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May 7, 2008

Prof. Giorgos Psycharis
University of Athens
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Dear Professor Psycharis:

The 11th International Conference on Mathematics Education (ICME-11) will take place July 6–13, 2008, in Monterrey, N.L., Mexico. The scientific program for the congress contemplates various types of activities, the main ones being: Plenary Sessions, Survey Teams, Topic Study Groups, Discussion Groups, Affiliated Study Groups, and Regular Lectures.

The International Program Committee is very pleased by your acceptance to participate as Contributor in TSG22: **New technologies in the teaching and learning of mathematics** with the paper ***Exploring angle through geometrical constructions in a simulated 3D space***. We have already announced the work your group will be developing at the ICME 11 web site (http://icme11.org), and the program of your group's presentation will be included in the Final Program distributed at the conference upon registration. The full text of your presentation *Exploring angle through geometrical constructions in a simulated 3D space* will also be made available at the web site before the conference and will appear in the proceedings we plan to publish after the conference.

We thank you for including in your agenda for July 2008 your participation in ICME 11. Let me also say how much we appreciate the willingness you have shown to share your professional work with fellow mathematics educators from all over the world.

Sincerely,

Marcela Santillán
Chair of the International Program
Committee for ICME-11

Exploring angle through geometrical constructions in 3D space

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Abstract. The construction of meanings for the notion of angle in 3D geometrical space is studied during the implementation of specially designed geometrical tasks in the classroom. 13-year-old pupils were engaged in exploring the mathematical nature of angles while controlling and measuring the behaviours of geometrical objects in 3D simulations. Pupils worked in pairs using a specially designed computational environment that combines symbolic notation (by means of programming) and dynamic manipulation of graphically represented geometrical objects. The constructionist theoretical perspective of the present study is based on the assumption that programmable geometrical constructions designed to help children abstract the notion of turtle movement in the 3D space provide a useful environment for developing their conceptualizations of geometrical objects, like angles. In the findings it is reported the progressive conceptualization of angle as a dynamic entity interrelated with the recognition of the different planes in the 3D space and the simultaneous sensual access to them through the dynamic manipulation of 2D geometrical representations using the available tools. The study is taking place as part of a cross-experimentation methodology developed in ReMath project [1] based on the design and the educational use of digital artifacts for mathematics in different educational and cultural contexts.

Introduction

This contribution reports classroom research aiming to explore 13 year-olds' construction of meanings for the notion of angle in 3D geometrical space during activity involving the construction and manipulation of programmable geometrical figures. The students worked in collaborative groups of two using *MaLT (MachineLab Turtleworlds)*, a constructionist (Harel & Papert, 1991) microworld environment for geometrical constructions which combines symbolic notation - through a specially designed version of Logo- with dynamic manipulation of graphically represented mathematical objects (Kynigos & Latsi, 2007). In MaLT, the elements of a geometrical construction can be expressed with the use of variables and dynamically manipulated by dragging on the 'number line'-like representation of these variables using a specially designed computational tool.

Angles and turns are critical to 3D geometrical knowledge and since they are related to students' everyday experience they can be considered as a rich conceptual field (Vergnaud, 1996) for mathematics meaning-making, not systematically studied up to now. Student's difficulties in the angle domains are well documented in the literature. Research results strongly suggest two major factors influencing students' use of angles: the physical presence or absence of the lines which make up its arms and the conceptualisation of turning as a relationship between two headings of segments (Clements et al., 1996). The situation is aggravated in 3D space by the apparent difficulty of students to recognise the basic elements of an angle. For example, an 'opening-closing door' situation incorporates a more obscure similarity of the standard angle domain since the arms of the embedded 3D angle are rectangles. Moreover, turning in 3D space can be considered as a sophisticated concept since the turning motion itself usually does not leave a trace and the 'heading' must be reconstructed from memory. It is also clear from previous research that students have great difficulty coordinating the various facets of the angle embedded in various physical angle contexts involving slopes, turns, intersections, corners, bends, directions and openings (Mitchelmore & White, 1998). Taking this into account Mitchelmore and White (2000) highlight that a mature abstract angle concept depends essentially on learning to link the various physical angle contexts together through "the systematic attempt to investigate our spatial environment mathematically" (p. 233).

In the research reported here we were motivated to relate parts from different physical angle situations reminding the ones that an individual experiences in everyday circumstances in 3D space where such situations need not be distinguished. We have selected the version of Logo developed in MaLT as one context to explore students' ideas around the concept of angle in 3D space based on turning and directionality. Our purpose was (a) to relate children's construction of meanings for the notion of angle in 3D space explicitly to their physical angle experiences and (b) to offer a framework in which to account specifically for their difficulties in coordinating different aspects of the notion of angle as well as to throw light on the paths by which students might come to integrate their various angle concepts in 3D space. More specifically, the main focus of the study concerned the ways by which students conceptualise angle as a spatial visualisation concept representing turn and measure through the construction and dynamic manipulation of 2D geometrical objects, e.g. angle as a change of direction and planes in 3D space, angle between two different planes defined by 2D geometrical objects.

The computer environment

MaLT is a programmable environment for the creation and exploration of interactive 3D geometrical constructions integrating representations and functionalities from both of groups of computational environments for geometry developed in the mathematics education community, i.e. programming tools (e.g. Logo-like microworlds) and expressive tools (e.g. Dynamic Geometry Systems). Such functionalities are symbolic expression inherited from Logo as a programming language and dynamic manipulation of graphically represented mathematical objects. Our theoretical perspective on learning with MaLT suggests that interaction with multiple representations of geometrical objects can be a fruitful domain to challenge student's intuitions and ideas concerning spatial thinking come into play. The joint use of visual, symbolic and dynamic manipulation representational registers (Kynigos, 2004) can be considered as a means to connect body movement with geometrical problem solving which might go beyond the simple visual recognition of spatial relations to their expression and further elaboration. MaLT is consisted of the following interconnected components (see Figure 1).

• *Turtle Scene (TS).* This is a 3D grid-like projection of a simulated 3D space in which a 3D turtle is visualised when a command (or a procedure) is executed. Whenever the turtle moves it leaves behind it a trace which is a selectable 3D cylindrical line. Initially the turtle is situated at a plane parallel to the horizontal one (surface) and directed to the 'interior' part of the TS.

• *Logo Editor (LE).* This is the Logo-like programming interface which is linked to the TS. The user is able to write, run and edit Logo procedures to 'drive' the construction of geometrical objects by the turtle using a specially designed version of Logo. A distinctive feature of this specific version is that it provides an extension of Logo commands in 3D space including two new kinds of turtle turns: *'UPPITCH/DOWNPITCH n degrees'* ('*up/dp n degrees*') which pitches the turtle's nose up and down and ‘*LEFTROLL/RIGHTROLL n degrees*’ (‘*lr/rr n degrees*’) which moves the turtle around its trunk/vertical axis.

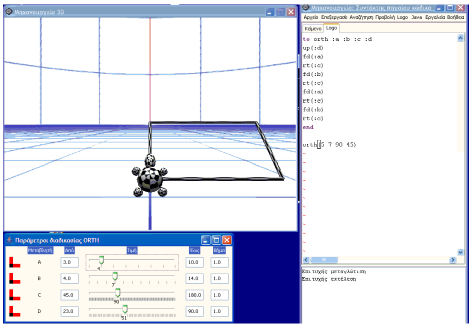

Figure 1: A rectangle with 4 variables.

- *Uni-dimensional Variation Tool (1dVT).* The main part of this component consists of ‘number-line’-like sliders, each corresponding to one of the variables used in a Logo procedure. After a variable procedure is defined and executed with a specific value, clicking the mouse on the turtle trace activates the tool, which provides a slider for each variable. Dragging a slider has the effect of the figure dynamically changing as the value of the variable changes sequentially.

Research setting and tasks

The work reported in this paper is part of a design-based experiment (Cobb et al., 2003) concerning the exploration of the mathematical nature of angles in 3D space by groups of 13-year-old pupils who work collaboratively in the classroom setting using MaLT to construct, control and animate 3D geometrical objects often encountered in everyday situations (e.g. doors, revolving doors, staircases). These objects have been selected because they seem to provide a meaningful context for bringing in the foreground the notions of angle based on turn and directionality in 3D space in multiple ways. For example, in a ‘door’ simulation the arms of the defined dihedral angle are rectangles while in a ‘spiral staircase’ simulation the arms of the defined angle are an equilateral triangle and a rectangle. The tasks are designed to integrate different angle domains (e.g. intersecting, turning, sloping) related to physical angle experiences in everyday circumstances (e. g. corner, slope and turn) as well as to the main definitions of angle. The constructionist theoretical implication is that the intrinsic geometry of move in 3D space is closely related to real world experiences as well as to body-syntonic activities such as walking or observe something flying. Under this perspective our central research aim is to study how students used the available representations in MaLT to construct meanings for the concept of angle: (a) as a *geometric shape*, i.e. formed between two geometrical objects which can be segments (in 2D geometrical figures) or 2D geometrical figures (in the 3D space, e.g. dihedral angles); (b) as a *dynamic amount*, indicating a change of directions both as a turn and as the result of a turn which can also be represented by a variable; and (c) as a *measure* represented by a number.

After a familiarisation phase with the basic Logo commands (Introductory phase), students were engaged in building rectangles using parametric procedures in at least two different planes of the TS (Phase 1) and experimenting with variable procedures designed to create 3D simulations like doors, revolving doors and staircases (Phase 2). The experiment took place in a multicultural secondary school in Athens with one class of twenty 7 grade pupils (13 years old) and one experimenting teacher who acted also as a researcher in the classroom. In each session a second teacher-researcher supported data collection acting as co-researcher. The class had totally 18 teaching sessions with the experimenting teacher over two months. Our methodological approach adopted ethnographic and inductive methods for the observation of human activities taking place in real time. We used a specially designed screen capture software -called Hypercam- which allowed us to record student’s voices and at the same time to capture all their actions on the screen. For the analysis we transcribed verbatim the audio recordings of three groups of students (focus groups) throughout the teaching sequence and we also selected significant learning incidents from the work of all groups in the classroom.

Method of analysis

In the analysis a phenomenological approach was adopted according to which researchers' interpretations were focused on the student's interactions with the available representations and the meanings they attached to these from their own points of view aiming to describe the progressive creation of the conceptual space emerging through the use of the available tools ("lived-in space", Nemirovsky & Noble, 1997). In our view, it is by working with integrated representations and the relationships between them that the student constructs his/her own meanings in the wider conceptual web of ideas and concepts (Noss & Hoyles, 1996). Particularly, we were interested to extract episodes combining student's bodily engagement (e.g. gestures, use of objects) with the development of mathematical meanings while describing the turtle's navigation in 3D space and struggling to conceptualise the role of 2D representations in forming angular relationships in 3D space. Here we use data from two of the focus groups concerning the introductory phase of the study which lasted four teaching sessions in total.

2D planes in 3D space

The move of turtle in MaLT is interrelated with the a conception of angle integrating two schemes based on turning: (a) angle as a turn indicating both the act of body turning and the result of it, which inevitably involves directionality (dynamic scheme) and (b) angle as a turn represented by a number (measure scheme) (Clements et al., 1996). During the introductory phase, pupils' conceptualisation of 3D space consisted mainly of the recognition of specific 2D planes in the 3D scene of MaLT. This kind of activity appeared as a result of the pupils' visual and kinaesthetic experiences during the initial navigation of the turtle. The horizontal plane, indicating the initial position of the turtle, was conceived by the students as the 'ground plane', although it is actually a horizontal one parallel to the 'flagged plane' visualised at the bottom of the scene. Despite the researcher's request to navigate the turtle at different points in the 3D scene (e.g. simulation of an airplane taking off), during the first two hours of the introductory phase both groups of students have chosen to move the turtle either on the 'ground plane' by constructing simple crooked lines (group A) or on the 'screen plane' (vertical to the 'ground plane') by constructing familiar geometric shapes such as a model of the letter T (group B). Apart from the essential familiarisation with the new kinds of turtle turns (uppitch/downpitch, leftroll/rightroll) this preference could possibly be interpreted in the light of the fact that pupils who were accustomed to work with 2D representations of geometrical figures might have had difficulties in understanding the conventions used to represent a 3D object in the computer screen (Lowrie, 2002). Pupil's experimentation concerning the notion of angle in this phase was developed around how to find out the measure of turtle turns for the construction of 2D figures without an explicit reference to the changes of planes, let alone how these might be related to the available commands. An indication of this was pupil's difficulty to conceptualise the role of the rotation commands -especially rightroll/leftroll). Though they had tried these commands and observed the turtles' turning around its trunk/vertical axis they could not understand how they might use them in a geometrical construction. Only after combining their experimentation with the simulation of the turtle turns using their hands did they realise that the use of these commands might affect a geometrical construction in multiple ways. As we will see in the following paragraph the role of variables in the development of such kind of understandings was critical.

Angle as a dynamic entity for moving in different planes

The use of the two new kinds of turtle turns in MaLT by the students was at the core of their experimentation to construct rectangles at different planes in 3D space. This kind of activity appeared to provide a fruitful domain that challenged student's intuitions and ideas about angle as a spatial quantity come into play since the use of these specific turns signaled a dynamic passage from one plane to another. For example, both groups of pupils recognised that in order to construct a rectangle on the 'screen plane' (which in mathematical terms is perpendicular to the 'ground plane') it was needed to insert the command up(90) at the beginning of the respective procedures before the construction of the rectangle (e.g. Table 1, Procedures 2 and 3).

to rect :a :b :c fd(:a) rt(:c) fd(:b) rt(:c) fd(:a) rt(:c) fd(:b) rt(:c) end	to rect :a :b :c up(90) fd(:a) rt(:c) fd(:b) rt(:c) fd(:a) rt(:c) fd(:b) rt(:c) end	to rect :a :b :c :d up(:d) fd(:a) rt(:c) fd(:b) rt(:c) fd(:a) rt(:c) fd(:b) rt(:c) end
Procedure1	Procedure 2	Procedure 3

Table 1: Logo procedures for the construction of a rectangle in TS (Group B).

In one group of students (group B) this kind of activity lead to a dynamic conception of angle as a varying entity used to visualise the move of rectangle at different planes in 3D space. We will describe the episode briefly. Pupils initially constructed a rectangle with three variables on the 'ground plane' (Procedure 1) and then on the 'screen plane' (Procedure 2). A student of the group when trying to concretise the new position of the rectangle in 3D space she used her palm to simulate the move from the surface to the 'screen plane' (Figure 2).

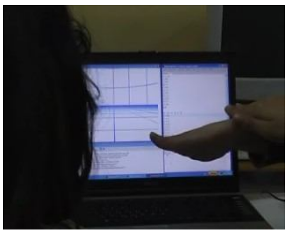

Figure 2

R: What happened to the turtle with up(90)?
S2: [*Showing with her palm the move from the surface to the screen plane*] It [*i.e.the turtle*] took it [*i.e. the rectangle*] that way.
R: If we didn't put 90 but put 45, what would have happened?
S2: It [*i.e. the rectangle*] would be nearly in the middle.
R: If we put 50;
S2: Ok, not in the middle. [*Showing with her palm*] A bit more than that.

The dynamic character of student's bodily engagement in the simulation challenged both of them to try to visualize it on the screen.

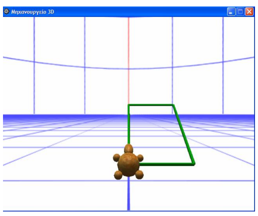

Figure 3

So, S1 afterwards had the idea to replace the value 90 in the command up(90) with a new variable :d to see what would happen (Procedure 3). Dragging on the slider of the variable :d in 1dVT had the effect of the figure dynamically moving upwards – downwards visualising in that way the dynamic move of the rectangle in different planes (Figure 3).

S1: [*Moving the slider :d*] Look! If we move it [i.e. *the slider :d*]

> upwards it [i.e. *the rectangle*] raises ... If we move it [i.e. *the slider*] downwards it [i.e. *the rectangle*] descends.

This episode indicates the conceptualisation of angle as a dynamic entity interrelated with the move to the different planes in the 3D space and the simultaneous visualisation of this move 'inside' the scene.

Angle as a dynamic entity for simulating 3D objects

Incidents like the above coupled with pupil's experience in using variables and handling variation with 1dVT facilitated further the extension of their experimentation around the different positions of a designed rectangle in 3D space. In contrast to the students of group A, who had not been able to incorporate the rotation commands in their experimentation during the introductory phase of the study, the students of group B related these commands with the possibility of turning the designed rectangle so as to simulate the move of real objects that include continuous turn in the space (e.g. a door). In the evolution of the above incident the researcher took the opportunity to ask pupils elaborate on how they could use the available tools to move the rectangle "like a door". Then the students immediately connected the move of the door with the use of the roll commands since the visualisation of the respective turtle turns that they had attempted earlier with their palms reminded them the move of a door. The sequence of what happened next is as follows. Initially one of the students substituted the command up(:d) (Procedure 3) with one of the roll commands (rr :d). Moving the slider :d then she realized that the direction of the axis of rotation was perpendicular to the screen plane ('it turns as a wheel' she said) (Figure 4, on the left).

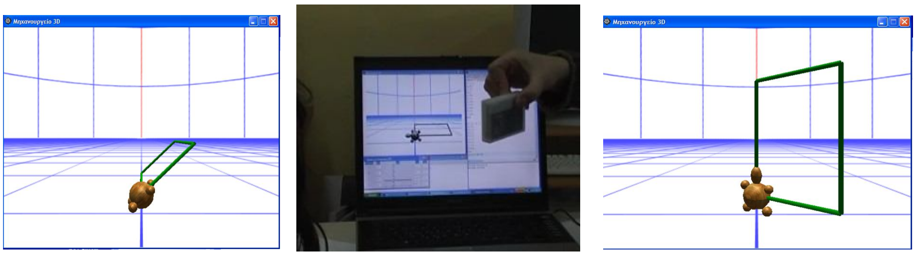

Figure 4

For the door simulation the students 'played' with the type and the sequence of Logo commands after rehearsing the move of the rectangle with the use of a concrete 3D object, in this case a video cassette, so as to visualise the change of planes of the rectangle as a result of the change of the initial position of the turtle in 3D space (Figure 5, in the middle). So, students realized that initially the rectangle needs to be raised up and then turned ('rolled') on the right. Modifying accordingly the Procedure 3 students used one more variable in the command rr(:e) that was inserted after the initial command up(:d). They subsequently achieved to simulate the 'opening-closing' door (Figure 4, on the right) by dragging the slider :e on the 1dVT after selecting the value 90 for the slider :d.

> S1: We changed this roll. When we put it [*i. e. the roll command*] at the beginning [*i.e. of the Procedure 3*] it [*i. e. the rectangle*] moves like that [*Shows with the cassette the move of the left Figure 5*]. So, initially we put it [*i.e. the rectangle*] in the vertical position.

Conclusions

Student's interaction with the available tools in this classroom activity appeared interrelated with the construction of a conceptual space of meanings around the notion of angle in 3D space. Under the constructionist theoretical perspective the above incidents concerning the work of group B in particular illustrate the dialectic relationship between the evolution of students' interaction with the available tools and their progressive focusing on angular relationships underlying the current geometrical constructions and its representations. As students were engaged in navigating the turtle in MaLT they gain a sense of the mathematical ideas related to the notion of angle through the construction and dynamic manipulation of 2D geometrical objects in 3D space by a process of hypothesising, experimenting and reflecting on the empirical observation of the graphical feedback on the screen. For example, the move of the rectangle using the available tools by the students brought in the foreground the notion of angle as a dynamic spatial entity interrelated with the recognition of different planes in 3D space and the simultaneous sensual access to them through the dynamic manipulation of a 2D geometrical figure. This kind of experience was critical for the subsequent door simulation which was based on angular relations underlying the links between 2D and 3D representations. The above analysis describes angle concept development in terms of interconnected stages of abstraction which can be considered as representing a progressively more refined classification of students' experience. It is necessary to include more cases in our analysis so as to account specifically for the learning processes that have taken place in this study.

Notes

1. "Representing Mathematics with Digital Media", http://remath.cti.gr, European Community, 6th Framework Programme, Information Society Technologies (IST), IST-4-26751-STP, 2005-2008.

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