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Meanings on the notion of curvature generated with a medium for dynamic manipulation and symbolic expression

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Abstract

This paper describes classroom research into aspects of mathematical meaning emerging from students' experiments while trying to construct the arcs of models of bridges of different sizes and shapes. They used a piece of software called "Turtleworlds" which through the use of the "variation tool" allows dynamic manipulation of turtle traces resulting from the execution of parametric procedures and thus combines functionalities of symbolic expression through a programming language with dynamic geometrical software. We investigated emerging meanings about the notions of curvature, arcs and angles and how these were constructed with the use of the variation tool. The ideas of mathematical continuity, decimals, curvature, angle properties, proportionality and generalised number were amongst those which the students grappled with through their use of the medium.

Keywords: curvature, dynamic manipulation, symbolic expression, meaning generation

1. Introduction

This paper describes on-going research into meanings generated by 13 year olds on curvature and arcs while working with "Turtleworlds", an environment integrating Logo programming with dynamic manipulation. In the field of learning geometry with exploratory software, the world of programming and symbolic expression of geometrical figures and relations has been surprisingly fragmented from that of direct manipulation. It may be that research interest in those two arose in different – almost sequential – times, or that the coherence and potential of properties of each type of corresponding software environment have been so exciting, that joining the two has not received much thought. Geometry, however, is a field where mathematical formalism and graphical representation of objects and relations are dynamically joined in interesting ways and have important potential for mathematical meaning making processes. A large body of research in the eighties and early nineties has shown that using a programming language to construct fixed and generalised geometrical figures afforded students with the opportunity to link symbolic and graphical representations and to engage in mathematical activity such as situated abstractions (Hoyles and Noss, 1992) and combined use of intrinsic and plane geometrical ideas (Kynigos, 1992). Working with dynamic geometry software such as Cabri or Sketchpad has provided students with the means to dissociate from perceiving figures as objects and work with dynamic entities focusing on how they evolve as dragging discriminates constant from variable features (Arzarello et al. 1998). "Turtleworlds" has been designed so that figures are constructed with Logo procedures in the old fashioned way. When these procedures are variable, however, the variable values can be dynamically manipulated resulting on the figure changing in Cabri style continuity. In this paper, we describe students investigating the ways in which an intrinsically constructed curve changes. In classical curriculum maths curves are found in circles, arcs, trigonometry and function graphics. In all these, there is not much focus on the nature of curvature. Curve seems to be the result of a mathematical property or relation. In old style Logo, students worked with the idea of intrinsic curve definition, i.e. polygon approximation. It was not possible however to gain some insight into how the curve evolves as one or another of its defining parameters change. In this paper, we describe a student project to build models of bridges, concentrating on how to construct the arcs as the height and the width of the bridge changes.

2. The Variation Tool

The students used a piece of programmable exploratory software called "Turtleworlds" built with e-Slate¹. E-Slate is a system for authoring exploratory software (Kynigos et al. 1997). It consists of a "desktop" environment and a set of software components. These can be connected in creative ways to make microworld environments and other exploratory software. Connections can be made through a prefabricated plug metaphor or through Logo which plays the role of a scripting language. Connecting scripts can also define aspects of the components' functionality. "Turtleworlds" is an E-slate Logo microworld involving turtle graphics. A special feature of Turtleworlds is the "Variation Tool" which allows students to dynamically manipulate the result of parametric procedures constructed in Logo. The tool provides a slider for each variable. Dragging a slider has the effect of the figure dynamically changing as the value of the variable changes sequentially. Turtleworlds also features a two - dimensional variation tool, where the user can drag the mouse freely on a plane representing a Cartesian plot of two variables of his choice. The effect of Turtleworlds is a combination of Logo² - like symbolic expression software and dynamic manipulation software for Geometry (such as Geometry Sketchpad or Cabri). Noss & Hoyles (1997) use the term 'autoexpressive' to characterize such technological frameworks i.e. 'the only way to manipulate or reconstruct objects is to express explicitly the relationships between them'. In Turtleworlds what is manipulated is not the figure itself but the value of the variable of a procedure. Dragging thus affects both the graphics and the symbolic expression through which it has been defined. The tool design is thus directly related to the visibility of exploration (Yerushalmy, 1999) facilitated by its use. The means of exploration are a combination of symbolic and graphical representation. In this study we were particularly concerned to investigate the role played by the available medium for the construction of meanings within and across the domains of children's interactions among the interdependent modalities that stemmed from Turtleworlds' multiple representational approach.

3. Curve concepts and representations

In the Greek mathematics curriculum (at least), the notion of curve is defined as part of a circle and there is no other introduction of the concept of curvature. The measure of an arc is considered to be the measure of the corresponding epicentric angle and its length is expressed as a formula relating it to elements of the circle (radius, degrees of the epicentric angle, π). As far as its construction is concerned an arc can be perceived as a plane geometrical curve (plane conception) by means of a radius of the corresponding circle or as a polygon-approximation definition of curve (intrinsic conception). In school, however, ruler and compass constructions are the only ones taught. Students are thus not exposed to generating meanings for the nature of curve and curvature. Logo environments are appropriate for the polygon approximation as a curve since their design is based on intrinsic geometry (Kynigos, 1992). However, in traditional versions of Logo a geometric figure is static whether it is perceived as the result of a turtle trip or as an object. In Turtleworlds figures resulting from the execution of variable procedures can be dynamically manipulated and treated both as turtle trips and as objects. In this paper, we discuss a Turtleworlds microworld for curvature involving the manipulation of a procedure describing an intrinsic definition of curve. In the corresponding classroom activity, the students were engaged in trying to build the arcs of models of bridges of different sizes and shapes using the following Logo procedure called 'mystery':

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to mystery :a :b :c
  repeat :a [fd :b rt :c]
end
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In this procedure the first variable changes the length of the arc, the second its width and the last its curvature. The graphical representation in itself is an instance of an arc under construction. Dragging any of the three sliders corresponding to the respective variable causes an effect resembling continual change of the

¹ <http://e-slate.cti.gr>

² D. Azuma's "Turtletracks" based on Berkeley Logo and continually extending with component connectivity primitives (www.ugcs.caltech.edu/~dazuma/turtle).

curve. The design intention was that students would engage both in dragging the sliders and in editing the procedure in order to construct a "tool" for building bridge models.

4. Research setting

An activity scenario was built around the curvature microworld, titled 'The Bridges of Europe' (BOE)³. The activity took place in five schools in Athens, Larissa, London and Rome. The plan was to engage students in searching for information on one or more local bridges, in building the model of a bridge through experimentation with curves, and then in communicating about themes related to bridges and aspects of the model construction. We began by providing a written framework for the activity outlining mathematical content and teaching and learning method, rather than some strict curriculum directive (for a discussion on integrating innovation in school settings, see Prawat, 1996). The activity framework did suggest that the students would build models of bridges using the 'mystery' procedure with respect to the notions of variable, arcs, curvature and arc size.

5. Method

Our objective was to study meanings generated in the classroom setting. Taking the role of participant observers we focused on one group of students, taking detailed data of their talk and actions and on the classroom as a whole taking observational notes. Our aim was to gain insight into a) the kinds of meanings constructed around the notion of curvature and arcs, b) the kind of mathematical activity in which the students would engage and c) the ways in which meaning generation interacted with the use of the computational medium. Our analysis focused on the meanings generated by the students themselves, rather than on the extent to which they "understood" the mathematics which was in our minds. We were interested in identifying the processes by which they constructed meanings in relation to the use of the tools. Our method thus builds on previous work such as Hoyles and Sutherland (1989), Noss & Hoyles (1997), Ainley et al. (2000). We carried out the analysis using a generative (Goetz and LeCompte, 1984) stance, i.e. allowing for the data to shape the structure of the results and the clarification of the research issues. At first, we read the data looking for incidents where mathematical meaning was discussed amongst students or those where we identified ideas in use. We then compiled those incidents, looking for commonalities with respect to the kind of mathematical process and the ways in which the microworld was used. In this paper, we selected three characteristic episodes each made up of a small sequence of incidents aiming to illustrate the kinds of issues emerging from the analysis. Classroom observations were conducted in all five schools (6 hours in each one) as well as interviews with teachers. A team of two researchers participated in each data collection session as participant observers. It was used one video camera and one microphone on the group of students who were our focus, occasionally moving the camera to capture other significant instances as they occurred. Background data was also collected (i.e. electronic communications, observational notes, students written works). Verbatim transcriptions of all audiorecordings were made.

6. Findings

Arc as a dynamic construction

Throughout the activity, an important characteristic in pupils' perception whenever the issue of arc was discussed was that of motion. Each change of one variable in the 'mystery' procedure and its immediate visualisation provided an evolving geometric object which represented a model of a bridge under construction. Previous studies on how students interpret graphics show that at first they seem to concentrate on the visual impact of the graphics as a shape ('shape-spotting activity', Ainley et al., 2000). Nemirovsky

³ NETLogo: the European Educational Interactive Site, European Community, Educational Multimedia Taskforce, Joint Call on educational Multimedia, MM1020, 1998-1999.

(1998) goes so far as to suggest that this initial stage of graph acquisition as a shape helps the students develop an 'overall sense' of it 'to share silent features with others, and to plan actions'. Other studies on the role of the provided dragging modalities in computational mediums such as Cabri and Sketchpad mention the importance of the dragging process in mediating the conceptual control in pupil's explorations of geometrical problems (Arzarello et al. 1998). In our study graphic and numeric interdependence mediated by the dragging modality of the variation tool and by the reference to and editing of the formal mathematical code, appeared to play a central role in pupil's interactions with the medium resources and was in close relation to their actions.

The first episode is about a group of three students firstly constructing two rectangular bases for their bridge model with a distance of 500 turtle steps between them, as instructed by their teacher. Subsequently, the students investigated how they could find a method to make an arc join the bases together if they wanted to be able to make bridges of varying widths with their model. Their plan was to use a procedure constructing a semicircle of varying size. They realised that the values of the iteration and turn variables would have to be constant. Instead of using the 'mystery' procedure as given inserting the appropriate values for iteration and turn and simply not dragging the sliders, they decided to edit the procedure by taking away these variables (:a and :b). It's as if they seemed more comfortable with perceiving these quantities as constant rather than as unchanging variables.

```
to arc :c
repeat 180 [fd :c lt 1]
end
```

They executed the procedure with a value of 1 and noticed that it fell short from joining up the two bases. After that pupils played for a while dragging the slider randomly observing the differences in the moving semicircle each time. Their dragging didn't remain in this mode (equivalent to that of 'wandering dragging', Arzarello et al. 1998) for a long time since they quickly saw that the appropriate value would be between 4 and 5. In fact, they saw that for :c=4 the moving edge of the semicircle fell before the upper left vertex of the rectangle representing the opposite base while for :c=5 it went past it. One of the students then suggested the use of decimals for the slider step. They then dragged the variation tool in the scale of 4 to 5 using decimals for the step but the arc still did not link the two bases neatly. Observing these changes one student suggested changing the step again.

S2: We need it [the arc] to be further along

S1: It doesn't fit [i.e. the other side]. Why?

S2: It needs one more step [i.e. a different step] / 0,01.

S2 suggests to change the step of the slider so as to get the arc to fit neatly, connecting the iconic representation ('the arc has to be further along') with a respective change in the numeric part (0,01 for the slider step). The way in which pupils 'see' the need to extend the shape of the arc is still in the stream of their exploration through dragging. The control of meaning is ascending (Arzarello et. al. 1998), i.e. they are manipulating the variation with precise intent and taking control on the continuity of the designed curve. However, they have integrated the idea of controlling the dragging feature in that they worked with the drag step factor. In that sense, not only did they find a use for the idea of decimals, but they may have formed some intuition of the notion of mathematical continuity and the idea of the discreteness of number. Had their teacher been there at the time and perceived this, they could have intervened to bring these ideas out in a more explicit way. The interaction with the features of the variation tool and the idea that it varied values of a procedure defined by the students themselves, however, had a crucial role to play in generating these ideas.

Proportionality - Fitting the changing semicircle

During their attempts to find a "theorem" (in the sense of Vergaud, 1987) to fit the semicircles on the bridge bases, the students' attention shifted from the curve itself onto the width of the bridge. Their idea was that maybe there was a proportional link between curve and distance between bases. What is interesting is that, in contrast to the way curve is presented in school as the result of a property of diameter and pi, here the students themselves investigated the relevance of diameter in their attempts to find a way to measure curves.

The teacher seemed to have adopted a 'mathematical discovery' view of the activity and perceived Logo use as a tool for mathematical experimentation, in order to find out mathematical relations. At the beginning of the session, he provided some guidelines asking the students to use variable procedures for semicircular bridges of different sizes. The students in the previous example first designed a semicircle of diameter 500 and found that the length of the step i.e. the value of variable :c, was 4,36. They then decided to make a longer semicircle linking the outside vertices of the designed rectangles of the base where the distance was 600. S2 suggested writing down the results on a proportion table (they were doing these in class at the time) and the teacher took the opportunity to introduce the idea of working out a general rule:

S2: We can always have a base for the bridge and make a proportion table. Let's say, when we have 100 we have this value for arc, when we have 30 how much is that?

T: It wouldn't be a bad idea for you to make a proportions table. What you've just said, proportion, leads to a general rule. I want you to be able to design a bridge no matter what the width of the bridge might be, which is the radius. You'd better construct another one with this distance 600. It may be helpful for you to find out if there is a relation.

S3 then undertook to calculate the value solving the equation from the following proportion table they had written in their writing book:

500	600
4.36	x

The other two students (S1 and S2) were trying it out perceptually on the computer. This time the exploration process became more 'focused' as if the dragging had found its path in a way that satisfied the necessity of the construction to become a semicircle.

S2: Now we need to find another arc. This [e.g. the value of c in the previous semicircle] is 4,36. Let's start with 4,5 and then we will play with the decimals.

After a while they found $c=5,23$ but S3 said the solution of the equation was 5,0234. The following dialogue took place:

S1: So, the distance between the bases is not 600.

S3: It is 600.

S2: [To S3] So you did not do the division correctly.

S1 and S2 seemed to accept that each specific distance corresponds only to one number which may be calculated on the computer or by solving the equation. S1 at first attributes the problem to a probable difference on the values of the distance they measured between the bases. When S3 confirmed that the distance on the computer is 600 S2, decided that there was a mistake on the calculation on paper and immediately repeated their dragging with the variation tool to check again the value of :c.

S2: (While dragging the slider) 5,24 stop for a while / Little very little // 5,24 was it near?

S1: Look, look.

S2: Now let's see 5,23 / look at it.

S1: We found it.

S2: So, for 600 is 5,23. Let's try that on paper to check out what's the output [of the equation].

When S3 checked again the equation he found that his calculation had been mistaken but he did not complete the calculations. He –like the others– seemed to accept that the final outcome would be the same as the one found on the computer. (In fact the exact result is 5,232). Near the end of the lesson they were ready to try with another number for the diameter following the same pattern: S3 would make calculations on the paper while the others would find the corresponding value of :c by the variation tool. This process was at the beginning when the teacher interrupted all of the groups and called them to present their works in a whole-class discussion.

There is a number of points of interest in this incident. Firstly, by associating the shape and size of the graphics with numerical and symbolic representations, their visual field was internalized (Marriotti et al.,

1997) in their attempts to find a property which could help them classify the figures into something they knew. In this case they discovered a way to 'produce' any semicircle. Their perception of the variable :c was that of the changing magnitude in proportional figures distinguishing it from the constant features of the construction. While trying to fit the changing semicircle to the fixed shape of the base each time pupil's conversation seemed to go back and forth between figural attributes and actions that had produced the graphical responses. The fact that they seem to trust the values derived from the visual effect on the computer more than their own capability to calculate. This may have been affected by their teacher's view of the computer as a tool for verification and checking of existing mathematical rules. It is, in any case, an interesting integration of formal and experimental mathematics and certainly a learning situation where discussion of the significance and the nature of each could have been brought into the foreground.

Investigating patterns emerging from direct manipulation

In the following section we describe selected incidents with the aim to highlight the emergence of the abstract from the concrete as a relational property. Wilensky (1991) suggests that such processes lead to a new kind of knowledge "The Connected Knowing" and relates it to the richness of the representations, interactions and connections between the objects and the learner.

The episode is about another group of pupils who decided they wanted to make horizontal arcs which were not so intensely curved as these created by semicircles. They realised that they had to make arcs which were smaller parts of a circle than half a circle. They used the mystery procedure with all three variables and they were dragging the sliders trying to discover some rule or invariant property so that they could change the curvature of the arc without losing its fit onto the bases. This phase was marked by fragmented comments stemming from the observation of the continuing changes in the variables.

S2: *I believe that we need to put a smaller step as well as a smaller size in the degrees of the turn.*

S1: *Let's try it with the variation tool.*

S2: *[After a while] It needs more degrees // And the step must be bigger. I don't think so. Maybe the degrees ...*

When they experimented by giving corresponding values to the iterations variable, they realised by observing the screen outcome that the shape seemed to be tilted towards the left and decided to insert a command to turn the turtle towards the right before it began to make the arc.

S2: *Just a minute. I've got an idea. Instead of having 180 degrees here, since we don't want to draw a semicircle... Yes, let's have 45 degrees here and 45 here and the rest of it -90- here.*

The students have thus identified a new feature influencing their construction and began to investigate whether there is some underlying property. In itself this is very important since they seem to have instinctively adopted a mathematical perception of the situation. Not only that, but, in contrast to the students in the first section, they decided to insert a variable value for the initial turn so that they could investigate by changing it with the use of the variation tool. They inserted the value of 45 degrees and then began to change the others so as to get a differing curvature for their arc.

```
to arc3 :a :b :c
  rt :a repeat :c [rt 1 fd :b]
end
```

Firstly, they moved the corresponding slider to a value of 45 for variable :a and then the slider for variable :c to 90. By dragging the variable :b slider, they 'found' the value which would give the right size for their arc. Their researcher asked them how they knew which values to give for variables :a and :c and one student's response was:

S2: *It's all part of the semicircle, i.e. the semicircle has 180 repeats inside it, let's say at the beginning we turned 45 degrees from the one side and assuming that there would be 45 on the other side, 45 plus 45 we have 90 degrees, and subtracting from 180 of the semicircle we have the part of the semicircle, the half / and we have 90 repeats.*

The students seem to have taken into consideration the symmetrical nature of the arc ('assuming that there would be 45 on the other side') and to have built on the previous properties they discovered. The researcher did not rest with this explanation, asking for a more elaborate one where he probed whether the students were able to generalize the description of their 'rule' to other values for variable :a. He found out that they in fact had already tried other values for :a and brought them in as examples to their explanation.

R: This is a rule which you've found.

S: Yes, in a way.

R: If I didn't turn 45 degrees and turned 30, what would have happened?

S: Yes, 30 plus 30 sixty, 120 times repeat, 120 times.

R: It would be 120 times repeat, did you try this?

S: [Pointing to the computer] Yes I've tried it here.

R: Did you try any other turns?

S: Yes, it worked. The more we turn at the beginning, the less we repeat. And the step changes.

It is particularly interesting in the above excerpt that the student's descriptions switch from referring to specific sets of values in concrete cases to attempting a more generalized kind of language explaining the ways in which the figure changes as the values of the variables change. An indication supporting this suggestion is their reference to interdependence represented by the variables in the last sentence. This type of generalization is in accordance with Noss & Hoyle's (1996) notion of situated abstractions since it was extracted from the actual experiment in that setting and did not move further to the formal generalization of the relationships between the variables neither in the explanations nor in changing their procedure to incorporate a relationship between variables (e.g. express :c as a function of :a or vice versa). The above excerpt thus indicates that through this 'theorem in action' (Vergnaud, 1987), pupils tapped onto the relatively deep mathematical idea of the way in which an object evolves as the value of a variable changes (i.e. the notion of a mathematical property) and onto the notion of co-variation between two values.

We can find some interesting elements also in the whole class discussion which immediately followed the activity. Further probing by the teacher – researcher aimed at drawing out any meaning created by the students of the group which related to the way in which the figure evolved as the value changed.

R: What do you change with this slider?

S: I'm changing the step.

R: And what happens to the figure?

S: So, accordingly it becomes larger, it fits the corresponding distance.

R: In what way does it fit?

S: The step also grows, each time it grows more and more.

R: Does the shape of the arc change (changing the step)?

S: Yes.

R: In what way does it change?

S: In its size, its opening.

R: What happens exactly?

S: The diameter becomes bigger.

R: If you change the third slider what would happen?

S: So as to be proportional again we have to change this slider too, correspondingly (points at variable a).

This excerpt indicates how the dynamic nature of the pupil's exploration was mirrored in the arguments they used in their descriptions. These students recollected what they had just found out reversing the exploration process by connecting it to when they referred to the constructed objects from a detached point of view (Marriotti et al., 1997) mentioning qualitative properties of them such as interdependence. It is important to underline the elements presented in their words.

1. Changing the step → the figure becomes larger → the step grows more and more

2. (The step changes →) The size changes → the diameter becomes bigger

3. Proportional (magnitudes): Changing :c ⇔ Changing :a

In the first one student begins with the changes in variable :b, after that mentions the outcome on the figure and then connects this process with the continual change of the step using twice the word more. In the second one the student mentions the change in the size when the step increases and describing the nature of

the change refers to the diameter of the circle which is not a visible part of the construction. In the third one the changes in the variable α are immediately related to respective 'proportional' changes to the variable α as necessary to keep the normality of the figure.

7. Conclusions

Some interesting meanings around curvature seemed to have emerged in this classroom activity, through the students' engagement with the graphical and symbolic interdependence of Turtleworlds. Amongst these were the idea of continuity and proportionality, the dynamic nature of mathematical relations, using the curve as starting point in generating arc properties, discovering unexpected properties of arc positioning and engaging in experimental and formal maths in the same activity. The use of symbolic and graphical notation in conjunction with the dynamic manipulation of the way the figures evolved as variable values changed, played an important part in the generation of these ideas which was interwoven with the activity and the use of the tools. The kinds of understandings supported by such media in varying mathematical activities warrants further research. More generally, now that new technologies are available which make the expression of mathematical meaning not so dependent on formal symbolisation and static graphics, it is interesting to reassess which – if any – aspects of this kind of representation of mathematical ideas are important for the generation of mathematical meaning. In that sense, computational environments weaving multiple types of representation together in a meaningful way may provide us with insight into which aspects of these representations may support creativity and the generation of mathematical meaning in students.

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