants and the TE's contributions (Fig. 7). In all these graphs, the y-axis depicts the types of quality (reported in Table 4), and the *x*-axis depicts the talk-turn number in the reflective discussion in that extract. Each data point represents a coded participant's talk-turn. When a talk-turn is not marked with a data point, this turn was not related to noticing.

4 Results

This section comprises three sub-sections following our three research questions. Each section presents findings for each research question.

4.1 Characterizing PTs' noticing of students' thinking

The systematic analysis of the reflective discussions brings forward three main characteristics of PTs' noticing: the *quality* of the PTs' talk-turns as they analyzed the CI in the discussions, the *links* the PTs draw between the students' activity and other issues concerning learning and teaching mathematics (e.g., affective issues, socio-cultural issues inside and outside the classroom), and the *sources* the PTs' use to support their analysis during the discussions. Table 7 depicts the quality of the PTs' talk-turns during their CI analysis.

As mentioned above, the participants made 108 talk-turns. Further, two turns have been coded twice, with two different codes. To allow continuing with the analysis, we counted them as two additional PTs' talk-turns. Thus, we received the total number of talk-turns to be 110.

In terms of the *quality* characteristic of the PTs' talk-turn, the most prominent type of quality is the interpretive stance (43 out of 110: 39%), followed by descriptive type (30 out

Table 7 Frequency of the PTs' talk-turns in terms of quality types

Quality type	Number of talk-turns	% of talk-turns in terms of quality type (out of 110 turns)
Descriptive	30	27%
Evaluative	9	8%
Interpretive	43	39%
Alternate explanation	16	15%
Questioning	12	11%
Total (*N*)	110	100%

of 110: 27%), alternate explanations (16 out of 110: 15%), and questioning (12 out of 110: 11%). Evaluative talk-turn was the least prominent type (9 out of 110: 8%). Yet, to have a broader view of the characteristics of the PTs' talk-turn, we present the different links and different sources the PTs used when analyzing the students' activity during the reflective discussions (Table 8).

Within the 110 talk-turns coded, there are twenty-seven times when these talk-turns did not have any links or sources, and 102 times where the PTs used a source or a link to analyze the CIs (one turn can include more than one link and/or one source, Table 8). This resulted in a total of one hundred and twenty-nine instances where we coded for links and sources. In 91 out of these 102 times, the participants made links between students' activity and broader aspects of learning and teaching mathematics. The most prominent links are students' activity and the teaching practices used in the CI (24 times), students' cognitive aspects (23 times), and the mathematical content (23 times).

In 53 out of these 102 times, the participants used various sources as evidence to support their claims. Table 8 shows that the main sources are classroom evidence, including the CI itself (30 times) and the speakers' own prior educational experience (11 times). Additionally, Table 8 brings forward the complexity inherent in the participants' talk-turns in the discussions. For example, although 49 times (out of 76), the participants did not use any source to support their analysis, they did make links between students' activity and various aspects of mathematics learning and teaching: students' activity and the mathematical content of the CI (17 times), students' activity and their cognitive aspects (15 times). Another example is found in classroom evidence. Within the 30 times participants brought evidence from the classroom to support their analysis, 14 contained links between their analysis of the students' activity and teaching practices. Conclusively, most talk-turns were characterized as interpretive and descriptive turns that contained links between students' activity and either teaching practices, students' cognitive aspects, or the mathematical content, and used classroom evidence as the main source for the reflections. Based on these results, in the following sections we elaborate on the evolution of the reflective discussion's characterization in terms of quality, links and sources, and the TE's role in promoting the discussion's evaluation.

Table 8 Links and sources used by PTs while analyzing the CI in the discussions

		Links: student activity								
		No link	Affective aspects	Cognitive aspects	Socio-culture aspects in the classroom	Socio-culture aspects outside the classroom	Math content	Teaching practices	Research findings	Total
Source	No source	27	2	15	3	5	17	7	0	76
	Prior educational experience	2	1	4	2	0	1	1	0	11
	Prior experience as a private tutor	1	0	2	1	0	0	0	0	4
	Textbook and other curriculum resources	1	0	0	0	0	1	1	0	3
	Classroom evidence	6	1	2	3	0	4	14	0	30
	Teacher/mentor explanations	1	1	0	0	0	0	0	0	2
	Math education research	0	1	0	1	0	0	1	0	3
	Total	38	6	23	10	5	23	24	0	129

4.2 The reflective discussion's evolution in terms of quality, links, and sources

Throughout the dataset, the discussions evolved in terms of quality, links, and sources. A discussion usually starts with a talk-turn of descriptive quality and then evolves to more profound types of quality. We use extract 4 from the dataset to exemplify this finding (here and in the following section). Extract 4 is a discussion that took place around one PT - Katia's[1]—chosen CI in the seventh university session (Table 3). Katia had designed the task at the center of the CI (Fig. 2) and she identified the CI from a lesson she had taught as a part of her fieldwork. The task asked students to determine "Which is bigger, Figure A (top figure) or Figure B (bottom figure)?" The CI was that students came up with many different solutions to the task, which Katia had not anticipated. Figure 3 depicts the evolution of the discussion around the CI in terms of the type of quality of the talk-turns.

The discussion starts with a descriptive type of quality. Katia is providing many details about the CI (turns 1 and 3). Additionally, she emphasizes what is critical for her: "The important thing … was that I got many different solutions for the 1st plot." Then, in the fifth turn, she moves into a questioning quality type, saying: "My concern here was *why there were so many different solutions*. I wasn't prepared for that." Katia seemingly did not anticipate that the task she had designed would raise so many different solutions. In turns 5 to 13, Katia continues to bring additional quality types to the discussion. Then, from talk-turn number 15 to 18, the discussion goes back to the question Katia raised about the possible reasons for the many students' solutions in the CI (Fig. 4). In these talk-turns, other participants bring different quality types.

This excerpt demonstrates how descriptive talk-turns propel the discussion into other quality types. Katia's CI and her reflection on it, in the form of questioning in talk-turn number 5, are a part of her internal reflective cognitive process. Then, as shown in the excerpt above, it becomes a collective experience within the community setting. Through the social and cultural context of constructing a shared meaning, Leonidas, Angela, and Stella shape their interpretations based on what is raised in the discussion. In talk-turn 15, Leonidas makes an analogy between having different solutions and having different ways of mathematical understanding. To support his reflection, he says, "it is something I have noticed myself." In turn 16, Angela *expands* Leonidas' interpretation saying that even the same person can understand the same mathematical content differently depending on the person's perspective at that time. Then, Stella *suggests a different* analysis of interpretive quality while linking students' activity with the teachers' teaching practices. In talk-turn 22, after a couple of turns, Katia focuses on a particular student and articulates a specific solution strategy he used to solve the task. It seems that the other participants' analyses of the possible reasons for the many students' solutions influence her noticing as she begins to focus on a particular student's thinking.

Looking at the discussion around Katia's CI (Fig. 3), we argue that there is an interplay between the descriptive talk-turns and the more profound quality talk-turns. We show this using the rectangles in Fig. 3. These rectangles divide the talk-turns into chunks, starting with a descriptive talk-turn and ending with the other types—evaluative, interpretive, alternate explanation, and questioning. They show the back and forth between the descriptive and the other types of quality. It could be that the descriptive talk-turns feed the conversation with more details of the CI, thus enabling PTs to build on

[1] Note: All PT names are pseudonyms.

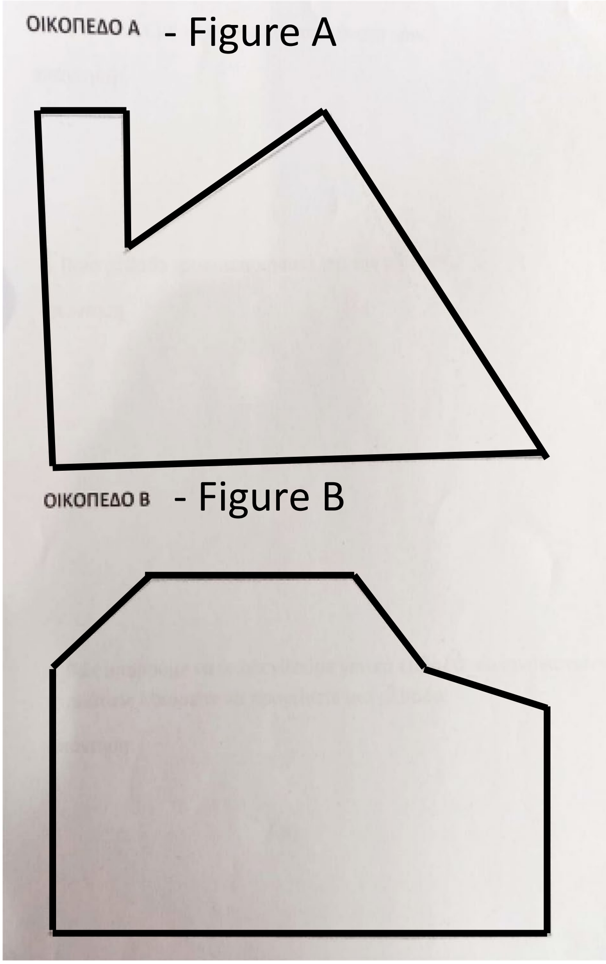
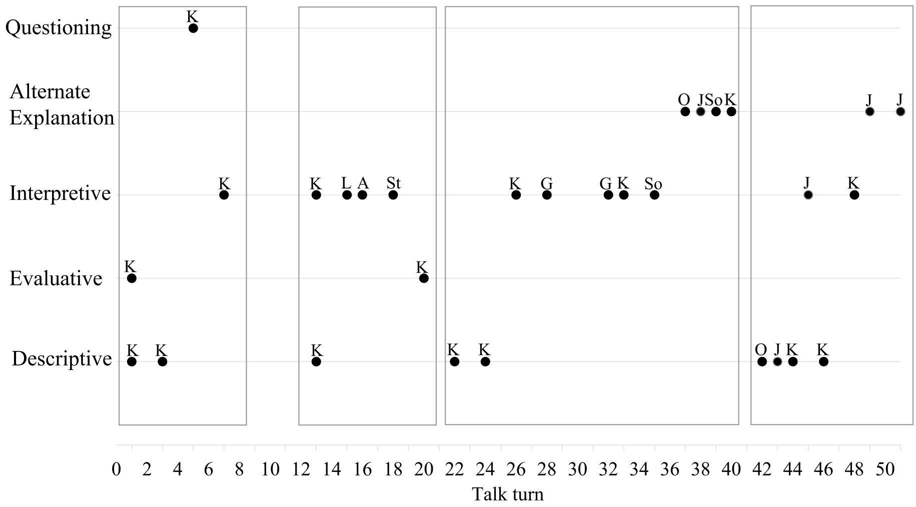


Fig. 2 The first task in Katia's worksheet

them and carry out their analysis. For example, in turns 43–45 James asks for descriptive details, which Katia provides. Only then does James offer his interpretation (Fig. 5).

After Katia provides additional details, James says "Nice! Okay" which may indicate that this information helped him to better understand the CI, as demonstrated by the analysis he provides in his next sentence.

Overall, in terms of the quality of the talk-turns, we recognize two ways in which the discussion is evolving. First, Katia, the PT who identified the CI deepens her analysis. Such deepening is expressed by variation in the types of quality characterizing her talk-turns: descriptive (turns 1, 3, 13, 22, 24, 44, 46); evaluative (13, 20); interpretive (7, 26, 33, 48); alternate explanation (40) and questioning (5). Second, it is not only the PT who identified the event who deepened their analysis, but also other PTs who took part in the discussion and changed their talk-turns quality. For example, Stella added an interpretive analysis to the discussion (34) followed by an alternate explanation type of quality (39) (Fig. 3). James



Note: This figure shows the evaluation of the quality of the participant's talk-turns in Extract 4. Each data point represents a participant's reflective talk-turn using abbreviations for their pseudonyms: K-Katia; G-George; St-Stella; A-Angela; L-Leonidas; O-Orestis; J-James; So-Sophia.

Fig. 3 Evaluation of the quality of the participants' talk-turns in extract 4

added descriptive (43), interpretive (45), and alternate explanation (49, 51) types of quality. These changes were not only a manifestation of the participants' internal cognitive processes; instead, they were also influenced by the other participants' talk-turns, as shown in the example above.

As mentioned above, the discussions around the CIs evolved not only in terms of the quality of the participants' talk-turns, but also in the links they made and sources they referred to during their analysis. Figure 6 illustrates the evaluation of the participants' links and sources around Katia's CI.

As shown by the figure, the discussion starts with descriptive and evaluative quality turns without links or sources. Then, in talk-turn 3, sources appear, and from there onward most turns included either links and sources, or just links between the students' activity and different aspects of learning and teaching mathematics. Table 9 depicts the links and sources made in this discussion.

Most of these links were made between students' activity and: the CI's mathematical content (12 times out of 30 turns made by the participants); the teacher's teaching practices (10 times out of 30); and the students' cognitive aspects (9 times out of 30). In terms of sources, the PTs used classroom evidence from their own fieldwork experience or the CI (15 times out of 30) and their own experience as students (4 times out of 30) to support their CI analysis. These results demonstrate the overall results described earlier, particularly how the type of the PTs' turns (Table 7) interact with the different links and sources (Table 8).

All in all, the evolution of the discussion around the CI occurs as the PT who brought the event starts with descriptive turns and then moves to alternate explanation and questioning. Sometimes, the questioning is based on participant observations from the event. In other cases, it is because of the discussion, or personal experience. Further, other PTs build on the descriptive contributions to suggest their analysis—interpretive, alternate explanation, and questioning—while using links and their personal experience. These two ways in which the

Leonidas: “I want to say that the variety of different solutions is quite interesting because it is something I have noticed myself. That is, in mathematics each individual thinks about a problem differently. Each person’s perspective on things is completely different!”

Angela: “Even the same person could take a different perspective for the same thing along time.”

TE: “Do we often see so many different solutions in the classroom, similar to what Katia showed us?”

Stella: “… it has to do with the type of the teacher’s questions and guidance. A task that is too closed does not allow the emergence of different solutions… we don’t need to ask “yes-no” questions... we have to ask why!”

….

Katia: “There was a kid … [who] I didn’t think would give me any answers. [Yet] he said to me “this looks like a chimney [Figure 2(A)]. I will cut it. I will put it as a roof.” That is, he found the shape a little strange. I didn't expect he will engage with the task, but I saw that he had measured some things!”

Fig. 4 PTs’ discussion around the many solutions raised in the CI (talk-turns 15–18, 22)

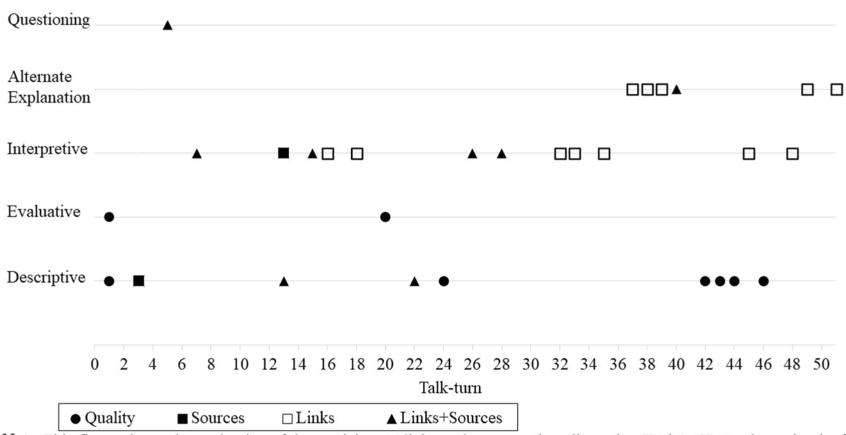
discussion evolves, building on the descriptive talk-turns, are explained through the coordination of the cognitive-psychological and socio-cultural perspectives on teacher noticing. As seen by the examples shared in this section, the PTs deepen their analysis by reflecting on their classroom and personal experiences together while being influenced and influencing the social and cultural context of the discussion when constructing a shared meaning around the CI. Furthermore the TE, who has a part in the social and cultural context of the discussion, also plays a role in prompting this evolution. In the next section, we present the results for the third research question: What is the role of the TE in these discussions?

James: “May I ask something? What grade were the children in?”

Katia: “2nd [year of] high school.”

James: “Nice! Okay. They are relatively young. Nevertheless, we are not smarter than the children. They were using exactly the same data that we are using [with this task] and give the same quality answers. We do not know more about [it] than they do!

Fig. 5 Asking for descriptive details to offer interpretation (turns 43–45)



Note: This figure shows the evaluation of the participants' links and sources when discussing Katia's CI. As shown by the figure, the discussion starts with descriptive and evaluative quality turns without links or sources. Then, in talk-turn 3, links and sources appear, and from there onward most turns included either links and sources, or just links between the students' activity and different aspects of learning and teaching mathematics.

Fig. 6 Evaluation of the participants' links and sources when discussing Katia's CI

Table 9 Links and sources used by the participants in the discussion around Katia's CI (Fig. 6)

Links: students' activity and						
	No link	Affective aspects	Cognitive aspects	Socio-culture aspects in the classroom	Math content	Teaching practices
No source	6[a]		7		7	2
Prior educational experience			1		1	2
Classroom evidence	1	1	1	2	4	6
Total	7	1	9	2	12	10

[a]A talk-turn can include more than one link or source; thus, the numbers in the table indicate how many times a particular link or source appeared

4.3 The role of the TE in these discussions

Overall, the analysis revealed that all of the TE's contributions in the discussions can be characterized using van Es et al.'s (2014) framework with our modifications (Table 6). Table 10 depicts the characterization of the TE's contributions in the discussions throughout the entire data set.

From Table 10, it emerges that the TE used many types of contributions in the discussions, and it is difficult to discuss a prominent one. This may indicate that in order for the discussion around the CI to evolve in terms of quality, links, and sources, the TE should use various types of contributions. Nonetheless, to show what usually happened in our data we demonstrate the four most prominent categories: *offering explanation* (20 out of 106: 19%), *pressing* (19 out of 106: 18%), *contextualizing* (14 out of 106: 13%), and *clarifying* (11 out of 106: 10%). When offering

Table 10 Characterization of the TE's contributions in the discussions

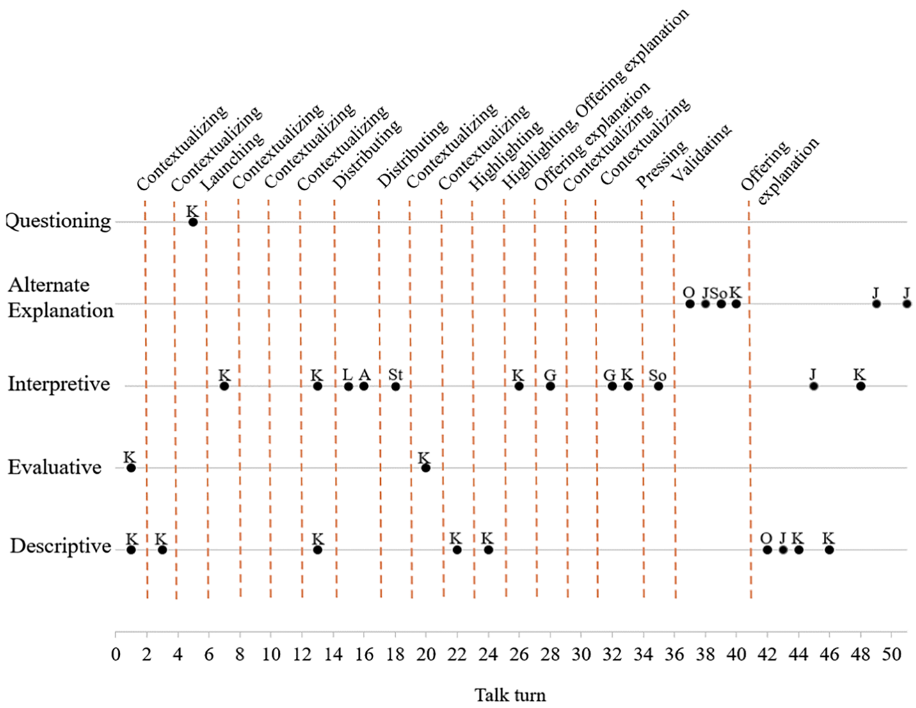
	Number of talk-turns	In percentage (out of 106 turns)
Launching	2	2%
Contextualizing	14	13%
Highlighting	6	6%
Lifting up	5	5%
Pressing	19	18%
Offering an explanation	20	19%
Countering	2	2%
Clarifying	11	10%
Redirecting	3	3%
Pointing to evidence	8	8%
Connecting ideas	1	1%
Standing back	1	1%
Distributing participation	6	6%
Validating participant ideas	8	8%
Total (*N*)	106	100%

explanations, the TE suggests some explanation for the issue raised in the discussion. When pressing, the TE pushes the PTs to reason about the argument or issue raised in the discussion. Contextualizing is when the TE gives or asks for further information of the context of the CI, and clarifying is when the TE clarifies the details, issues, and ideas raised in the discussion.

Figure 7 illustrates the TE's role (dashed lines) in prompting the discussion around the CI by including the information in Fig. 3 with the TE's contributions.

Figure 7 demonstrates the interplay between the descriptive talk-turns and the more profound quality—evaluative, interpretive, alternate explanation, and questioning. Some of this interplay is manifested through the PTs talk-turns and some is due to the TE's prompting as she contextualizes the CI. In talk-turns 2–4, both Katia and the TE give descriptive details which Katia builds on in her questioning in talk-turn 5. This occurs again in talk-turns 8–13 where both the TE and Katia provide descriptive details. In turns 19, 21, 22, and 24, Katia and the TE provide more descriptive details, after which the TE highlights a specific issue for the discussion followed by Katia and George taking interpretive turns (26, 28). Then again, the TE brings more details into the discussion that George and Katia build on in their interpretive turns (29-33). In turn 29, the TE is contextualizing the task that the students had worked on. The TE mentions that usually in similar tasks, the irregular shapes are already divided, but here the students had to divide the shape themselves (31). Building on this TE contribution, George addresses the effect of this task's feature on the students' work: "it is important that [the figure] was not given separately! So that the students had to measure the perimeter" (32). This is followed by Katia's interpretive contribution in which she frames the feature of the undivided irregular shapes as an important task feature, as it requires the students to measure the irregular shapes. Further, Katia addresses the connection between her teaching strategies and the task—the undivided irregular shapes (33).

Besides contextualizing, the TE uses more types of contributions to push the discussion forward. For example, following Katia's interpretive talk-turn 33, the TE pressures and validates the participants' analysis (turns 34, 36). Then Sophia and Orestis offer alternate explanations (Fig. 8).



Note: This figure shows the TE's contributions in extract 4 together with the quality of the different PTs' turns. By including the information in Figure 3 with the TE's contributions, it illustrates the TE's role (dashed lines) in prompting the discussion around the CI.

Fig. 7 TE's contributions in extract 4 together with the quality of the different PTs' turns

In this part of the discussion, the TE has a few contributions—pressing (turn 34), validating (turn 36), and offering explanation (turn 41)—and the PTs carry out the discussion with few inferences from the TE's. Hence, in terms of the discussion evolution, we see that in the beginning, the PTs stay close to the classroom observations either from the CI script or their own observation in the classroom. Then, due to the TE's various types of contributions, they start to detach from the classroom as a source, moving forward with the analysis and making more connections between students' learning and teaching.

33	Katia:	Yes, I didn't want to put... At first, I thought putting some [numbers on the] sides, to give some numbers myself. But I saw that if I gave some numbers, I had to break it up somehow. That is, the heights would not come out as they should, if I gave the sides around...
34	TE:	Why did this activity work? Why really?
35	Sophia:	It invites the students to get involved and act on their own, they need to react!
36	TE:	Nice! Yes;
37	Orestis:	Because it built on something they knew and could do. They have a number. Now, they could go further. They could do it, and they weren't far off from what they could do, so they found it interesting.

Fig. 8 The TE's contributions that support the PTs in carrying out their analysis

Summarizing, throughout the discussion the TE provides various contributions to prompt the discussion's quality. Particularly, the TE's contextualization brings descriptive details through which the discussion around the CI evolves. In a way, it functions similarly to the participants' descriptive turns.

5 Discussion

This paper explores the characteristics and evolution of PTs' noticing of students' mathematical thinking during reflective discussions triggered by CIs they chose in a fieldwork-based university course, while considering the TE's role. We answered three research questions: (1) what characterizes the PTs' noticing of students' thinking during the reflective discussions; (2) how the PTs' noticing of students' mathematical thinking evolves during these discussions; and (3) what is the TE's role in these discussions.

The first question takes the cognitive-psychological perspective of noticing and examines the PTs' individual noticing. The findings reveal that the PTs mostly provided contributions in which their reflection was interpretive and descriptive while making links between students' activity and (1) teaching practices, (2) the students' cognitive aspects, and (3) the mathematical content. Further, they did so while using various sources as evidence to support their reflection, with mainly classroom evidence, including the CI itself, and their prior educational experience as students.

For the second question, to grasp the process in which the PTs' noticing develops in the interaction, we use socio-cultural methodological approaches (Bragelman et al., 2021) and show the discussion's evolution. Indeed, the PTs' noticing evolves through and by the interaction where different PTs express their individual noticing. The social and cultural context of the discussion enables them to build on each other's talk-turn and construct a shared meaning of the CI (Section 4.2). The findings indicate the importance of descriptive talk-turns in the discussions. By searching for more details about the CI (descriptive turns), the PTs seemed to understand the phenomenon better. So, their analysis evolves and introduces evaluative, interpretive, alternate explanations, and questioning talk-turns. Further, most of these interpretive, alternate explanations, and questioning talk-turns included links between the students' activity and different aspects of learning and teaching mathematics.

For the third question, the study also shows the TE's various contributions when forming a community of inquiry (Jaworski, 2006). The TE's four most prominent contributions are offering explanation, pressing, contextualizing, and clarifying through which she facilitated the discussions' evolution in terms of quality, links, and sources. We suggest that the TE's contributions such as contextualizing and pressing enable the discussions' evolution. Indeed, by contextualizing the CI, the TE brings more of the CI's details into the discussion, triggering the PTs to move into interpretive, alternate explanation, and questioning talk-turns while making links and using sources to discuss students' mathematical thinking. In a way, the TE's contextualizing functions as the PTs' descriptive turns, which triggers the PTs to deepen their analysis. We see this as another indication of how the TE's individual contributions, combined with other participants' comments, influence the process by which the PTs' noticing evolves.

Most studies that have examined teachers' noticing of students' thinking utilize videos to facilitate the teachers' noticing (e.g., Amador, Bragelman, et al. 2021; Roth McDuffie et al., 2014), suggesting that the video offers a shared view of the classroom. Here, we suggest a different structure by using authentic CIs identified by PTs in fieldwork activities. It can be claimed that the descriptive talk, whose role is emphasized in the study's findings, is meant to support the participants' shared viewpoint of the classroom context. However, even when all the participants watch the same video, they might attend to different things (Star & Strickland, 2008). Furthermore, recent theoretical and empirical works argue that noticing, as a general construct, is affected by the teacher positioning and sense-making of the CI (Scheiner, 2021; van Es & Sherin, 2021). Thus, we argue that our analysis of both PTs and TE's contributions shows the evolution in which descriptive talk supports further reflection, which in turn stresses the importance of reconstructing the CI while using as many details as possible to enable nuanced reflection (e.g., Carpenter et al., 1999; Coles, 2013; Sherin & Star, 2011).

Further, most frameworks used to explore, measure, and foster teacher noticing perceive descriptive talk as the lower level of noticing (e.g., Jacobs et al., 2010; van Es, 2011). Such frameworks thus imply a hierarchy between noticing levels in which previous levels of noticing should be accomplished before moving into higher levels. Following this view, researchers have claimed that PTs' development of noticing is shown by indicating shifts into higher levels of interpretations (e.g., Bragelman et al., 2021). We argue that our results show that noticing levels are not linear in the sense that after describing the students' thinking in the CI, the PTs can move into more nuanced reflection. The PTs have to go back and forth to the details of the CI to generate their further reflection. Hence, we see this study's theoretical contributions to the theory of teacher noticing: it provides further empirical results to the descriptive talk role as a vehicle for more profound reflection and adds to the accumulated knowledge supporting contemporary views of the non-linearity of noticing (Scheiner, 2021; van Es & Sherin, 2021).

Further, our study coordinates the cognitive-psychological and the socio-cultural perspectives in a unique structure where the PTs identify CIs (cognitive-psychological perspective). Each PT shares a CI in the discussion where their individual noticing is shaped by the community and vice versa (socio-cultural perspective). At the same time, we explore the characteristics and evolution of their noticing. Thus, our study provides one possible variation for a structure that coordinates individual teacher noticing and the community's. This demonstrates possible ways collective noticing evolves (Amador, Wallin, et al. 2021) when considering the different participants' positioning (Scheiner, 2021).

Our study also has methodological and practical contributions. From a methodological perspective, our research provides modified frameworks of noticing students' mathematical thinking that combine the PTs' reflective talks (van Es, 2011) with the TEs' role (van Es et al., 2014). Moreover, Bragelman et al.'s (2021) methodological approaches allowed us to demonstrate the interplay between the PTs' and the TE's talk-turns, showing the development of the community of inquiry throughout the reflective discussions. To the best of our knowledge, the relationship between the TE and PTs in the context of students' noticing has previously only been reported once (Bragelman et al., 2021). Our combination of van Es' (2011) and van Es et al.'s (2014) frameworks to explore the relationship between PTs' reflection and the TE's role can be used by other researchers who wish to use authentic CIs to study professional noticing and community of inquiry growth. This can be achieved in contexts other than our own, such as in-service teachers' professional development programs. From a practical perspective, teacher educators can develop a community of inquiry using CIs in their mathematics teacher education courses. In particular, they can use the structure presented here, as well as the combined frameworks of this study (van Es, 2011; van Es et al., 2014) to facilitate discussions around CIs identified during fieldwork activities.

Although this study provides evidence of the underrepresented, yet important issue of how individual teacher noticing is shaped by the community and vice versa, it has some limitations. First, after almost each PT's talk-turn, the TE responded (Table 3, Fig. 7). Further, not all participants participated in all the discussions around each CI (see the example in Section 4). The characterization and evaluation of the discussions could have been different if all participants had participated more frequently. We assert that this requires more exploration of the TE role and considering how TEs can prompt the PT's participation and engagement to foster even more in-depth reflections by PTs.

Second is the variation in the setting from which the PTs identify the CIs. Some come from observing lessons, some from small group teaching, and others from teaching the whole class. This variation did not allow us to make conclusions about how the setting from which the PTs identify the CI and the discussion evolution are connected. However, this setting reflects the complexity of the authentic nature of fieldwork activities. Thus, we point to the need for more explorations into how researchers and teacher educators, can better support PTs in this complex setting of authentic fieldwork activities composed of various experiences. Specifically, future investigations could explore how PTs can be supported to bring more research findings as evidence and link students' activity with research findings they learned in university courses.

Appendix

Examples of data analysis

Here we use an example to illustrate phase 2 data of the analysis process. We exemplify this using Katia's and the TE's turns taken from extract 4 used as an example in Section 4. Figure 9 shows turns 7–10 of the extract.

We coded the complete turn 7 as *quality-interpretive*, *links-teaching practice*, *and sources-classroom evidence*. We demonstrate how we coded this turn, using the schemes presented in Tables 4 and 5. For convenience, we segmented Katia's words with numbers in brackets. Nevertheless, in the analysis, it was treated as a single unit.

In this turn, Katia tries to explain the diversity of student answers to the task. In segments [1], [3], and [4], she reasons why this could happen: the students worked hard during the lesson [1]; they worked freely, without guidance [4] which allowed them to calculate the areas of the two figures in different ways. Katia links what the students did to her teaching practice. She claims that the reason for the students' diverse approaches is because she gave the time to work [2] without guidance [6]. Further, she supports her

7	Katia:	[1] I think everyone worked hard enough… Especially in the 1st task [2], where we gave them more time. [3] And they tried to calculate it. [4] I thought that if we let the students work freely, they will bring very different things, [5] because we had given them nothing! Just a ruler!... [6] So, if we don't guide them, we can see exactly how they think. [7] They measured and added all the sides together and saw that it didn't work. Then they considered breaking it into pieces and measured the height and other dimensions…
8	TE:	In the discussion!
9	Katia:	Yes
10	TE:	In the discussion! After you have worked through the problems.

Fig. 9 Turns 7–10 from extract 4

Table 11 van Es et al.'s (2014) modified framework for TE's contributions

TE's contributions	Examples from our data
Launching	"And what do you say, huh?" (E4T6)[a]
Contextualizing	"Is this a classroom lesson?" "Among how many children?" (E4T2; E4T4)
Highlighting	"And I also see at the discussion level that some interesting topics have arisen from a mathematical point of view. The children question 'which shapes are irregular?' and 'what does it mean irregular?' …" (E4T23)
Lifting up	"…he remembers! It may just be a recall. Let's discuss further the cognitive actions he does over here and his mathematical activity." (E2T58)
Pressing	"Why don't you like this answer?" (E2T46)
Offering an explanation	"Looks like the kids… were concerned about the area of the trapezium that was visible in there…" (E4T25)
Countering	(PT saying: "I believe that Dimitra belongs to the group of students who are … although do have to study a lot, can think more globally and deeply.") TE: "We could hypothetically say that perhaps Dimitra's knowledge is more formal than Demetris' knowledge. But it does not mean that it is always that the formal knowledge is also deeper." (E5T14)
Clarifying	(PT saying: "Of course, what he says is not correct, i.e., he just remembers something about the rest.") TE: "Right. He remembers something about the remainder, again conceptually and says, 'I remember something: remainder, quotient, smaller...'" (E2T37)
Redirecting	The TE reads from the CI script: "…'The mind as a tool is fine.' 'Fine! We have learned something maybe in the 1st chapter.' What references does the teacher make here?" (E2T24)
Pointing to evidence	"So here it seems at this point that the student gives a correct answer. But the teacher does not want to see only this" (E2T8)
Connecting ideas	The TE connects several PTs' interpretations with research "Here we could say that the guidance itself… gives some hints which help the student to recall, to make a connection on one level of course… We don't know if it goes deeper. We might also see things that perhaps were discussed theoretically in the course Mathematics Education I about Vygotsky and the zone of proximal development... (E3T22)
Standing back	Such an example can be found in extract six after the TE's turn (14), and then the discussion continues without The TE's intervention.
Distributing participation	The TE and Katia discussed the incident. Then the TE asks: "What do you say? What do you have to observe? What do you ask?" After this, Leonidas starts participating in the discussion. (E4T14)
Validating participant ideas	"Nice! Yes." (E4T36)

[a]Brackets refer to the extract number and the talk-turn for the data set. For example, E4T6 means that the example is taken from talk-turn 6 in extract 4

reflection with evidence from the classroom. She articulates different student approaches [7] and says she provided only a ruler as evidence that she did not guide them [5].

Then in turns 8 and 10 the TE contextualizes (Table 6 in Section 3.3 and Table 11) Katia's interpretation by suggesting additional information about when things happened in Katia's CI. Table 11 depicts van Es et al.'s (2014) framework as we refined it with further examples from our data.

Data availability Unfortunately, in order to protect the privacy of the study's participant, our data cannot be shared openly.

Declarations

Competing interest The authors declare no competing interest.

References

Amador, J. M. (2022). Mathematics teacher educator noticing: examining interpretations and evidence of students' thinking. *Journal of Mathematics Teacher Education*, *25*(2), 163–189. https://doi.org/10.1007/s10857-020-09483-z

Amador, J. M., Bragelman, J., & Superfine, A. C. (2021). Prospective teachers' noticing: A literature review of methodological approaches to support and analyze noticing. *Teaching and Teacher Education*, *99*, 1–16. https://doi.org/10.1016/j.tate.2020.103256

Amador, J. M., Wallin, A., Keehr, J., & Chilton, C. (2021). Collective noticing: teachers' experiences and reflection on a mathematics video club. *Mathematics Education Research Journal*, 1–26. https://doi.org/10.1007/s13394-021-00403-9

Ball, L. D., & Forzani, F. M. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education*, *60*(5), 497–511. https://doi.org/10.1177/0022487109348479

Barnhart, T., & van Es, E. (2015). Studying teacher noticing: Examining the relationship among pre-service science teachers' ability to attend, analyze and respond to student thinking. *Teaching and Teacher Education*, *45*, 83–93. https://doi.org/10.1016/j.tate.2014.09.005

Borko, H., Koellner, K., & Jacobs, J. (2014). Examining novice teacher leaders' facilitation of mathematics professional development. *The Journal of Mathematical Behavior*, *33*, 149–167. https://doi.org/10.1016/j.jmathb.2013.11.003

Bragelman, J., Amador, J. M., & Superfine, A. C. (2021). Micro-analysis of noticing: A lens on prospective teachers' trajectories of learning to notice. *ZDM–Mathematics Education*, *53*(1), 215–230. https://doi.org/10.1007/s11858-021-01230-9

Carpenter, T. P., Fennema, E., Franke, M. L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*. Heineman.

Coles, A. (2013). Using video for professional development: the role of the discussion facilitator. *Journal of Mathematics Teacher Education*, *16*, 165–184. https://doi.org/10.1007/s10857-012-9225-0

Dyer, E. B., & Sherin, M. G. (2016). Instructional reasoning about interpretations of student thinking that supports responsive teaching in secondary mathematics. *ZDM–Mathematics Education*, *48*(1), 69–82. https://doi.org/10.1007/s11858-015-0740-1

Gibbons, L. K., Lewis, R. M., Nieman, H., & Resnick, A. F. (2021). Conceptualizing the work of facilitating practice-embedded teacher learning. *Teaching and Teacher Education*, *101*, 103304. https://doi.org/10.1016/j.tate.2021.103304

Goodwin, C. (1994). Professional vision. *American Anthropologist*, *96*(3), 606–633. https://doi.org/10.1525/aa.1994.96.3.02a00100

Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. *Teachers and Teaching: Theory and Practice*, *15*(2), 273–289. https://doi.org/10.1080/13540600902875340

Hiebert, J., Berk, D., Miller, E., Gallivan, H., & Meikle, E. (2019). Relationships between opportunity to learn mathematics in teacher preparation and graduates' knowledge for teaching mathematics. *Journal for Research in Mathematics Education*, *50*(1), 23–50. https://doi.org/10.5951/jresematheduc.50.1.0023

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202. https://doi.org/10.5951/jresematheduc.41.2.0169

Jacobs, J., Seago, N., & Koellner, K. (2017). Preparing facilitators to use and adapt mathematics professional development materials productively. *International Journal of STEM Education*, *4*(1), 1–14. https://doi.org/10.1186/s40594-017-0089-9

Jaworski, B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education*, *9*(2), 187–211. https://doi.org/10.1007/s10857-005-1223-z

Jaworski, B., & Potari, D. (2009). Bridging the macro-and micro-divide: Using an activity theory model to capture socio-cultural complexity in mathematics teaching and its development. *Educational Studies in Mathematics*, *72*(2), 219–236. https://doi.org/10.1007/s10649-009-9190-4

Karsenty, R., & Arcavi, A. (2017). Mathematics, lenses and videotapes: A framework and a language for developing reflective practices of teaching. *Journal of Mathematics Teacher Education*, *20*, 433–455. https://doi.org/10.1007/s10857-017-9379-x

König, J., Santagata, R., Scheiner, T., Adleff, A. K., Yang, X., & Kaiser, G. (2022). Teacher noticing: A systematic literature review of conceptualizations, research designs, and findings on learning to notice. *Educational Research Review*, *36*, 100453. https://doi.org/10.1016/j.edurev.2022.100453

Leikin, R. (2008). Teams of prospective mathematics teachers: Multiple problems and multiple solutions. In T. Wood & K. Krainer (Eds.), *International handbook of mathematics teacher education: Vol. 3. Participants in mathematics teacher education: individuals, teams, communities, and networks* (pp. 63–88). Sense Publishers.

Lincoln, Y. S. (1995). Emerging criteria for quality in qualitative and interpretive research. *Qualitative Inquiry*, *1*(3), 275–289.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. Routledge. https://doi.org/10.4324/9780203471876

Nemirovsky, R., & Galvis, A. (2004). Facilitating grounded online interactions in video-case-based teacher professional development. *Journal of Science Education and Technology*, *13*(1), 67–79. https://doi.org/10.1023/B:JOST.0000019639.06127.67

Op't Eynde, P., Corte, E. D., & Verschaffel, L. (2006). "Accepting emotional complexity": A socio-constructivist perspective on the role of emotions in the mathematics classroom. *Educational Studies in Mathematics*, *63*(2), 193–207. https://doi.org/10.1007/s10649-006-9034-4

Phelps-Gregory, C. M., & Spitzer, S. M. (2021). Prospective teachers' analysis of a mathematics lesson: Examining their claims and supporting evidence. *Journal of Mathematics Teacher Education*, *24*(5), 481–505. https://doi.org/10.1007/s10857-020-09469-x

Potari, D., & Psycharis, G. (2018). Prospective mathematics teacher argumentation while interpreting classroom incidents. In M. E. Strutchens, R. Huang, D. Potari, & L. Losano (Eds.), *Educating Prospective Secondary Mathematics Teachers, ICME-13 Monographs.* https://doi.org/10.1007/978-3-319-91059-8_10

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM–Mathematics Education*, *40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Rezat, S., & Sträßer, R. (2012). From the didactical triangle to the socio-didactical tetrahedron: artifacts as fundamental constituents of the didactical situation. *ZDM–Mathematics Education*, *44*(5), 641–651. https://doi.org/10.1007/s11858-012-0448-4

Rotem, S. H., & Ayalon, M. (2022). Building a model for characterizing critical events: Noticing classroom situations using multiple dimensions. *The Journal of Mathematical Behavior*, *66*, 100947. https://doi.org/10.1016/j.jmathb.2022.100947

Rotem, S. H., & Ayalon, M. (2023). Constructing coherency levels to understand connections among the noticing skills of pre-service mathematics teachers. *Journal of Mathematics Teacher Education*, 1–27. https://doi.org/10.1007/s10857-023-09574-7

Roth McDuffie, A., Foote, M. Q., Bolson, C., Turner, E. E., Aguirre, J. M., Bartell, T. G., … Land, T. (2014). Using video analysis to support prospective K-8 teachers' noticing of students' multiple mathematical knowledge bases. *Journal of Mathematics Teacher Education*, *17*(3), 245–270. https://doi.org/10.1007/s10857-013-9257-0

Santagata, R., Zannoni, C., & Stigler, J. W. (2007). The role of lesson analysis in pre-service teacher education: An empirical investigation of teacher learning from a virtual video-based field experience. *Journal of Mathematics Teacher Education*, *10*(2), 123–140. https://doi.org/10.1007/s10857-007-9029-9

Schack, E. O., Fisher, M. H., Thomas, J. N., Eisenhardt, S., Tassell, J., & Yoder, M. (2013). Prospective elementary school teachers' professional noticing of children's early numeracy. *Journal of Mathematics Teacher Education*, *16*(5), 379–397. https://doi.org/10.1007/s10857-013-9240-9

Scheiner, T. (2021). Towards a more comprehensive model of teacher noticing. *ZDM–Mathematics Education*, *53*(1), 85–94. https://doi.org/10.1007/s11858-020-01202-5

Scheiner, T., & Kaiser, G. (2023). Establishing and emerging theoretical perspectives on teacher noticing. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel, & M. Tabach (Eds.), *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 104–108). PME 46.

Schwarts, G., Pöhler, B., Elbaum-Cohen, A., Karsenty, R., Arcavi, A., & Prediger, S. (2021). Novice facilitators' changes in practices: From launching to managing discussions about mathematics teaching. *The Journal of Mathematical Behavior*, *64*, 100901. https://doi.org/10.1016/j.jmathb.2021.100901

Sherin, B., & Star, J. R. (2011). Reflections on the study of teacher noticing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes (pp. 66–78)*. Routledge. https://doi.org/10.4324/9780203832714

Star, J. R., & Strickland, S. K. (2008). Learning to observe: Using video to improve preservice mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education*, *11*(2), 107–125. https://doi.org/10.1007/s10857-007-9063-7

Tekkumru-Kisa, M., & Stein, M. K. (2017). A framework for planning and facilitating video-based professional development. *International Journal of STEM education*, *4*(28), 1–18. https://doi.org/10.1186/s40594-017-0086-z

Thanheiser, E., Melhuish, K., Sugimoto, A., Rosencrans, B., & Heaton, R. (2021). Networking frameworks: A method for analyzing the complexities of classroom cultures focusing on justifying. *Educational Studies in Mathematics*, *107*, 285–314. https://doi.org/10.1007/s10649-021-10026-3

van Es, E. A. (2011). A framework for learning to notice student thinking. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 164–181). Routledge. https://doi.org/10.4324/9780203832714

van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education, 10*(4), 571–596.

van Es, E. A., & Sherin, M. G. (2021). Expanding on prior conceptualizations of teacher noticing. *ZDM–Mathematics Education*, *53*(1), 17–27. https://doi.org/10.1007/s11858-020-01211-4

van Es, E. A., Tunney, J., Goldsmith, L. T., & Seago, N. (2014). A framework for the facilitation of teachers' analysis of video. *Journal of Teacher Education, 65*(4), 340–356. https://doi.org/10.1177/0022487114534266

Warshauer, H. K., Starkey, C., Herrera, C. A., & Smith, S. (2021). Developing prospective teachers' noticing and notions of productive struggle with video analysis in a mathematics content course. *Journal of Mathematics Teacher Education*, *24*(1), 89–121. https://doi.org/10.1007/s10857-019-09451-2

Wenger, E. (1998). Communities of practice: Learning as a social system. *Systems Thinker*, *9*(5), 2–3.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458–477. https://doi.org/10.5951/jresematheduc.27.4.0458

Yang, Y., & Ricks, T. E. (2012). How crucial incidents analysis support Chinese lesson study. *International Journal for Lesson and Learning Studies*, *1*(1), 41–48. https://doi.org/10.1108/20468251211179696

Zeichner, K. (2012). The turn once again toward practice-based teacher education. *Journal of Teacher Education*, *63*(5), 376–382. https://doi.org/10.1177/0022487112445789

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