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EDITORS: HILDA BORKO & DESPINA POTARI

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Editors:

Hilda Borko
Stanford University
Stanford, CA, USA

Despina Potari
National and Kapodistrian
University of Athens
Athens, Greece

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STUDYING PRIMARY AND SECONDARY TEACHERS' COLLABORATIVE DESIGN OF RESOURCES FOR ALGEBRA

Giorgos Psycharis
National and Kapodistrian University of Athens, Greece
gpsych@math.uoa.gr

Jana Trgalová
Claude Bernard University, S2HEP4148, Lyon, France
jana.trgalova@univ-lyon1.fr

Mohammad Dames Alturkmani
Ecole Normale Supérieure de Lyon, France
mohammad.alturkmani@ens-lyon.fr

Elissavet Kalogeria
National and Kapodistrian University of Athens, Greece
ekaloger@math.uoa.gr

Maria Latsi
National and Kapodistrian University of Athens, Greece
mlatsi@ppp.uoa.gr

Sophie Roubin
Ampère middle school
Lyon, France
sophie.roubin@ens-lyon.fr

This paper reports research on primary and secondary teachers' collaborative design of resources aiming at fostering students' algebraic thinking. The study took place in the context of a professional development program implemented in France and Greece aiming to explore the issue of algebra teaching and learning in primary and secondary mathematics classrooms. Analysis of selected examples of interactions at the boundaries of the two educational levels highlights the evolution of teachers' collective documentation work as well as their professional learning.

Introduction

The reported study took place in the context of the PREMaTT[1] project targeting the development of algebraic thinking in primary and secondary mathematics classrooms through the collaboration of teachers and researchers. The project focused on the processes of designing and sharing digital and non-digital resources as new forms of teachers' professional development (PD) towards better understanding of what algebra is and what kinds of tasks foster its development in primary and secondary grades. This resource approach to teacher collaboration raises issues such as how teachers from both educational levels conceptualize what algebra is, how these conceptualizations are brought to the fore through interactions between sets of resources coming from different collectives and how they are imprinted in the designs resulting from teachers' collaboration (Gueudet & Trouche, 2012). Our focus is on the process of collective design and implementation of resources by two communities of researchers, primary teachers and secondary mathematics teachers from different schools established in the context of project implementation in France and Greece.

Over the years, teaching and learning of algebra has been figured as a prominent research area in mathematics education. Considering arithmetic as a prerequisite for algebra, algebra has been traditionally taught in secondary school. Thus, algebra has been excluded, until recently, from many primary mathematics curriculums around the world. In light of students' difficulties at the secondary level and the need to address them, it has beed suggested a progressive introduction to algebra in primary grades (Carraher & Schliemann, 2007). Thus a need has arisen to characterize the nature of algebraic thinking and consider tasks in which younger students could be engaged. Radford (2014) suggests three features of algebraic thinking: (1) indeterminacy: the existence of unknown quantities (e.g., variables, parameters); (2) denotation: the need to name and symbolize

[1] http://ife.ens-lyon.fr/ife/recherche/groupes-de-travail/prematt

these indeterminate quantities in different ways (with algebraic symbolism, alphanumeric signs, natural language, gestures, or a mixture of these); and (3) analyticity: the manipulation of indeterminate quantities (e.g., addition, multiplication) as if they were known. Therefore, algebraic symbolism is neither a necessary nor a sufficient condition for algebraic thinking. At the same time, growing research on early algebra has recently raised issues such as: characterization of "early algebraic thinking" as there is no clear-cut break between early algebra and algebra; forms of curricular activity that support early algebraic thinking; nature of PD programs that support teachers' capacity to foster early algebraic thinking; and use of digital tools in the teaching and learning of early algebra (Kieran et al., 2016). Existing research focuses mainly on early algebra at primary level (e.g., tasks, teaching approaches, students' difficulties) as well as on the introduction of algebra at secondary level. However, the transition from arithmetic and early algebra to algebra within and between the two levels remains problematic as (a) the teachers at one level have limited knowledge of the issues related to the other and (b) PD programs struggle to engage teachers from the two levels. Therefore, how primary and secondary teachers collaborate in PD programs to develop resources aiming to develop students' algebraic thinking is an area that requires further research. In our study, we use the boundary crossing approach (Akkerman & Bakker, 2011) to explore primary and secondary teachers' collaborative work to develop resources fostering the development of students' algebraic thinking, as well as their professional learning.

Theoretical Framework

We use the documentational approach to didactics (Gueudet & Trouche, 2009) to study teachers' PD, focusing on choice, appropriation and transformation of resources, either individually or collectively by groups of teachers. The term resource describes a variety of artifacts: textbook, software, discussions with colleagues etc. Teachers modify existing resources to adapt them to their pactices/contexts, giving birth to documents that can further evolve over time. This process integrates practice and knowledge and combines elements of stability and evolutions (adapting to new context, new curriculum…). The process of gathering, creating and sharing resources within a community, called community documentational genesis, results in a community documentation composed of "shared repertoire of resources and shared associated knowledge (what teachers learn from conceiving, implementing, discussing resources)" (Gueudet & Trouche, 2012, p. 309).

Since our study involved teachers from both primary and secondary education, between which a clear boundary exists, we use also the boundary crossing approach (Akkerman & Bakker, 2011). A boundary is defined as "a socio-cultural difference leading to discontinuity in action or interaction" (ibid., p. 133). Boundary crossing refers to an individual's transitions/interactions across sites in professional situations where participants may need to enter onto unfamiliar territories/contexts, by negotiating and combining their ingredients so as to create hybrid situations (Suchman, 1994). Boundary objects are artifacts facilitating crossing of boundaries, fulfilling a bridging function (Star, 2010). Situations requiring crossing of boundaries can generate learning through four mechanisms: (a) *identification* of a boundary, entailing "a questioning of the core identity of each of the intersecting sites, that leads to renewed insight into what the diverse practices concern" (Akkerman & Bakker, 2011, p. 142). This can be achieved by a dialogical subprocess of "othering", i.e., defining one practice in light of another and highlithging differences between them; (b) *coordination* of activity flow: partners find means/procedures allowing diverse practices to cooperate efficiently, dialogue between partners is established only as far as necessary to maintain the work flow; (c) *reflection* on the specificities of two sites and the existence of a boundary between them involving processes of perspective making (i.e., redefining one's own perspective in relation to the other) and perspective taking (i.e., taking a new look at one's own perspective through the other's eyes); (d) *transformation*: actors from different sites engage in constructive activity by crossing boundaries and addressing boundary objects. Transformation processes start with a confrontation/problem "that forces the intersecting worlds to reconsider their current practices and the interrelations" (ibid., p. 146) often resulting in recognizing a shared problem space. This, in turn, engages participants in the process of hybridization by combining ingredients from different contexts into something new and unfamiliar. Our research question thus is: How does the teachers' collaborative work at the boundary between primary and secondary education (1) influence their documentational work targeting the introduction to algebra, and (2) contribute to the development of a shared view of what algebraic thinking is through the design and use of digital and non-digital resources?

Methodology

Our methodology refers to design-based research (Cobb et al., 2003), which is a collaborative and iterative approach to the design (in our case of educational resources) conducted in ecological conditions (close to teachers' practices). The collaboration between and with the teachers is at the heart of our research. In the Greek context, the collaboration consisted of PD meetings and classroom implementations including (a) initial discussions on what algebra is, how it is introduced in the curriculum and textbooks (with respective examples) and ideas for developing classroom tasks, (b) design of digital and non-digital resources individually or in subgroups, presentation and discussion of the designs, (c) classroom implementation, and (d) reflection and possible adaptations. In the French context, the collaboration took two different forms. At the level of schools (primary and secondary) involved in the project, groups of teachers accompanied by one or more researchers, called 'factories', worked on the design of classroom activities, their implementation and re-design. At the project level, specific sessions were organized, managed by a pedagogical engineer mobilizing agile methodologies fostering collaboration. During these sessions, all members brought their expertise to the common endeavour of designing resources that respond to teachers' needs. At this level, more general issues of interest for all factories were discussed, such as how to orchestrate classroom activities, what kinds of strategies can be expected from students at various school levels, or what algebra is. The collected data in both contexts are: designed resources (lesson plans, worksheets, digital files); video records/transcripts of PD meetings or collaborative work sessions; teachers' e-mail messages; teachers' interviews; and videos from classroom implementations.

In the following section, we analyze episodes selected from two different contexts - Greek and French - as representative of interactions between primary and secondary teachers occurring during collaborative task design and community meetings. Our analysis attempts to identify the evolution of the collective documentation and utterances indicating the activation of learning mechanisms.

Analysis of Two Cases

The Greek Case

Context. The Greek PREMaTT community involves a group of 7 secondary, 2 primary teachers working in different public schools and 4 researchers. Both primary and secondary teachers were experienced and the majority of them had been involved in several research projects and PD programs. Although algebraic thinking is targeted in the primary mathematics curriculum through specific tasks (e.g., patterns in the last grade, 12 year-old pupils), most of the primary teachers consider algebra as an 'unknown' terrain belonging exclusively to secondary education. This view, held also by secondary teachers, is reinforced by the fact that primary (6-12 years old, 1st-6th grades) and secondary (13-18 years old, 7th-12th grades) school levels constitute two distinct cycles and there are not official initiatives promoting their connection (e.g., through PD programs). In this paper, we focus on a subgroup of two qualified teachers: Anna (PhD in teaching mathematics with digital tools, primary) and Tom (Master degree in mathematics education, secondary) who collaborated in the design and implementation of one task involving the use of eXpresser (Noss et al., 2009), a specially designed microworld that had been presented in the PD meetings. eXpresser allows students to create figural patterns through repeated building blocks of square tiles, to use (iconic) variables to reproduce their constructions for different number of repetitions, to express generalization and to check their correctness through appropriate feedback.

A case of reflection and transformation (Anna and Tom). In the initial group discussions, identification and coordination constituted the dominant mechanisms. Teachers exchanged views as to how algebra appears in different grades of primary or in the fisrt grade of secondary education. The presentation of concrete examples from curriculums and textbooks triggered the emergence of coordination across the two sites. Reflection (mainly perspective making) and transformation processes were brough to the fore as the teachers were progressively engaged in developing their didactical designs through analyzing, justifying and adapting their choices. The case we analyse below highlights such processes.

Perspective making and taking around curricular and digital resources. During the first PD meeting, Anna presented the 'flower bed' task from the primary school textbooks (6th grade) in which students have to design the next instance of a geometric pattern representing a flower bed (the green part) framed by square yellow tiles and fill in a table (Fig. 1). Inspired by Anna's presentation, Tom chose this task (perspective taking) and adapted it for the 7th grade (first grade of secondary education). By making explicit the rationale of his choices (perspective making), Tom presented his first version of the task in which students were asked to: (a) calculate the number of the blue square tiles for instances with 1, 2, 3, 10, 100 red tiles (Fig. 2); (b) find out how many tiles would be red and how many would be blue if the pattern was created with 150 tiles in total; and (c) answer the previous questions by creating the pattern in eXpresser. It seems that Tom's approach for introducing algebraic notions is decontextualized and rather abstract while it is not clear how students are supported to build bridges between early algebraic thinking and algebraic generalization. The use of the digital tool only in the final task indicates that it is exploited to support students in expressing the generalization formally using the eXpresser's notation (Fig. 2). This approach is consistent with typical teaching approaches followed by the majority of the Greek secondary mathematics teachers.

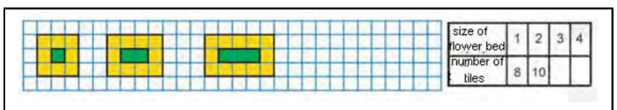


Figure 1: The 'flower bed' task

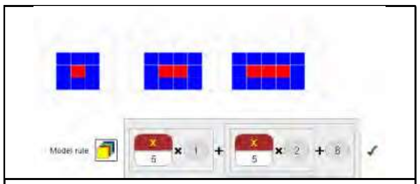


Figure 2: Tom's initial design

Transformation: Confrontation of task design as a space of negotiation. The design of Tom operated as a boundary object that helped him transform his documentational work taking into account Anna's perspective and crossing the boundaries between primary and secondary education. Reflecting on Tom's design, Anna brought to the fore aspects of her knowledge and her assumptions concerning early algebraic thinking by proposing to him to: (a) contextualise his tasks engaging students in constructing kinesthetically the first instances of the pattern with the use of eXpresser; (b) promote a more recursive view of the pattern – at least in the first instances - that would scaffold students' attempts to identify regularities and work with functional situations; (c) give students the chance to express mathematical concepts and relations verbally in everyday language before moving to symbolisation and abstraction. Anna's constructionist perspective towards the use of digital tools became explicit by her suggestion to Tom to develop together a pattern in eXpresser that would be given to students to build on it, change it or decompose it. This way students would not start working on eXpresser from scratch and it would be easier for them to focus on the pattern's properties. The digital tool here operates as a boundary object forcing the two teachers to reconsider their practices.

In parallel, Anna and Tom discussed how to use tables of values in worksheets to guide students' exploration and help them discern the pattern. Tom had initially thought of using tables to guide students' exploration and help them investigate relations among sets of numbers. By reflecting with Anna on the role of table in students' solutions/approaches, he finally decided to exclude it so as to preserve the openness of the task. It seems that in the process of recognizing a shared problem space around the use of tables the dialogical mechanisms of perspective taking and making were also present. Reflecting around boundary objects such as the use of digital resources and the integration of tables in the task, they both came to know what the diverse practices are, they recognized shared problem spaces while boundaries were encountered and contested. This is more evident in Tom's redesigned activities.

Hybridization through collective documentation. Tom redesigned the task by attempting to bridge the different perspectives. Specifically, he: (a) used a realistic frame according to which the students had to help a mayor to colour the pavements of a town; (b) introduced the whole activity through the use of eXpresser by engaging students in constructing specific instances of the pattern; (c) asked students to generalise through the construction of the general rule and the use of algebraic notation in eXpresser. This way the digital tool is meant to be used by students both as an experimentation space and a means of generalization. Tom's new worksheet comprises only two questions: (a) How many blue square tiles are needed in a pavement with 500 red square tiles? and (b) If 1557 square tiles of both colours are used in total in a pavement, how many tiles are blue and how many tiles are red? In both questions students had to explain their line of thought.

The collaborative task design by Anna and Tom brought to the fore various boundaries such as: a more constructive/geometric vs a more instructive/arithmetic approach to patterns; a step-by-step approach of generalization vs a formal approach towards the general rule. Although these discontinuities are still present, the evolution of Tom's design indicates a process of community documentation. This process led to a shared body of resources and shared associated knowledge at the boundaries of primary and secondary education related to how algebraic thinking can be approached at each educational level. Anna's and Tom's professional knowledge influenced the design of activities and at the same time their collective documentation work operated as a boundary object facilitating the extension of their existed knowledge.

The French Case

Context. The French PREMaTT community involves a group of secondary mathematics teachers from three different schools, groups of teachers from three primary schools, several researchers and a pedagogical engineer. The project took place in a period of a curricular reform that resulted in establishing four cycles of schooling. Cycle 3, comprising grades 4-6 (last 2 years of primary and first year of secondary school, 9-11 years-old), creates a 'bridge' between primary and secondary education and this transition poses a new challenge for teachers from the two levels.

The group of four secondary teachers who formed the SESAMES factory (S2 in the sequel), is involved for several years in research on teaching algebra in Cycle 4 (last 3 years of lower secondary school, 12-14 year-old). It has produced a number of resources shared on a platform and is involved in PD of mathematics teachers. S2 joined the project to reflect on the transition within the Cycle 3 and between Cycle 3 and Cycle 4 in relation with (early) algebraic thinking. Lamartine factory (L1 in the sequel) comprised two experienced primary (Grade 5) teachers, one having a university degree in mathematics. The analysis reported below focuses on S2 and L1.

Adaptation of an existing resource: coordination, reflection and transformation. From the outset, S2 presented some of their resources designed for teaching algebra in Cycle 4. One of the resources (Fig. 3) was chosen for adaptation to different school levels, from Grade 4 to 9.

With matchsticks, I construct triangles as in the figure. 	How many matchsticks are needed: at stage 1 (to construt 1 triangle)? At stage 2? At stage 5? At stage 10? At stage 100? At stage 265? Find a way to calculate the number of matchsticks needed depending on the number of triangles constructed.

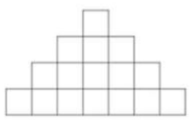
Figure 3: Resource offered by S2 chosen for the first design and implementation

This task was implemented by all factories (coordination). Sharing experiences of these first implementations sparkled a reflection about patterns: what is/what is not a pattern, and consequently what is/what is not related to algebra. Patterns thus acted as a boundary object that led teachers from both levels to reflect on cognitive processes involved in solving pattern tasks and how they connect to algebraic thinking. Questions were raised about stages of algebraic thinking development that students at different shool levels can reach, delineating a shared problem space recognized collectively (transformation): when are they able to create formulas? How pattern generalization manifests itself in primary pupils not yet introduced to algebraic symbolism?

Design of a new resource: coordination, identification and reflection mechanisms. Reflecting on patterns led the teachers to suggest other examples. The patterns considered were often of the order of recurrence. To avoid installing a didactical contract following which the pupils are excpected to manipulate symbols, a new problem was brought by a teacher from S2: a pattern of 'pyramids'. A collaborative session was organized to study the potential of this problem (coordination). Whereas L1 found the problem relevant with respect to the generalization, S2 was not convinced as they anticipated the difficulty to reach the formula, which is n^2 (n being the number of cubes in the middle column), that requires deconstructing and reconstructing the figure. Other possible solutions are 1+3+5+…+(2n-1) when considering stages of the pyramid, or 1+2+…+(n-1)+n+(n-1)+…+1 when considering its columns; these solutions are complex as they require establishing a link between the rank of the pyramid and the number of cubes in the middle column. This session was a key moment of the project: from a same problem, the opposite views about it led to the design of two different activities by L1 and S2 respectively (Fig. 4).

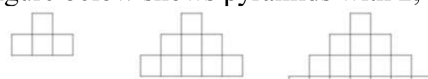
Figure 1 Figure 2 Figure 3

How many cubes are there in figure 4? Explain.



Phase 1. Find different ways how to calculate the number of squares needed to construct this pyramid.

Phase 2. The figure below shows pyramids with 2, 3, and 4 stages.



1. Calculate the number of squares for pyramids with 2, 3, 5 stages.
2. How many squares for a pyramid with 10 stages?
3. How many squares for a pyramid with 100 stages?

Figure 4: Different tasks designed by L1 (top) and S2 (below) from a same problem

Instead of looking for commonalities in the sequence of pyramids, S2 decided to start by focusing on the structure of the pyramid with four stages and search for differents ways of calculating how many squares it comprises (phase 1). The goal was to let students (Grades 6-8) to manipulate, modify the structure of the pyramid, and accept that the new figure has no relation to the initial problem anymore. The emphasis on the calculation (rather than counting) led to a more focused work on numbers and their properties. Pattern generalization came in phase 2 with a question about the number of squares in a pyramid with a quite big number of stages (100) so that the students are not able to represent or draw the pyramid and count the squares but are forced to reason. L1 rather chose to progressively guide younger students (Grades 4-5) toward generalization by working with pyramids having smaller number of stages (1, 2, 3, 4 and for the 'quick' ones 6 and 10). These designs brought to the fore various boundaries (identification): different routes toward algebraic thinking - through figure construction and pattern generalization (L1) vs figure deconstruction and working on the structure of the figure (S2); different stance to the level of difficulty in problem solving – choice of variables to guide pupils toward the solution and avoid failure (L1) vs choice of variables to let pupils face difficulty leading to the evolution of strategies (S2). Identification of such boundaries was critical in reflecting on the essence of algebraic thinking.

Conceptualization of algebraic thinking. At the end of almost two years of collaboration, a work session was organized aiming at coming back to the project central issue: what agebraic thinking (embracing early algebra) is. Based on experiences of all factories, groups mixing researchers and primary and secondary teachers were invited to express and share their ideas (perspective making). These discussions revealed the necessity to clarify and agree upon meanings of words such as modelling, generalizing or pattern (identification of boundaries and coordination). Only then the community was able to come up with two directions leading toward algebraic thinking: (1) pattern generalization (independent of symbolic writing) requiring modelling the structure of figures and allowing prediction, and (2) working on numbers: structure, equivalence, relations, and properties.

Conclusion

In this paper we focused on the boundaries and learning mechanisms activated in the context of primary and secondary teachers' community documentation targeting students' algebraic thinking. We studied two distinct contexts: French, where a recent curricular reform established a cycle bridging primary and secondary levels; and Greek, where a clear cut between the two educational levels exists. In the Greek case, the two qualified teachers were 'ready' to work with digital resources. The analysis brough to the fore various boundaries (e.g., step-by-step vs formal approach to generalization) but indicated the potential of collaboration between teachers of the two levels: the collective documentation operated as a boundary object favoring (a) the recognition of a shared problem space related to how algebra can be approached by each one of them and (b) the development of hybrid documents resulting from teachers' collaboration that combined views/ingredients from both educational levels. In the French case, collaborative design of resources revealed boundaries in perception of algebra (i.e., primary teachers were novices whereas secondary teachers, although having already been teaching algebra, had limited insight on processes of algebraic thinking). Collaboration among the teachers raised the need to make these perceptions explicit and agree upon common definitions of key terms like modelling, generalizing or patterns. These terms acted as boundary objects enabling negociation of meaning and resulted in both the design of classroom activites for Grades 4-8 and a shared view of algebraic thinking and the kinds of tasks fostering its development since primary grades. In both cases, resources from primary and secondary teachers' collaboration appeared as a lever for their professional development. This design was orchestrated and monitored by researchers. Their role and effects on the reported outcomes will be analyzed at the next stage.

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