Chapter 5
Tools and Resources Used/Designed for Teacher Collaboration and Resulting from Teacher Collaboration

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5.1 Introduction

This chapter, corresponding to Theme D in the conference, is dedicated to tools and resources used in, designed for and resulting from teacher collaboration, and is composed of six more sections after this brief one. Section 5.2 introduces the

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essential issues to be addressed. Sections 5.3 through 5.6 are dedicated to tools and resources: designed for teacher collaboration (Sect. 5.3); for learning to improve teaching practice in collaboration (Sect. 5.4); for fostering collaboration (Sect. 5.5); for studying collaboration (Sect. 5.6). Section 5.7 looks to the future, considering not only the discussion at the conference, but also other possible topics of interest for research.

5.2 Resources, Tools and Collaboration: Essential Issues

5.2.1 Presentation of the Chapter, in Continuity with the Past

This chapter describes the past experiences (Sect. 5.2.1)—mainly previous ICMI studies and the ICME 2016 survey on teacher collaboration (Robutti et al., 2016)—and the contemporary experiences—other chapters in the book and plenaries (Sect. 5.2.2)—and progressively guides the reader to the main issues at stake: what tools and resources support teacher collaboration or are derived from teacher collaboration (Sect. 5.2.3). To express this continuity adequately over time and themes, this section aims to get readers' attention to the matter of keywords and their role and contextualisation in theoretical frames and methodological issues (Sect. 5.2.4). Meanings of keywords, paper presentations and the modalities of working in collaboration among the Theme D participants group were discussed in person and remotely before, during and after the conference as described in Sect. 5.2.3.

This sub-section essentially frames this present chapter on two elements: the previous studies that led to the theme of the present ICMI conference, on teachers working collaboratively and the continuity with past ICMI Studies that contain some seeds of this theme, in relation to collaboration and use of tools and resources. Since the ICME survey on mathematics teachers working and learning through collaboration (Jaworski et al., 2017), there has been a continuous increasing "interest in exploring and examining different activities, processes, and the nature of differing collaborations through which mathematics teachers work and learn" (Robutti et al., 2016, p. 652). And the proposal of the present ICMI Study 25 is in continuity with that interest. In particular, the present chapter, on tools and resources for/from teacher collaboration, finds a continuity in previous Studies, such as the following.

1. The 15th ICMI Study: The Professional Education and Development of Teachers of Mathematics, 2005, for the attention on mathematics teachers' knowledge (Liljedahl et al., 2009) as a component of teacher education, and on the methods and forms inside the institutions that support learning in collaboration, for example Lesson Study (da Ponte et al., 2009), with some focus on the tools and resources for teacher collaboration.
2. The 17th ICMI Study: Digital Technologies and Mathematics Teaching and Learning: Rethinking the Terrain, 2006, for the theoretical frameworks emerged (e.g. the instrumental approach—Guin et al., 2005), and the ways in which

technology can mediate, support and influence the teaching and learning of mathematics, especially for their relation to collaborative practice, as a precursor of the different sections present in this chapter, as we can see in the corresponding book (Hoyles & Lagrange, 2010).

3. The 22nd ICMI Study: Task Design in Mathematics Education, 2013, for its focus on the design, and on the issue of relating the tool-specific discourse representation to mathematical knowledge (Watson & Ohtani, 2015), interpreting tools as physical or virtual artifacts with the potential to mediate between mathematical experience and mathematical understanding (Leung & Bolite-Frant, 2015).

The presentation of the topic of tools and resources for and from this collaboration in the more general context of teachers' collaboration in working and learning, and in continuity with previous Studies, gives us the possibility to enter into the topic of Theme D, in the following sub-section.

5.2.2 This Chapter in the Book

This sub-section identifies the research aims and questions described in the Discussion Document and references the other chapters, or the Themes and the Plenaries, organised within the conference.

This chapter is the result of the work and paper presentations that took place both in person and remotely—before, during and after the conference—and of participants from the Theme D group. This chapter (Theme D) is focused on *tools* and *resources for* teacher collaboration and *from* teacher collaboration: "Resources *for* and *from* teacher collaboration can be considered as two ingredients of continuous processes: *adopting* a resource leads always to *adapting* it, and that is more the case in the context of teacher collaboration" (Borko & Potari, 2019, p. 9; *italics in original*). The research questions addressed in the call for contribution are:

> *What resources are available to support teacher collaboration? With what effects, both on the collaboration and on the resources themselves?*
>
> *What resources are missing for supporting teacher collaboration? How and to what extent can teachers overcome these missing resources?*
>
> *To what extent and under what conditions do digital environments (e.g., mobile devices, platforms, applications) constitute opportunities for teacher collaboration? How have these resources been used to support teacher collaboration?*
>
> *Which resources can be used (and how) to sustain and scale up collaboration over time?*
>
> *How are teachers engaged in the design of resources in collaboration? What are the outcomes of these collaborations?* (Borko & Potari, 2019, p. 9; *italics in original*)

This chapter presents a new way to speak of tools and resources: seeing them as products of teachers' collaborative work or means to support teachers' collaboration and their possible evolution in various settings. The papers submitted examined

these issues (and other possible sub-issues) in different ways. We will present and weave different threads over the different sections of this chapter, considering the different topics involved, their interaction, and the theoretical and methodological issues and approaches. In particular, we will look at tools and resources not as static products, but as evolving objects, which can be the products of teachers' collaboration, and can support and mediate collaboration. Since tools/resources are seen in this chapter as two sides (means and products) of a coin, using and designing them are then to be considered as two possibly interrelated processes. In the presentation of the glossary (Sect. 5.2.4), we will describe the shared meaning of the terms used in this chapter.

The chapter is connected not only with past research, but also with the other chapters of this book:

- with Theme A for theoretical frames and approaches for studying collaboration, and for the particular tools, artifacts and resources involved in studying collaboration;
- with Theme B for exploring different forms of collaboration and their outcomes, the different number of collaborators, the subjects involved in collaboration and the timing of collaboration (e.g. synchronous or asynchronous modes);
- with Theme C for the different roles, identities and interactions of the various participants engaged in collaboration (e.g. lead teachers, facilitators, mathematicians, researchers, policy makers).

A continuity can be seen particularly with Chap. 9 (this volume) by Karin Brodie (who made a plenary talk in the Study Conference) and Kara Jackson (her reactor), investigating teachers' collaboration with the use of resources. The authors give a framework for systematically studying professional learning communities and propose various kinds of resources to be studied: knowledge, material/logistical, affective and human.

5.2.3 Participants and Collaboration in Theme D

This sub-section shows the participation to the Theme, in terms of papers, participants, and kind of working—both in-person and remotely—to present, discuss and contrast the different studies. The sub-section also highlights the different collaborative modalities used to organise this work and provides examples.

The participants were: Ornella Robutti and Luc Trouche as leaders of the group, and Karin Brodie and Kara Jackson as plenary speakers (respectively lecturer and reactor). Twenty-two additional scholars presented studies from a number of countries, for a total of 26 participants, who arrived up to about 28–30 for the turnover of observers from the Scientific Committee of the Study and from the plenary speakers. The 26 participants were from the following countries: Canada (one), China (one), Colombia (two), France (two), Greece (one), India (one), Israel (two), Italy (five), Japan (one), South Africa (one), Spain (one), Taiwan (one), Turkey (one) and the

United States (six). They presented a total number of 18 papers, divided in six sessions, to give time in the last two sessions for a discussion in preparation for this chapter.

The collaborative modalities of the Theme D participants included both in-person and virtual formats. During the conference, paper presentations and discussions took place in person. However, the Theme D participants also collaborated virtually before, during and after the conference, via a platform. Participants communicated at all hours and a large amount of contributions and level of synergy was noted. Materials collected in the platform were: papers, presentation slides, reactions, schedules, a common list of references and—last but not least—a shared *discussion sheet* (totalling 73 pages). The discussion sheet was organised by the nine sessions. Each session consisted of the session theme, a session chair, two secretaries, who prepared a brief report of the session, and the paper presentations and reactors that corresponded to the theme.

We must also note that—due to the pandemic—colleagues from China, specifically Shanghai, were unable to attend the conference in person. Therefore, during the conference, our colleagues were invited to present virtually and we paid special consideration to our communication and responses. In Sect. 5.7.2. we will describe how we, as teachers and researchers, are reconsidering the roles of tools and resources as a result of the COVID-19 pandemic.

The participants to Theme D collaborated face-to-face and at distance as a community of practice (Wenger, 1998), and, more specifically, a community of inquiry (Jaworski, 2006), made by researchers who use collaboration to share goals and methods and to study tools/resources in teachers' activities from a theoretical perspective (see Sect. 5.2.4). But there is something more: the discussion sheet was used not only as a passive repository of materials of the discussion, but properly as a resource (see Sect. 5.2.4) enriched by participants in a collective way. Therefore, this discussion sheet had the role of a tool in the collaborative work of participants, and a meta-tool for reflecting on their practices of inquiry on the theme.

5.2.4 Towards a Shared Glossary on Tools and Resources

This sub-section shows and motivates the main terminology choices made in this chapter and contextualises these terms within their theoretical frames, or from a general point of view. The reflections made on the glossary call attention to the fact that using a term with a specific meaning may be contextualised in a theoretical frame and gives sense to how the term is used in research, or the same term can be used in a more generic way, embracing a meaning not directly linked to one frame.

Reporting here we group the terms into four sets:

(a) objects used by subjects (teachers, researchers, students);
(b) modalities of working together as subjects in a community;
(c) interaction among teachers' while collaborating in communities;

(d) work done by teachers, involving processes of teaching and of learning.

(a) The fundamental contrast between *artifact* and *instrument*, *tool*, *resource* and *document* is presented, according to the theory of instrumental approach (Verillon & Rabardel, 1995). An artifact is an object—with its characteristics and affordances—that can be transformed into an instrument, with the introduction of the subject's schemes of use. The process of transformation is called instrumental genesis and has a double side: instrumentation (Trouche, 2020b), in which someone acquires an instrument in order to perform a given activity, and instrumentalization, as "*adapting a tool for adopting it as a support of one's mathematical activity*" (Trouche, 2020a, p. 392; *italics in the original*).

Tool, as well, is an object, but in a broader sense, noticing that "the development of mathematics has always been dependent upon the material and symbolic tools available for mathematics computations" (Artigue, 2002, p. 245). The term 'tool' comes principally from Vygotsky, and it is used in the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008). From a Vygotskyan perspective, an activity is composed of a subject and an object and mediated by a tool (material tools as well as mental tools, including culture, ways of thinking and language). While the subject is engaged in an activity, the object is held by the subject and motivates activity, giving it a specific direction (Vygotsky, 1978). The role of instrument-mediated activities (Rabardel & Bourmaud, 2003) can be considered in its different kinds: as *mediation to the object of the activity*, aimed at getting to know the object and also at acting upon it; as *interpersonal mediation*, oriented toward others; and as *reflexive mediation*, towards the subject, in her/his relation with her−/himself, mediated by the instrument (Sect. 5.4).

Resource is intended in the sense of Adler (2000), as something to re-source the teacher's practice. If Adler also includes human resources, the *documentational approach* (evolved by instrumental approach—Trouche et al., 2020a, b) intends objects, with a lesson plan attached (explaining how to use it for teaching, including didactical objectives), that through schemes of use (Vergnaud, 1998) introduced by a subject evolve into documents. The resources can be either textual (e.g. textbooks, curricular guidelines, student worksheets), or digital (e.g. digital textbooks or websites) (Trouche et al., 2020a, b).

Applying a specific theoretical approach, as instrumental or documentational, or the theory of semiotic mediation, these terms are to be intended as specified above. To describe and contrast studies from different theoretical frameworks, in this chapter we prefer to choose a shared meaning for tool and resource, sufficiently framed in literature but not linked to a specific frame, in order to be flexible enough in using them. For this reason, we consider artifacts and resources not synonymous, but one larger than the other: resource is conceived in a larger significance than the notion of artifact, which can be avoided.

The term 'resource' will be used to indicate what is used by the teachers in their teaching activity: a lesson plan, a mathematical problem, a digital animation (in this case digital resource), and so on, of the material, socio-cultural or didactic–methodological type. We may also reference human resources, as

determined by the papers we cite. The term 'tool' will be used to indicate something allowing to find and/or manipulate a given resource—a browser, the email, a word processor—or to guide its usage—a theoretical framework, a national curriculum, an assessment system of a school, and so on.

(b) This chapter is relevant to the modalities of working together as subjects engaged in a *community of practice*, conceived as a joint enterprise with a shared repertoire and mutual engagement (Wenger, 1998), eventually *online* (Johnson, 2001). Subjects may also be engaged in a *community of inquiry*, early introduced by Dewey (1902)—as any group of individuals involved in a process of empirical or conceptual inquiry into problematic situations—and used in mathematics education by Jaworski (2006) (see also Sects. 5.3 and 5.4), as a community that brings inquiry into practices of teacher education in mathematics—where inquiry implies questioning and seeking answers to questions and problems.

The *professional learning communities* are centred on shared learning (Jaworski, 2014) and aim "to enhance teacher effectiveness as professionals for students' ultimate benefit" (Stoll et al., 2006, p. 229), and have an organising structure development inside a broader community that acts as a reference point for teachers' professional learning, based on systematic reflection, inquiry into one's own practice and collaboration with colleagues (Brodie & Borko, 2016). In this chapter, we will refer to communities of practice and/or more specific communities, according to the studies mentioned above and to other approaches to communities.

Referring to teachers organised in communities and engaged in professional development (PD), different theoretical frames need attention if we want to describe teachers' knowledge or their learning as a process:

- CK—*Content Knowledge* and PCK—*Pedagogical Content Knowledge* (Shulman, 1986) and its derivations frames, applied to mathematics education, as MKT—*Mathematical Knowledge for Teaching*, defined as "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Hill et al., 2008, p. 499), and particularly the Mathematics Teacher Specialized Knowledge (Carrillo-Yañez et al., 2018) to refer to the knowledge of teachers (Sect. 5.5.3), or TPCK—*Technology, Pedagogy and Content Knowledge* (Koehler & Mishra, 2005; Thomas & Palmer, 2014), more related with the use of technologies;
- MDT—*Meta-Didactical Transposition* (Arzarello et al., 2014; Robutti, 2020) to refer to teachers, professionally engaged in PD process, who are learning in a community of colleagues, in relation to a community of researchers, and are evolving in their meta-didactical praxeologies;
- DAD—*Documentational Approach to Didactics* (Pepin et al., 2013) evidences the dialectic relationship between the development of a community of teachers, and the development of a shared repertoire of resources—giving also a social aspect to the process (Pepin & Gueudet, 2020): the approach has been precisely developed for considering which learning occurs when many people interact with many resources.

(c) Teachers' interactions can be observed by the lens introduced by Akkerman and Bakker (2011), who identified different mechanisms—identification, coordination, reflection and transformation—that sign the boundary crossing across communities (Sects. 5.4 and 5.5). The theme is particularly useful to study evolution in terms of processes and of products (see Sects. 5.4.4 and 5.5.3), intending them as boundary objects (Star & Griesemer, 1989; Star, 2010) upon which communities of researchers and of teachers act on the boundary, and it has been applied to mathematics teacher education (Robutti et al., 2020).
(d) Speaking of the work done by teachers in/for the class, it is important to consider the use of terms such as *learning path*, *lesson plan*, *learning sequence*, *teaching sequence* and *learning trajectory* (Simon, 2014), conceived as proper resources for teachers in their collaborative work. Some examples for researchers and teachers have been identified, for example referring to:

- practical resources for teachers that might become conceptual tools to work collaboratively on at a meta-level;
- tasks that have multiple solutions;
- activities, objects, methods that can be boundary objects;
- evaluating tasks for the purpose of class.

5.2.5 *Structure of the Chapter*

The structure of the different sections of the chapter, presented in the following, came out by the discussions in Theme D sessions during the ICMI Study 25. We could structure the sections according to a categorisation of the different kinds of tools/resources used in the studies, or according to the protagonists: teachers, researchers, teacher educators, ... However, we present here something more than just a categorisation: we want to give the readers possible ways to interpret the complexity in the using tools/resources in relation to collaboration and with respect to communities of teachers in various contexts. Therefore, we identify a set of threads running throughout the papers that can properly give sense of that complexity. Each one of the next sections describes one of the threads and reflects the ideas that pass across the papers presented and discussed in Theme D.

Starting from the shared meaning of the terms *tools* and *resources*, as declared in Sect. 5.2.4, we accept that they do not only represent material objects, but also represent symbolic abstract objects. Then we intend them as tools/resources that serve *for* teacher collaboration and that come *from* teachers' collaboration. This double sense (*for* and *from* collaboration) of intending tools/resources as part of the collective work of teachers/researchers/educators ... gives the main idea for starting to find the different threads:

> In Sect. 5.3, there are examined studies on resources for teaching mathematics that are particularly designed and developed through collaboration among teachers, researchers and knowledgeable others;

In Sect. 5.4, the focus is on the tools and resources that support teachers' collaborative inquiry into teaching;

In Sect. 5.5 there is a possible classification of different tools and resources for fostering teachers' collaboration, in the sense that may structure and mediate teachers' collaborative activities and support reflection on teaching;

Section 5.6 presents how researchers examined/analysed the teacher collaboration organised by the tools or how teachers interacted with the tools and resources themselves;

Section 5.7 weaves together the themes discussed throughout the chapter to highlight the main research questions that Theme D aimed to address, and examines these themes in the light of the pandemic period, to address issues of equity and inclusion.

The various sections illustrate these perspectives using research from different projects and one project may be discussed in multiple sections. The first time a project is introduced we describe the following characteristics: the country, the teachers involved (pre- or in-service), the number of people involved (if it is known) and the specific context (inside the institution or outside). Other aspects related to the project (e.g. the type of collaboration; the interactions between participants; the resources used) are then discussed in subsequent sections, according to their different foci.

5.3 Resources for Teaching That Develop and Evolve Through Collaboration

This section pertains to resources for mathematics teaching that are specifically developed through collaboration among teachers or/and researchers and knowledgeable others. The sub-sections are organised by the purpose for which these resources are developed: implementing a new curriculum (Sect. 5.3.1); teaching complex mathematics topics (Sect. 5.3.2); supporting teachers to develop teaching (Sect. 5.3.3).

5.3.1 Supporting the Implementation of a New Curriculum

Implementing a curriculum reform is a complex process, requiring a change in teachers' beliefs and practices (Clarke & Hollingsworth, 2002). Remillard (2005) identifies the central role that teachers play in drawing/interpreting/participating with the curricular text and thus "constructing the enacted curriculum" (p. 224). In this sub-section, we discuss the central role that teachers play in creating their own versions of resources, by adapting and adopting them according to their own needs defined as *documents* through the process of documentational genesis which may be individual or collective. We discuss how old and new curricular resources (traditional and digital) are mobilised by teachers in collaboration with knowledgeable

others to evolve into new resources either through their implementation or by redesigning post reflection on its implementation. We will rely on three contexts—a Greek one, a French one and an Algerian one—all of which involve the evolution of resources by teachers for classroom use to support this discussion.

The resources considered in all these contexts are a mix of traditional textbooks, drill exercises or new digital repositories like Sesamath or micro-world (eXpresser) created by others but transformed by in-service teachers in their use. The discussion with other collaborators is also considered as a resource in these cases. These collaborators are teachers in the Greek and Algerian cases, and also include researchers in the French case. These are termed *factories*[1] having the responsibility of designing new resources, their implementation and their redesign. The evolution of new resources occurs through the "community documentational genesis" (Gueudet & Trouche, 2012, p. 309), which involves discussing, designing, implementing, reflecting and adapting the design collectively.

Psycharis et al. (2020) highlight how the community documentation in both the Greek and the French PREMaTT[2] community is able to bridge the divide between the primary and secondary grade teacher's approach and thinking towards the teaching of algebra. The Greek case provides an example of a primary and a secondary teacher working together towards developing algebraic thinking using a microworld eXpresser (Noss et al., 2009). The suggestions given by a primary teacher to contextualise tasks, encouraging recursive view of the pattern and verbalising in everyday language led to its redesign by the secondary teacher and results in the hybridisation of the document.

The French case provides evidence of collective documentation through implementation and reflection of predesigned Sesamath resources by *factories* (Psycharis et al., 2020), and reflection on stages of development of algebraic thinking and generalisation in primary and secondary pupils. The primary teachers design a problem to identify different ways to calculate the number of cubes needed to construct the pyramid, making students focus on number properties. Phase 2 involves focusing on pattern generalisation through the reasoning for the stage 100 pyramid structure. The secondary teachers focused the discussion on such tasks provide shared space for both primary and secondary teachers, to make their "perceptions explicit and agree upon common definitions of key terms like modelling, generalizing or patterns" (Psycharis et al., 2020, p. 675). Thus, the evolution of resources (Trouche et al., 2019) involves the process of identification of *boundary objects*, othering, reflection and transformation (Akkerman & Bakker, 2011).

Sayah (2020) presents an Algerian case, connected to the French one through the use of Sesamath resources (https://www.sesamath.net/) created by the Sesamath community to develop textbooks collaboratively and make them freely available online. The new Algerian curriculum is structured on mathematical competencies, problem solving and the usage of information and communication technology (ICT),

[1] Collaborative groups of researchers and teachers to generate resources.

[2] http://ife.ens-lyon.fr/ife/recherche/groupes-de-travail/prematt

Fig. 5.1 Meriam's schematic representation of her resource system. (Sayah, 2020, p. 649)

and lack of corresponding resources led some teachers to try to appropriate resources from abroad, particularly from the French-speaking countries.[3] Sayah presents a case of a teacher "Meriam", a middle-grade teacher, who uses and adapts Sesamath textbooks from French to Arabic institutional context, while also mobilising other resources and colleagues in her network to create a resource system.

Figure 5.1 represents the drawing of Meriam's *resource system* made at the request of the researcher. It highlights how interactions with her school colleagues (Nadine, Adam and Youcef) lead to the identification of resources and renewing her resource system. Sayah's work illustrates the interactions that develop in the frame of small *communities of practice* (Wenger, 1998), where the participants engage in learning together and have shared purposes and objectives. Youcef and Meriam (as teacher colleagues) constitute a community working mainly for integrating ICT (*GeoGebra* and Interactive whiteboard) in mathematics teaching, and Adam and

[3]French was spoken in Algerian schools from the beginning to the end of the colonial period (1830–1962), leading to the use of a mix of languages. French is still spoken for mathematics teaching in Universities, and most Algerian researchers in the field of mathematics education participate, each 3 years, to the conference "Espace mathématique francophone" (http://emf.unige.ch/), one of the ICMI regional conferences.

Meriam work mainly for the adaptation/translation of Sesamath resources. Thus, the teachers' resources evolve from the interactions of small communities sharing the results of their work leading to a joint evolution of teachers' communities and teachers' resource systems (Gueudet & Trouche, 2012). Sayah proposes a model of teacher resource system based on these evolutions, distinguishing *mother resources* (textbooks, Sesamath resources), *intermediate resources* as results of teachers' collaborative work, and *stabilised resources*, once integrated into teachers' practices.

In all the cases discussed in this sub-section, the need for new resources emerged in the context of the implementation of a new curriculum. This need leads teachers to collaborate for sharing and adapting the pre-existing resources, resources provided by the institutions (as instructions) or from external organisations/platforms. The appropriation of the standardised resources outside the 'designers' circle' is illustrated in Sayah's work through the evolution of a resource system of a teacher in Algeria and by adaptations of Sesamath and eXpresser by *teacher workshops* to address teachers' own needs in France and Greece. The collaborative process of the generation of resources and being available online in open access form plays a critical role in its appropriation and further evolution.

In all cases, the use of language constitutes a critical resource, as a support and a result of teachers' collaboration. In Psycharis et al.'s (2020) study, a shared understanding of terms related to Algebra emerges through the collective documentation work and reflection and transformation of practices across the primary and secondary grades; in Sayah's (2020) study, the interaction between Arabic and French language leads teachers to deepen the mathematical knowledge at stake, while facing the problems of cultural and institutional transposition (Mellone et al., 2019; see also Esteley et al., Chap. 3, this volume). In the next sub-section, we discuss further how the evolution of resources through collaboration is inextricably related to the transformation of teachers' knowledge of mathematical concepts.

5.3.2 *Supporting the Deepening of Teachers' Mathematical Content Knowledge*

Development of understanding of mathematical concepts can be viewed as a result of a complex and complementary interplay between developing an understanding of the structural aspect as mathematical *objects* and operational aspects as *processes* (Sfard, 2012). In this sub-section, we discuss two cases from Japan (Ohtani et al., 2020) and India (Kumar, 2020), to understand how collaboration supports the evolution of resources for a difficult concept like functions and integers. In both cases, collaborators are in-service teachers (both elementary and secondary teachers in the case from Japan), designers and researchers and additionally, in the case of Japan, ICT specialists and Lesson Study experts engaging in design research.

The intervention is organised and designed by researchers in both cases who are also authors (Ohtani et al., 2020; Kumar, 2020). The teachers play the role of both co-designers as well as implementers. The units developed as a product of collaboration are implemented by the in-service teachers. The tools to promote collaboration are knowledge of collaborators engaged in designing, as well as the awareness of inconsistencies and gaps in the discourse, artefacts like textbooks, representations and variability of meanings of the focused concepts and pedagogical strategies for teaching them. The critical resources generated *from* the collaboration is a unit on functions (Japan) using an ICT-based learning environment in form of *GeoGebra* applets (designed by Nunokawa) and representations, models, contexts and activities included in the individual unit plans developed by teachers through collaborative investigation (India).

The variety of evolving resources that come together to support the discursive practices is discussed with respect to their role in leading to the development of collaborators' teachers' in-depth understanding of these complex topics. The setting for collaboration is outside of the school space where collaborative discussions take place. However, Ohtani et al. also discuss the insights gained from classroom implementation of the designed environment. They designed an ICT-based learning environment that fosters an understanding of functions through reification (Sfard, 2012), by converting an operational procedure into a mathematical object on which subsequent operations can be performed.

Considering the problem of classroom discourse focused only on calculations rather than referring to functions as a 'mathematical object', Ninjas are proposed as a metaphor for functions. The other representations of functions like numbers, tables, graphs and algebraic expressions are then considered as shadows of Ninja, which gives a glimpse of the existence of Ninja and through which several properties of Ninja (function) can be derived to identify how Ninja moves. This distinction between the representation of a function as Ninja and its representations as its shadows is proposed to help students focus on key aspects of functions like co-varying quantities, rate of change and expanding the range of variables in case of direct, indirect, linear and quadratic functions. The students are expected to make conjectures about functions, "saying something like [...] 'the Ninja moves much slower when far from the origin of the coordinate plane'; 'this linear function Ninja moves faster than this linear function Ninja'" (p. 664).

To ensure that classroom discourse is consistent and supports reification, the focus is on developing teaching units and features of *GeoGebra* applets to engage students purposefully to investigate covarying quantities, represent their properties and talk about functions as existing objects. Researchers analysed classroom discourse to identify the use of low-level discourse (focused on calculations) or high-level discourse (treating functions as objects and referring to its property of co-varying quantities). Analysis by researchers indicates the need to maintain consistently the high level of discourse in the classroom as the concept of functions, as "change in variables" become the object of the talk at the beginning of the unit, but not in its latter part. This insight is presented in the Ohtani et al., paper as an implication for the further redesign and implementation of the unit.

In Kumar's (2020) work, the tool for collaboration was the framework of meanings of integers and their operations. The meaning of integers can be interpreted as state, change and relation in different real-life or *realistic* situations (van den Heuvel-Panhuizen & Drijvers, 2020) while the meaning of negative sign as unary, binary and symmetric function is evident in mathematical expressions. The situations depicting combine, change and comparison of quantities are represented by addition and subtraction of integers in the framework (Kumar et al., 2017). The analysis of meanings using contexts, models and symbolic representations for teaching integers lead to discussions of the criteria for selecting, using and designing these resources for teaching.

For example, the teachers from Mumbai (with a tropical climate) felt that it was difficult for students to make sense of negative temperatures as students do not experience it, and therefore selected the negative scores on tests to indicate the negative state. However, the decrease in (negative change) temperature was used to represent the change using negative integers. The relation between the temperature of two different days can be represented by negative or positive integers depending on which day is taken as the reference point. The addition or subtraction of integers can be further represented as change or combination of their states or cumulative changes or as the difference between the two related states in contexts like scores, change in baby's weight, change in height and so on.

The analysis and correspondence of meaning of integers and operations in different contexts, models and numerical expressions led to making the implicit criteria for selecting, using and creating representations explicit in the teachers' collaborative discourse. These criteria are expressed at a surface level or deeper level, depending on the level of concern for meanings and consistency. The 'translatability' criterion refers to mapping in representations like a number line or numerical expressions, when the quantities or their change is represented mathematically through them.

A surface-level concern might be focusing on showing equivalence between numerical expressions (3–4) and (−4 +3) using symbols, while a deeper level of concern is indicated when teachers acknowledge that taking away a larger number from a smaller number may not make sense to students. Not considering the difficulties faced by students due to implicit + (positive sign) in expressions like $3–4 = 3 + (−4)$ indicates a surface-level concern for meaningfulness criterion while being sensitive to students' difficulties indicates a deeper level of concern. When the meaning attributed within contexts and models are consistent with mathematical meanings (e.g. equivalence in numerical expressions through commutativity), the criterion for mathematical consistency is reflected at a deeper level while focus on rules indicates surface-level concern.

Both cases focused on developing resources for concepts considered abstract. There were inconsistencies in the discourse related to both concepts among teachers, in textbooks, language and even in the meanings held by researchers. The collaborative discourse during designing of unit plans for teaching functions and integers engaged members to delve deeper into meanings of concepts, understand difficulties faced by the students and identify inconsistencies in the discourse for teaching them,

with the aim of it influencing the movement of discourse at the descriptive level in the classroom to the deductive level (Ohtani et al., 2020). Therefore, the issue of coherence and consistency in the discourse across grades for teaching became an important one for developing an understanding of the concepts among students as was also observed in Psycharis et al.'s (2020) study in Sect. 5.3.1.

To analyse what changed as a result of designing the unit plans collaboratively, Ohtani et al. focused on classroom discourse adopted by the teacher and the students, while Kumar focused on the change in the discourse of the teachers during the collaborative investigation and designing of plans. Ohtani et al. found that discourse that refers to functions as objects was discerned in the classroom discourse when students compared the rate of change of linear function and inverse function dynamically on *GeoGebra* applets. However, students may also construct "idiosyncratic meaning to Ninja movements" (p. 666) using the *GeoGebra* applet and focus on "only surface relationships between those expressions and graphs" (p. 666).

Kumar observed the teachers' discourse within the workshop interactions reflected gradually deeper concerns for translatability, meaningfulness and mathematical consistency, and they reported increased use of contexts and models, and reflected on the importance of using them as representations rather than rules on symbolic expressions to develop the meaning of integers and their operations (Kumar et al. 2017).

Ohtani et al. and Kumar both analyse how the deliberations and interactions between the team, associating researchers and teachers, using the classroom discourse as a tool for and from the collaboration are important for achieving the outcomes in form of teacher's more meaningful use of resources in the classroom. The tasks were collaboratively designed by both researchers as well as teachers while the classroom implementation was led by the teacher and researchers played either supporting or observer roles. Ohtani et al. highlight the need "to establish a transparent context between researcher and practitioner" (p. 667), as necessary for collaborative engagement of all actors.

Kumar highlights the discursive nature of collaboration and discussions about the suitability of the representations as one of the ways that teachers seem to develop consistent discourses for teaching the concept. Thus, in both cases, the collaboration focused on specific mathematical abstract concepts led to the outcome of the development of discourse practices in the workshop and the classroom settings. In the following sub-section, we discuss how these collaboratively designed tools and resources can even support the development of mathematics teaching practices.

5.3.3 Supporting the Development of Mathematics Teaching Practice

The collaboration between teachers and knowledgeable others (Huang, 2020) in professional development settings involves resources that are directly or indirectly

related to teaching practice. The resources discussed in the previous two sub-sections are ideas or materials that are integrated with teaching. So, the nature of the tool for collaboration is some form of material resource (like a digital resource) or a cognitive resource (framework of meanings) that can be used for a specific purpose in teaching and therefore gets redefined as a 'tool' (e.g. in the form of lesson plan).

In this sub-section, we discuss the tools for collaboration that are "representations of practice" (Herbst & Kosko, 2014) in the form of scenarios that may occur in the classroom (Cusi et al., 2020) or video-recording of actual teaching (Uzuriaga et al., 2020) that were used in a professional development setting with the purpose to develop new teaching practice itself, such as giving more room to students for participating to the advancement of knowledge. Such tools and resources have been used with the purpose of developing ideas of new ways of teaching, creating substantive learning opportunities for teachers by promoting visualisation of several possibilities and interventions to support student learning in different scenarios (Cusi et al., 2020) or by promoting reflection on teaching through observation and analysis of classroom teaching through identification, analysis, collective discussion, and systematisation (Uzuriaga et al., 2020).

In this sub-section, we take up the case of Cusi et al. from Italy and then Uzuriaga et al. from Colombia to analyse how the evolution of tools or resources, collaboratively driven in the form of representations of practice, contribute to developing mathematics teaching. The case from Italy involves four in-service teachers in six meetings over an eight-month period in a professional development program working on scenario design taking the example of a task aimed at promoting students' exploration of the relationships between elements of a varying figure. The Colombian case illustrates collaborative action research among 15 primary and secondary teachers participating in a Master's-level course in mathematical methods course for 2 years through the design and redesign of didactic, inquiry-based unit plans in workshops, and implementation and analysis of the teaching of didactic unit plans in schools.

In both cases, the teacher educator and/or researchers played a supportive role during discussion and validation, while the task of designing scenarios or didactic plans were done by the teachers. Cusi et al. focus on analysing the teachers' interactions during the workshop and change in praxeologies evident in the transactions, while Uzuriaga et al. focus on analysing the teachers' implementation of didactic plans using tools for observing teaching practice and analysis matrix which was developed and validated by the researchers during the course of 2 years Masters' program.

The work of Cusi et al. involves teachers "not only in designing the tasks for students and the teaching methodology, but also in hypothesising possible students' answers to the tasks and hypothetical excerpts of classroom discussion, containing teachers' interventions" (p. 605). This results in *an ordered set of scenes* called *Scenario*, which includes not only teachers' and students' interventions in a class setting represented in a storyboard format (with characters depicted as teachers and students), but also thought balloons for teachers, in order to express their rationale behind the actions depicted in the scenario.

Every scenario refers to a specific class situation depicting a mathematical task and the teachers discussed the tree of possibilities in response to students' conjectures and teacher actions. In one such discussion, the teacher educator introduced the possibility of all students either agreeing or disagreeing with a conjecture to make teachers rethink and redesign the scenario considering the 'tree of possibilities' and, thus, including different "ramifications" and introducing "thought balloons" to depict teachers' thinking behind her response (Cusi et al., 2020). Thus, teachers' re-designed scenario reflected the evolution of their didactical praxeologies at the classroom level (Chevallard, 1985/1991) by identifying gaps between teachers' intentions and student thinking, adopting ways of questioning and becoming more flexible in their practice.

At the same time, the re-design also reflects the evolution of "meta-didactical praxeologies" (Arzarello et al., 2014) in PD context when facing new educational paradigms and engaging in shared reflection (see Sect. 5.4.3). The meetings in the PD setting involved the development of theoretical ideas and connecting them to practical aspects by playing the role of the learner and then engaging in designing scenarios for using a particular task with students, revising it based on feedback received from researchers and peers. How teachers' reflections on the classroom experiences contributed to the collaboration and the revision of scenarios will be discussed in Sect. 5.5.3.

Uzuriaga et al. used collaborative action research to make teachers question their practice and develop practices related to inquiry methodology in their teaching practice. Both inquiry methodology discussed in the course and the instruments for observation and analysis of teaching practice are used as tools for supporting collaboration, and the outcome from the collaboration is also the development of inquiry methodology as practice. The inquiry methodology had four phases of practical investigation: triggering event, exploration, integration, and resolution (Bustos, 2011), and involved self-evaluation and co-evaluation for regulation of learning. The researchers present the analysed transcribed video records of classroom teaching to identify the extent as well as occurrence and co-occurrence of teachers' adoption of different practices related to inquiry methodology.

Some of the highly appropriated practices included 'connecting the content with the daily life situations' (e.g. making a poster of favourite foods to teach fractions), 'searching for hypotheses for the proposed problem' and 'resolution of solution through teacher evaluation'. The practice of promoting student argumentation co-occurs most during the phase of 'resolution confirmation' during the practical inquiry illustrated with the example of organising the teams for football championship based on average goals scored. Interactivity was promoted by involving students in the construction of meanings illustrated by the example of rewriting repeated addition in the form of multiplication. Teachers developed an understanding of inquiry methodology, as well as reflecting on their teaching practices in the process of engaging in analysis (for further discussion, see Sect. 5.4.3).

Cusi et al. (2020) and Uzuriaga et al. (2020) provided evidence of how both anticipating the classroom events and analysing them collaboratively led to the evolution of teachers' understanding and practices. Thus, the tools for collaboration

represented, anticipated and supported practice, as well as became tools for reflection on classroom practices during both pre- and post-implementation. In the case of Cusi et al., the deliberations in the PD setting worked directly on developing teachers' thinking using anticipated students' responses in scenarios. Teachers made the reasons behind their moves explicit, using the thought balloons as well as the different possibilities that may occur as a result of the move. In this way, these deliberations allowed the teacher to make reasoned and explicit choices in terms of interventions and choices on teaching practices and praxeologies, and even get feedback from knowledgeable others about the intervention, thus promoting learning about adopting new practices.

In the case of Uzuriaga et al., the deliberations occurred at multiple levels—first between the peer teachers working together in class and analysing teaching through video records and also with feedback given by faculty on didactic unit plans as well as the tool for analysis of classroom practice. The variability in the appropriation of different aspects of the practice by the teachers points to the possibility that collaboration with the teacher educators and peers in a professional development context impacts teachers' images of practice in different ways. These alternative images of practices get realised in the classroom and reflection on them further revises these images. Thus, the representation of practice that is being used as a tool for collaboration is the old practice adopted by teachers, as well as the idea of inquiry and the practice associated with it. The classroom practice and video records also work here as a tool for the collaboration of teachers as they analyse it.

Both cases presented in this sub-section had used the representations of practice to foster collaboration and to delve deeper into the understanding of teaching practice. Here, representations of practice are revised in several iterations, and therefore are simultaneously resources *for* and *from* collaboration. Further discussion on using tools for collaboration for teacher engagement in activities are discussed in Sect. 5.5.2. Next, we reflect on the three previous sub-sections to identify the transversal issues we have raised.

5.3.4 Transversal Issues and Perspectives Around Living Resources

The three previous sub-sections illustrate the different ways of collaborating (diversity of scale, of agents and of settings) and different purposes for collaborating: facing a curriculum change, facing the teaching of complex topics and developing teaching practices. In this final sub-section, we summarise these findings and identify the transversal strands across the papers. We discuss the diversity of theoretical frameworks, the diversity of resources for/from collaboration and the interconnections between these resources.

The theoretical frameworks used in the papers are diverse, but essentially come from socio-cultural paradigms highlighting an aspect of the collaboration established

between different participants including teachers. The documentation approach to didactics (Gueudet & Trouche, 2009), used by Psycharis et al. (2020) and Sayah (2020), emphasises the dialectical relationships between resources and teachers working in collaborative contexts. Ohtani et al. (2020) use the Cultural–Historical Activity Theory (Engeström, 1990) to highlight the dynamicity of the collaboration and the relationship between the participants, while Cusi et al. (2020), using the framework of Meta-Didactical Transposition, focus on how collaborative work fosters teachers' meta-didactical praxeologies through a double dialectic between the didactical (teaching) and meta-didactical (learning in a PD context) level.

Uzuriaga et al. (2020) focus on the appropriate practices by the teachers and the different phases of the inquiry process. Kumar (2020) has teachers engaged in the analysis of representations of integers based on a framework of integer meanings (Vergnaud, 1982), and arrives at an emergent framework of criteria of translatability, meaningfulness and mathematical consistency used for determining representational adequacy for teaching integers through analysis of discourses in PD context. The commonality in all these frameworks is the focus on how interactions between the collaborators bring about a dialectical change in teachers or their community, while what exactly changes might be focused differently.

The nature of the resources *for* and *from* collaboration may vary across the projects.

- The resources *for* collaboration can be classified into two categories: the first one corresponds to the material adaptive resources (Sayah, 2020) or the digital resources (Ohtani et al., 2020; Psycharis et al., 2020) that are adapted or used in teaching; the second one corresponds to resources focusing on planning or reflecting on teaching, highlighting the key ideas to be focused on (Uzuriaga et al., 2020). The collaboration plays a critical role in developing teacher competencies through reinterpretation and recontextualisation within collaborative discussions using either a cognitive resource like a theoretical framework of integer meanings in Kumar (2020) or material tools such as storyboards and resources such as scenarios (Cusi et al., 2020).
- The resources *from* collaboration can be classified into three categories: the first one concerns material resources like digital resources (Ohtani et al., 2020) or didactic plans (Uzuriaga et al., 2020; Kumar, 2020) that can be directly used in classrooms with students; the second one consists of human resources in the form of individual teacher's competencies as mathematical knowledge (Ohtani et al., 2020; Kumar, 2020) or appropriated practices (Uzuriaga et al., 2020); the third one consists in resources which can be considered as community resources in the form of a resource system (Sayah, 2020), collaborative units like factories (Psycharis et al., 2020) or shared criteria among the designers (Kumar, 2020).

Although we have discussed the resources for and from collaboration as distinct categories, both types of resources influence each other as anticipating practice, sharing the experience of practice and reflecting on practice co-occur in interactions in professional development settings. Section 5.3.1 underlines these close interconnections, within collaborative settings, between material (e.g. digital resources),

human (e.g. agents and their roles) and cognitive resources (e.g. language) (Psycharis et al., 2020; Sayah, 2020) when addressing a curriculum change. Similarly, Sect. 5.3.2 underlines the connections between the cognitive resources in the form of mathematical meanings and connections between representations in discourse, the interactions with human resources leading to the development of material resources for classrooms for teaching complex topics (Ohtani et al., 2020; Kumar, 2020).

Section 5.3.3 also underlines the connection between the scenarios or didactic units as material resources, the interactions with human resources which led to changes in the material resource using reflection on classroom experiences as cognitive resources (Cusi et al., 2020; Castro Superfine & Pitvorec, 2020). Because of these interconnections among the diverse resources, collaboration and interactions play a major role in establishing and highlighting these connections, incorporating ideas and experience from diverse resources and settings, and bridging the context of professional development with the school context having teachers as collaborators and incorporating their experiences. Thus, collaboration appears as an essential means for producing or making living resources for teaching, in order to support mathematics teaching in various settings.

5.4 Resources and Tools for Inquiring Collaboratively into Teaching

In the context of PD (for both pre-service or in-service teachers, within institutional contexts or outside institutions), involving teachers in collaborative activities could give them the opportunity to learn about teaching and to find out ways for improving teaching practices. While the focus of Sect. 5.3 is mainly on resources (for teaching) as a *product* of these collaborative activities, in this section we focus on those tools and resources that are specifically designed by researchers and teacher educators, with the aim of triggering and supporting a fundamental *process* that characterises teachers' collaborative work—the collaborative inquiry into teaching.

In other words, the focus of this section is on the different ways in which, within PD settings, tools and resources are designed and used to give teachers the opportunity to reflect deeply upon their own teaching, with the aim of promoting their shifts of attention toward constructs, theories and practices that can inform and guide their future choices (Mason, 1998, 2008). We, therefore, refer to those research studies that are focused on PD settings that can be characterised as *communities of inquiry* (Jaworski, 2006—see Sect. 5.2.4).

In this section, we reflect on possible ways of supporting collaborative inquiry into teaching, through the identification of tools and resources to foster and sustain teachers' collaborative work in: designing and redesigning teaching (Sect. 5.4.1); analysing different kinds of data from school practice (Sect. 5.4.2); representing teaching to reflect collectively upon it (Sect. 5.4.3). In order to reflect upon the ways

in which these tools and resources are used to support teachers' inquiry into teaching, we will refer to Rabardel and Bourmaud's (2003) categories of orientations that characterise instrument-mediated activities: (a) the *mediation to the object* of the activity, aimed at getting to know the object and also at acting on it; (b) the *interpersonal mediation*, oriented toward others and aimed at both knowing others and acting in interaction with them; (c) the *reflexive mediation*, through which the subject's relation to him−/herself is mediated by the instrument.

Final remarks on the reflections developed in Sects. 5.4.1, 5.4.2 and 5.4.3 are proposed in the final Sect. 5.4.4, which is aimed at discussing how the effects of teachers' collaborative inquiry into teaching, in terms of teachers' learning about teaching, could be investigated and highlighted. Since collaborative reflective inquiry could be seen as a common characteristic of most of the research studies presented within the theme D group at ICMI 25, in this section we will refer to a plurality of voices, discussing the different ways in which teachers' reflective practices are supported within different PD settings.

5.4.1 Supporting Teachers' Collaborative Inquiry into Teaching Through the (Re)Design of Resources

The design and redesign of resources for planning teaching, such as learning trajectories (Simon, 2014) or lesson plans, within communities of inquiry could represent a fundamental opportunity for teachers for collaborative inquiry into teaching. While the focus of Sect. 5.3 is on the product of these processes of design and redesign, here we focus on the ways in which PD settings are organised, around the use of specific tools and resources, with the aim of supporting teachers' collaborative design and redesign of teaching resources.

In this sub-section, the focus is, therefore, on the organisation of PD settings, interpreted as a particular combination of tools and resources aimed at fostering teachers' inquiry into teaching. This combination concerns: (a) the choice of resources upon which to focus teachers' design; (b) the identification of a proper environment within which design and redesign processes are fostered and implemented; (c) the tools provided to teachers to support their design or redesign processes.

The activities within which teachers are involved are initially aimed at fostering the two first categories of orientations that characterise instrument-mediated activities (*mediation to the object* and an *interpersonal mediation*), since teachers, first of all, have to know the objects they are working on (and with) and have to know each other to become able to act together on these objects. Here, the word 'object' refers both to the resources on which teachers' design and redesign processes are focused and to the tools provided to teachers to develop these processes. Moreover, the choice of the environment within which design and redesign processes are realised is fundamental in supporting (or not) the interpersonal mediation.

In the institutional setting introduced by Cusi et al. (2020), for example, in-service teachers have the opportunity to work collaboratively within a face-to-face environment. The object of their design work is the creation of a *scenario*, a specific resource representing different moments within classroom situations and their possible development (see also Sect. 5.3.3). The initial stage of teachers' collaboration is aimed at understanding what scenarios are and what are the main criteria according to which they could be designed.

Another example in which in-service teachers collaboratively work within a face-to-face environment is presented by Chang et al. (2020). These authors provide a PD program (Lin & Chang, 2019) characterised by a mutual collaboration among mathematics teachers, teacher educators and researchers. The resources upon which mathematics teachers' design is focused within this program are what Lin and Chang call mathematics-grounding activities. During the design process, teachers' collaborative inquiry into teaching is fostered by making them reflect upon how the activities they design could be implemented to enhance students' motivation and conceptual understanding in mathematics.

The environment within which teachers' design or redesign is developed could also be online or blended, as in the cases presented by Albano et al. (2020), who introduces a PD setting, characterised both by face-to-face and online interactions, where in-service teachers collaboratively design learning trajectories. In the study presented by Segal et al. (2020), the environment for teachers' design is completely online. These authors present a digital platform (http://RAMZOR.sni.technion.ac.il) designed with the aim of providing teachers with a communal environment where they can collaboratively design, share and preserve their teaching resources (lesson plans, teaching programs, assessment items).

Within the projects presented in the four examples, the step toward the third category of orientations that characterise instrument-mediated activities (the *reflexive mediation*), occurs when teachers are involved in subsequent activities aimed at making them carry out a collaborative reflective inquiry on the resources they have designed and on the process of design itself. During the project presented by Cusi et al. (2020), for example, teachers, after having worked in small groups at the scenario design, are led, during meetings that involve the whole group of teachers and teacher educators, to discuss collectively the effectiveness of their initial scenario design. Within the PD program presented by Chang et al. (2020), teachers' reflections on their design and redesign of mathematics-grounding activities are not developed with the whole group of people involved in the program, but within different groups on different foci (general pedagogical issues, technical aspects related to the construction of tools to be used in the work with students, reflections on students' difficulties).

In the case of the PD program presented by Albano et al. (2020), teachers' collaborative reflexive inquiry is realised through two main steps. Within the first step, teachers have the opportunity to collaborate with researchers, while, during the second step, teachers are engaged in a peer-review process. Also, within RAMZOR (Segal et al., 2020), teachers have the opportunity to reflect upon their development and joint improvement of materials for teaching and learning (Movshovitz-Hadar,

2018), by sharing and discussing their knowledge about teaching practice and daily experience through online and face-to-face meetings between mentors (experienced mathematics teachers) and mentees (groups of teachers of the same school).

Teachers' reflexive inquiry during these kinds of activities is supported by the use of specific tools provided to them. The collective discussions on teachers' scenario design presented by Cusi et al. (2020), for example, are developed not only by means of teachers' spontaneous reflections, but also by referring to specific theoretical tools, which support teachers in their re-design of scenarios. This redesign process is conceived as an on-going process, developed thanks to continuously shared reflections on the most effective ways in which teachers could intervene within classroom discussions to support students' learning processes, to highlight and discuss their difficulties, and to foster their reflections.

Theoretical tools (specific constructs deepened through the course materials) are also used to support teachers' and researchers' reflections, during the face-to-face meetings organised within the program presented by Albano et al. (2020), and to guide teachers' redesign of learning trajectories (Simon, 2014) according to these reflections. In the example presented by Albano et al., a further tool is used to support teachers' reflexive inquiry, in combination with theoretical tools. It is a methodological tool: the peer-review process. Thanks to this process, in which the combination of theoretical reflections and other colleagues' feedback plays a central role, teachers have the opportunity to reflect upon and improve their own teaching, giving rise to a further redesign of learning trajectories.

The collaborative inquiry activities testified in the four examples presented in this sub-section highlight the effectiveness of specific organisations of PD settings, aimed at providing teachers with the opportunity to reflect upon their own teaching practice by designing and redesigning resources and teaching approaches to be developed in their classrooms. The reflections developed within the communities of teachers and researchers that interact within these PD settings could have different foci. Although the products of the design and redesign carried out within these settings could have different characteristics, the four examples highlight the key role played by these shared reflections in determining the on-going evolution of these products. Moreover, the examples highlight the role played by theoretical and methodological tools in fostering teachers' reflexive inquiry into teaching during their design and re-design of teaching resources.

5.4.2 Supporting Teachers' Collaborative Inquiry into Teaching Through the Analysis of Classroom Data

The focus on specific tools (with this term we also refer to theoretical tools, as indicated in Sect. 5.2.4) within PD settings is also aimed at supporting teachers in analysing school practice, by observing, discussing, comparing their own classroom activities and the activities carried out by other teachers, referring to different kinds

of data (concerning both students' learning processes and products and classroom interactions). While Sect. 5.4.1 was devoted to the collaborative inquiry work *a priori* developed by teachers, that is before implementing the designed resources in their classes, this sub-section is therefore aimed at discussing the collaborative inquiry work that teachers, within different PD settings, develop *a posteriori*, that is after the implementation of specific resources in their classes or in other educational contexts.

An example of a setting within which in-service teachers have the opportunity to analyse data collected within a wide institutional context could be found in Ferretti et al. (2020). These authors introduce the use of Gestinv, an interactive database with structured information regarding Italian standardised assessment, aimed at creating a PD setting embedded in the institutional context of national assessment (Bolondi et al., 2017) to bridge large-scale assessment to the improvement of teaching and learning of mathematics at the level of the school system (De Lange, 2007). Within this setting, teachers develop, working in groups, a collaborative inquiry work focused on the exploration of Gestinv to find out, in relation to specific mathematical topics, items associated with the lowest percentages of correct answers.

The collaborative work is developed throughout different stages. At the first stage, when the collaborative activity fosters a *mediation to the object* and an *interpersonal mediation*, the focus of the inquiry work is on studying the collected items to start a reflection on typical students' mistakes and difficulties. At the second stage, teachers are provided with resources and tools to deepen the reflection engendered within the previous stage: they are asked to study specific resources, that is research materials regarding historical–epistemological and didactical aspects connected to the identified items and then to prepare a written presentation to be shared during the subsequent general discussion.

At this stage, *reflexive mediation* starts to be developed, since key elements, such as beliefs, convictions, reflections, emotions and agency, are brought to the forefront. The collaborative reflective inquiry is developed at a double level: the level of empirical analysis of typical items that have been difficult for students, aimed at making teachers identify the problem to be studied, and the level of theoretical analysis, focused on epistemological aspects (to support teachers' identification of possible origins of students' mistakes) and didactical aspects (to support teachers subsequent design of activities to help students in overcoming their difficulties). The construction of written presentations represents a key element in making teachers deepen their reflections. In fact, the need to communicate with others fosters teachers' explication of the results of their empirical and theoretical analysis and a consequent deeper awareness about the objects of the analysis itself.

In many PD settings the data that teachers analyse come from their classes or from the classes of other colleagues with whom they are collaborating. Pynes et al. (2020), for example, presented a web-based collaborative setting where in-service teachers, working in school-based teams, have the opportunity to analyse the written work of their own students. Also, in the example presented by Uzuriaga et al. (2020; see Sect. 5.3.3), the data under analysis comes directly from the classrooms. A group of in-service teachers, in fact, is involved, during the third phase of a two-year Master's

course, within an activity of observation and analysis of their own implementation of a didactic unit designed according to an inquiry approach to teaching (Wells, 2001). Teachers, who work in pairs, have to observe and analyse their own teaching practice, by focusing on the recordings and transcripts of their interaction with students during the implementation of the didactic unit.

As in the example by Ferretti et al. (2020), to develop the data analysis, also within the projects described in these last two examples, teachers are provided with specific tools that foster the *reflexive mediation* dimension. In particular, the teachers involved in the study presented by Pynes et al. (2020) are supported in self-facilitated collaborative inquiry through the use of a Collaborative Inquiry Tool (Pynes, 2018) aimed at supporting the development of complex skills foundational to noticing effectively children's mathematical thinking (Jacobs et al., 2010). The collaborative dimension of this activity supports teachers' development of deep awareness about the process in which they are involved, since, thanks to group discussions on students' written work, teachers are given the opportunity to share with others what they noticed about students' mathematical thinking. Consequently, they make this noticing explicit to themselves, developing, in this way, reflections on their own ways of noticing.

In the study by Uzuriaga et al. (2020), teachers' observation and analysis of the data collected in their classes is supported by the use of two main tools provided by researchers: a grid for observing the teaching practice according to the three categories of didactic sequence, scientific competence, and interactivity (González-Weil et al., 2012), and an analysis matrix. The authors highlight that, although observing and analysing practice was demanding for teachers, the use of the grid and of the analysis matrix has enabled them to develop a scientific attitude in the observation of their classes.

The three examples presented in this sub-section highlight the effectiveness of supporting teachers' collaborative inquiry work through the use of specific tools (theoretical materials, grids for observation, web-based tools, ...) that provide them with lenses that direct their attention during the *a posteriori* analysis of different data collected within local or national educational contexts (data from standardised assessment, videos of students work, students' answers or classroom interactions). Independently from the objects of the collaborative analysis carried out within the PD settings presented in this sub-section, the tools that direct this analysis put it from an empirical level to a more theoretical level, effectively supporting teachers' on-going reflections on the teaching and learning behind the data themselves.

5.4.3 *Supporting Teachers' Inquiry into Teaching Through the Representation of Mathematics Teaching Practice*

Within PD settings aimed at fostering collaborative reflective inquiry into teaching, the design and redesign of lesson plans (Sect. 5.4.1), and the analysis of teaching

practices or of other kinds of data from classroom activities (Sect. 5.4.2), are often developed through the use of different representational tools. An investigation of the tools used to support the representation of practice has been discussed also in Sect. 5.3.3, where it has been highlighted how these representations help in bridging the PD context and the school context. In tune with the focus of this section, in this sub-section the ways in which these tools could support a reflective interpretation of teaching are investigated.

The first example we discuss refers to the context of pre-service education. Weingarden and Heyd-Metzuyanim (2020) present a study in which pre-service teachers' analysis of real classroom data is supported through the use of a tool providing them with opportunities for collaboratively discussing and investigating the level of students' authority and the extent to which mathematical objects were treated within real whole-classroom discussions. This tool, the Realization Tree Assessment (in the following, referred to as RTA) (Weingarden et al. 2019), has been inspired by the commognitive theory of mathematical objectification (Sfard, 2008), according to which, since mathematical objects are discursive entities, students have to reify and alienate the different realisations of mathematical objects and to 'same' them. By talking about these realisations as the same thing, students become able to participate exploratively in the discourse about them. The RTA tool (Fig. 5.2) is aimed at visualising the realisations of mathematical objects that arise during classroom discussions and the extent to which students authored the different realisations.

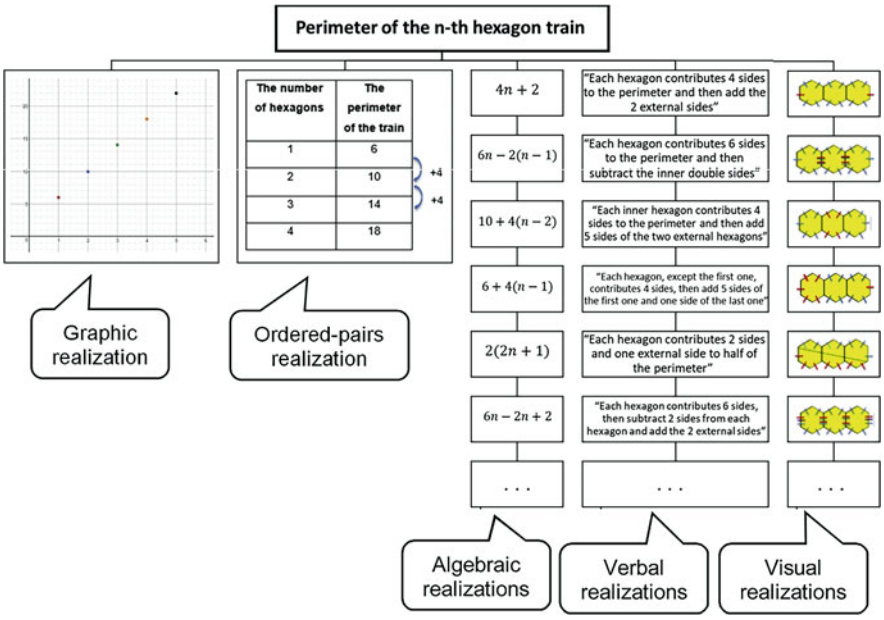

Fig. 5.2 An example of realization tree. (From Weingarden et al., 2019)

In the study presented by Weingarden and Heyd-Metzuyanim (2020), pre-service teachers are asked to work collaboratively at the coding of an empty RTA based on videos of lessons and at comparing and discussing different RTA images. After a phase in which they start exploring this tool (*mediation to the object*) to become able to develop collaboratively the required coding (*interpersonal mediation*), the visualisations realised thanks to RTA support pre-service teachers' in focusing their reflections both on the different types of links that can be made between realisations and on the importance of students' authority in mathematics lessons. Therefore, it is this visualisation that supports a *reflexive mediation*, since it fosters pre-service teachers' observations of the extent to which students authored narratives about the mathematical object and its different realisations during the lesson.

Also, Yuan and Huang (2019) present an approach characterised by a collaborative work focusing on representations of actual teaching, with the aim of making teachers reflect on the ways in which they can activate what the UK National Centre for Excellence in the Teaching of Mathematics (NCETM) defines teaching for mastery (NCETM, 2019). Within this approach, developed within the UK–China Mathematics Teacher Exchange Programme, the teachers are involved in activities of observations of lessons and discussions with other teachers in post-lessons meetings, during which they delineate representations of teaching practices referring to a research-based model, which introduces "five big ideas" behind the construct of teaching for mastery (NCETM, 2019): coherence, representation and structure, mathematical thinking, fluency and variation.

In tune with some of the examples presented in Sect. 5.4.1, this approach is therefore focused on the use of theoretical tools that support teachers' collaborative inquiry into teaching, by making them focus on specific characteristics of the lessons that are examined. Other well-known, research-based models have been developed with the aim of providing teachers with sets of theoretical lenses to observe, represent and reflect on teaching practices. Among them, we mention, for example, Schoenfeld's (2013) model of Teaching for Robust Understanding, Hollingsworth and Clarke's (2017) five-dimensional observational framework, Karsenty and Arcavi's (2017) six-lens framework and Stein et al.'s (2017) quadrants coding scheme.

In other cases, as in some of the examples presented in Sect. 5.4.1, instead of focusing on the analysis of real data from teaching–learning processes, teachers are led to make hypotheses about ways of fostering effective teaching, through the design of specific resources, such as hypothetical lessons, that consists not only in creating classroom activities and their *a priori* analysis, but also in writing down hypothetical transcripts of classroom discussions to foresee the possible interactions between the teacher and the students that could be realised. Also, in these cases, the used representational tools provide teachers with specific lenses that enable them to focus their attention on particular aspects of teaching–learning processes, engendering a collaborative inquiry that makes them develop deep reflections on their own teaching (*reflexive mediation*).

The characteristics of the scenarios presented in Cusi et al. (2020), for example, make them powerful tools that enable teachers, through the representation of

hypothetical teaching interactions with their students, to focus their attention on the ways in which students' development of inquiry attitudes could be supported through the activation of specific teachers' interventions within classroom discussions. The shared reflections developed by teachers and researchers, while they carry out the collaborative work on scenario design, therefore boost the evolution of their meta-didactical praxeologies (Arzarello et al., 2014; see also Sect. 5.2.4), that is the specific tasks that teachers have faced in their daily teaching, the techniques used to face these tasks and the justifying discourses through which teachers explain the choices they made in terms of chosen techniques and ways of using them.

Another example in which teachers collaborate by being engaged in cycles of scripting, visualising and arguing about moves within a lesson—Story*Circles*—is presented in Herbst and Milewski (2018, 2020). The product of this collaborative work, which is carried out using specific resources (e.g. a task statement, records of student work), is lesson maps that are represented through partially ordered sets of storyboards. These representations can grow in complexity as practitioners identify new decision points and alternative courses of action in lessons or as facilitators bring in possible contingencies that participants may not have anticipated. When constructing these representations, teachers can deploy not only strategic knowledge (e.g. which problem to be used to lead students to a particular goal), but also tactical knowledge (e.g. how to respond to diverse students' contributions).

The examples presented in this sub-section highlight the effectiveness of using representational tools to support teachers' collaborative interpretation, analysis and reflection on teaching. Moreover, they enable us to highlight different categories of representational tools: (a) from tools aimed at generating visual representations of the ways in which mathematical objects are treated within classroom discussions; (b) to theoretical tools aimed at identifying and observing specific aspects of teaching practices; (c) to digital tools and environments aimed at supporting teachers' construction of representations of teaching episodes and lessons. As Herbst and Milewski emphasise, these tools are more effective in supporting teachers' inquiry into teaching, if compared with other traditional resources, which can leave out much of the tactical problem solving done while teaching (like tasks and lesson plans), or may make it hard to distinguish what is usable elsewhere and what cannot be disentangled from context (like the records of actual instances of lessons).

In summary, representational tools give strengths to the reflective dimension of collaboration (engendering a *reflexive mediation*), since they enable teachers to bring to light what usually is not made explicit, making it a tangible object of reflection.

5.4.4 Transversal Issues and Perspectives Around Inquiring Collaboratively into Teaching

The previous sub-sections enabled us to discuss the use of different resources and tools to foster teachers' collaborative inquiry into teaching, through the *a priori*

design and redesign of teaching resources (Sect. 5.4.1), the *a posteriori* analysis of different kinds of data from school practice (Sect. 5.4.2) and the use of various tools to support the construction of representations of teaching to reflect collectively upon it both *a priori* and/or *a posteriori* (Sect. 5.4.3).

Through the different examples we presented, we interpreted teachers' interactions with tools—theoretical, methodological, or technological—when they collaboratively work on specific resources or analyse different data from teaching practices, in terms of kind of mediation (*to the object*, *interpersonal*, *reflexive*) that the work with these resources and tools could engender. Although all the three kinds of mediation are jointly present within each instrument-mediated activity introduced in the previous sub-sections, *reflexive mediation* represents the key aspect that characterises the activities developed when teachers collaboratively inquire into teaching by means of different resources and tools. In fact, in our analysis, we highlighted how the different phases of teachers' collaborative inquiry work gradually foster the shift from the *mediation to the object* to the *reflexive mediation*.

The choice of the resources and tools that mediate the collaborative inquiry into teaching plays a central role in this gradual process. In particular, the combination of theoretical and representational tools (see Sect. 5.4.3) seems to be particularly effective in this sense, since it provides teachers with specific lenses that direct their attention during the *a priori* or *a posteriori* analysis that they develop within the PD programs we have presented in this section. Moreover, the different examples highlight the key role played by the settings within which these processes are realised. The combination of tools and resources that gives birth to these settings is, in fact, critical in fostering teachers' inquiry into teaching.

Another fundamental element in fostering the engendering of *reflexive mediation* is the collaboration between teachers and among groups of teachers and teacher educators or researchers. This collaboration, in fact, fosters the deepening of the reflections that are developed during the inquiry work, since the need of comparing and communicating ideas to others makes teachers bring to light what is usually not made explicit when they work alone. We can therefore observe that the examined studies highlight the influence that *interpersonal mediation* has on *reflexive mediation*.

After having examined the use of tools and resources, within the different examples described in the previous sub-sections, to support teachers in collaboratively learning about teaching, a spontaneous question is: how could the teachers' learning, as an effect of teachers' inquiry into teaching by means of different tools, be highlighted and investigated? The analysis of the studies previously discussed enables us to propose an initial categorisation of the ways in which this investigation can be developed.

A first way of investigating teachers' learning is to *look at the evolution of specific products of their collaborative inquiry work*, such as the resources that teachers collaboratively design and redesign. In the study presented by Ferretti et al. (2020; see Sect. 5.4.2), for example, teachers' learning is investigated by focusing on the evolution of specific components of the learning trajectories designed by them. Also, in Cusi et al. (2020; see Sect. 5.4.3), teachers' learning is investigated by

highlighting the evolution of their praxeologies, which is, in turn, highlighted by analysing the corresponding evolution of the products of the teachers' collaborative work, that is the scenarios.

A second category of approach adopted to investigate teachers' learning as an effect of their collaborative inquiry into teaching is *to look at the interactions that characterise the collaborative inquiry work that teachers are developing*. An example of this approach can be found in Psycharis et al. (2020; introduced in Sect. 5.3.1), who investigate teachers' learning while interacting with others for designing and sharing digital and non-digital resources. This learning is investigated not only by looking at the evolution of the products of the collaborative inquiry work, but also by analysing the evolution of processes, in terms of utterances in teachers' interaction, indicating the activation of the four learning mechanisms of boundary-crossing introduced by Akkerman and Bakker (2011; see also Sect. 5.2.4).

In some of the presented studies, teachers' learning about teaching is investigated also by *studying teachers' meta-reflections on their experience within the collaborative inquiry activity in which they are involved*. This third category refers to the idea of involving teachers in what we could call *inquiry on inquiry*. Examples of teachers' involvement into 'inquiry on inquiry' processes are presented in Segal et al. (2020), Uzuriaga et al. (2020) and Cusi et al. (2020). The proposed categorisation is obviously provisional and partial. A wide survey of the research on this field is needed to deepen the fundamental issue of categorising the approaches adopted by researchers to investigate teachers' learning as an effect of teachers' inquiry into teaching by means of different tools.

The remarks shared within this concluding sub-section enable us to stress upon the fundamental role played by the *reflexive mediation* that could be engendered when teachers collaboratively work by interacting with different tools (theoretical, methodological, technological) to inquire into teaching. The studies presented in the previous sub-sections highlight, in fact, that the ways in which collaborative settings are designed to give teachers the opportunity to reflect deeply upon their own teaching certainly foster teachers' learning about the teaching practices that are the object of their reflections. Further studies have to be developed to confirm these results and to deepen the investigation of the ways in which the use of specific tools and resources to inquire into teaching fosters and affects both reflexive mediation and teachers' learning.

5.5 Resources and Tools to Facilitate Teacher Collaboration

In this section, teacher collaboration is considered in itself with a particular focus on the tools and resources designed and/or used to facilitate it. In diverse contexts, such as PD and classroom, teachers, teacher educators and researchers exploit a multiplicity of tools and resources that structure and mediate teachers' collaborative

activities and support reflection on—their own or other teachers'—teaching (see Sect. 5.5.4).

In the following sub-sections, we consider the nature of these tools and resources by exploring their different categorisations (Sect. 5.5.1) and how they are used to support teacher collaboration (Sect. 5.5.2). By taking a broader view, we also explore what theoretical tools and professional practices can be oriented towards teacher collaboration and how (Sect. 5.5.3). Finally, we synthesise the findings and draw conclusions (Sect. 5.5.4).

5.5.1 Categorising Tools and Resources for Fostering Teacher Collaboration

The nature of tools and resources fostering teacher collaborative work is an important concern for designing, studying and understanding teacher collaboration. Over the last 20 years, technology has offered a variety of tools and resources that have been used to support teacher collaboration, ranging from specially designed environments to represent practice to online platforms allowing documentation and sharing of materials (Herbst et al., 2016).

Focusing on the types of these tools and resources, we categorise them in two broad categories: (A) tools and resources designed for teacher collaboration, and (B) tools and resources that were not initially conceived for teacher collaboration in an educational context but under professionals' (teachers'/teacher educators') intervention were adapted to operate as formal or informal environments for teacher collaboration. The second category is divided in two sub-categories: (B1) tools and resources that can be considered as designed for collaboration but not necessarily for educational purposes, and (B2) tools and resources designed for educational purposes but not necessarily for collaboration.

In category A, we identify digital environments designed to promote teacher professional development by supporting their collaborative activities. Pynes et al. (2020; see Sect. 5.4.2) present a web-based tool, the Collaborative Inquiry Tool, created to support teachers in self-facilitated conversations with colleagues regarding the mathematical thinking of their students. The tool allows participation of groups of teachers in Collaborative Inquiry sessions to discuss activities such as posing problems to students, analysing students' written work for a common problem type, sharing teaching artifacts and creating new problems based on specific students' understanding.

Similarly, Segal et al. (2020; see Sect. 5.4.1) explore the potential of a digital platform (i.e. RAMZOR) designed to facilitate teacher collaboration around the development and improvement of teaching materials such as lesson plans, teaching programs and assessment items. The environment allows sharing and transformation of materials through feedback comments and joint elaboration and, thus, it can serve as a pillar for the development of teacher communities of practice.

In category B1, we identify tools and resources designed for distance working (and eventually learning), storing/sharing and communicating, that allow the development of activities for subjects in general (not necessarily educational) contexts and can be used by teachers. One strand of tools in this category stems from the area of e-learning platforms (e.g. Moodle) that provide educators with integrated systems to create personalised learning environments. However, it is under question if and to what extent these systems can be used to support virtual collaborative activities among participants in e-learning courses. For instance, working with practicing secondary mathematics teachers in an e-learning PD course, Albano et al. (2020; see Sect. 5.4.1) report that specific tools of Moodle (i.e. Assignment and Workshop) allowing exchange and peer-review of teachers' submitted work (i.e. learning trajectories) can scaffold teachers' collaboration and promote re-design of learning trajectories and reflection on own teaching.

Another strand of tools in this category concerns shared drives of general use that provide access to the same object in a single cloud-based storage facility such as Google Drive and similar drives, clouds, etc. Such tools allow collaborative activities of teachers to take place (e.g. by sharing resources/materials and modifying them according to their needs) in different contexts. For instance, McKie (2020) explores the different ways by which in-service teachers participating in school-based professional learning communities in Canada can collaborate while sharing resources online through Google drives. Also, databases provide another type of tool belonging to category B1. For instance, Ferretti et al. (2020; see Sect. 5.4.2) build a model for designing activities for mathematics teachers' PD based on the use of the interactive national database Gestinv that involves structured information regarding standardised assessment and mathematics tests in Italy (1718 tests in total).

Another strand of tools falling into category B1 concerns social media such as Facebook and other online spaces, obviously not designed for educational purposes. The research interest in how such tools can foster teacher collaboration has been increasing. As teachers find self-directed, online learning opportunities more beneficial than required online experiences (Parsons et al., 2019) elevating teacher-initiated collaboration online is critical. Anderson's (2020) study of a public Facebook group (1738 members, USA) tailored to mathematics education indicates the potential of interactions among group members to promote professional collaboration. The Facebook group members were able to participate in PD through discussions of artifacts from members' practice which generated collaborative learning opportunities.

In category B2, we classify a range of digital tools and technological advances (e.g. video streaming, video-conference software, online forums) that promote the representation of teaching in new ways. These tools can be exploited in diverse educational activities for teachers and can be adapted for teacher collaboration. For instance, the Realization Tree Assessment (RTA) tool (Weingarden & Heyd-Metzuyanim, 2020; see Sect. 5.4.3) was originally designed to assess the extent to which students participate exploratively during the lesson (i.e. identifying different realisations of mathematical objects and authoring narratives about them). The

Table 5.1 Categorisation of tools and resources for fostering teacher collaboration

A. Designed for teacher collaboration	B. Adapted for teacher collaboration	
Web-based tools, (e.g. collaborative inquiry tool), Digital platforms (e.g. RAMZOR)	B1. Designed for collaboration but not necessarily for educational purposes Designed for distance working, storing/sharing and communicating e-learning platforms (e.g. Moodle) Shared drives (e.g. Google drive) Databases (e.g. Gestinv) Social media (e.g. Facebook)	B2. Designed for educational purposes but not necessarily for collaboration Designed for representing teaching in new ways Use of cartoons (e.g. lesson sketch) Web-based storyboarding tools Multiple representations of mathematical objects (e.g. RTA)

authors examine its potential for supporting prospective teachers' learning as participation in explorative pedagogical discourse.

Two other examples of tools and resources from category B2 concern specially designed pieces of software that allow representing classroom interactions through cartoons: Cusi et al. (2020) (see Sect. 5.3.3) engage teachers in scenario design through the use of Lesson Sketch, while Herbst and Milewski (2020; see Sect. 5.4.3) engage teachers in collectively creating a representation of a lesson through a web-based storyboarding tool (i.e. Story*Circles*). The emergence of digital environments supporting representation of teaching in new ways brings to the forefront the need to explore further how these new forms of representational and social/communication infrastructures (Hegedus & Moreno-Armella, 2009) might affect the design/study of classroom practice and teacher collaboration.

The types of tools presented in this sub-section (see Table 5.1) indicate that digital technologies have offered important tools and advances to promote teacher collaboration. Digital tools often allow a wide range of uses—not necessarily anticipated by designers—which provides teachers, teacher educators and researchers with an opportunity to adapt their use to serve the purpose of teacher collaboration in formal and non-formal settings. The potential of social media, online spaces and innovative representations of teaching for teacher collaboration appears to be an emerging field of research.

5.5.2 Designing for Supporting Teacher Collaboration

Even though the proliferation of tools and resources has broadened opportunities for teacher collaboration, there are still open issues about how this could happen. In this sub-section, we describe ways by which different tools and resources from the aforementioned categories (Sect. 5.5.1) are designed and used to support teacher collaboration in recent research studies. While in Sect. 5.5.1 we focused on the

nature of these tools and resources, here our focus is on their affordances that shape the design of collaboration, the kinds of activities in which teachers are expected to engage and are actually engaged, and the status/forms of mathematics in teachers' collaborative work. The titles below indicate the different categories/sub-categories presented in Sect. 5.5.1.

5.5.2.1 Tools and Resources Designed for Teacher Collaboration

In this category, we refer to the digital platform RAMZOR (Segal et al., 2020; see Sects. 5.4.1 and 5.5.1) that allows teachers to develop, share and jointly improve teaching materials (e.g. lesson plans). The PD project carried out with this tool in Israel involved engagement of groups of mathematics teachers—supported by mentors (24 teachers, 20 mentors)—in designing, and redesigning collaboratively learning plans and evaluation items in the platform, implementing the learning plans in their classes and participating in periodic meetings with the project staff. Using the platform, the teachers provided feedback to learning plans submitted by their peers and wrote new versions adapted to their own classes.

The analysis indicates that interaction with and employment of other teachers' lesson plans in RAMZOR promoted the development of teachers' mathematical and didactic knowledge, and enhanced their sense of belonging to a community of practice. Elements of the gained knowledge indicating the status of mathematics in teachers' collaborative work include new ways of proving a theorem, visual explanations, focus on mathematical details, a wide range of teaching approaches and different ways of solving mathematical problems.

5.5.2.2 Tools and Resources Designed for Distance Working, Storing/Sharing and Communicating

As regards existing research with tools in this category, we provide an example of a study (mentioned also in Sect. 5.5.1) involving the use of the e-learning platform Moodle. Albano et al. (2020) exploit two specific affordances from Moodle to support teachers' collaboration while carrying out the online activities of a PD course on research-informed mathematics instruction blending face-to-face lectures and an online part: *assignment* and *workshop*. Assignment allows a cyclic interaction between trainers and teachers in the form of 'feedback-responses' around an assigned task (e.g. design of hypothetical learning trajectories) and (re)submission of teachers' work. This affordance allows teachers to prepare their response collaboratively and submit it when they reach an agreement. Workshop allows teachers' engagement in reviewing other teachers' submissions on a task according to criteria given by the trainer. A distinct feature of the kind of collaboration in the two activities is that, in the first case, the teachers create together a product in response to a given task, while, in the latter, each teacher becomes a resource for each other by providing and receiving comments. As regards the status of mathematics in teachers'

work, the results indicate that anonymous redistribution of hypothetical learning trajectories and feedback through Workshop strongly influences re-designing and improvement of teachers' activities to enhance students' argumentative competence (i.e. exploring, conjecturing, justifying).

Along the same category, we also note the increasing research interest on how teachers use tools such as Google web-based applications, databases and social media which can be integrated into teacher education for communication and collaboration. McKie (2020; see Sect. 5.5.1) focuses on the collaboration of teachers in the context of their participation in professional learning communities in Canada, while sharing resources online through shared Google drives. She reports on the collaboration within a specific professional learning community focused on selecting pedagogical strategies that would best meet the needs of their students in relation to the Grade 9 mathematics curriculum. Resources promoting teacher collaboration include material resources (i.e. shared computer drives, social media platforms, textbooks, research articles) and human and socio-cultural ones (i.e. verbalisation, communication, time). Human and socio-cultural resources support and facilitate collaboration through sharing of beliefs, enhancing collegiality and evolution of the community of practice.

As regards the potential of social media for teacher professional development, Anderson (2020) investigates how contextually relevant teacher collaboration is mediated through a public Facebook group focused on mathematics education. The group involved 1738 members who interacted asynchronously. The affordances shaping the design of collaboration were Facebook posts indicating questions or requesting in-the-moment support, as well as artifacts (e.g. activities) from members' practice providing all groups members access to real classroom situations. The group's interaction led to four discourse structures: starting from commenters providing desired support (Desired), commenters offering different ideas than requested (Reframe), commenters challenging requested support or previous ideas (Challenge) and commenters working together to build a new understanding of desired support (Generate). The platform allowed for a lengthy collaboration time, permitting individuals to join the conversation at their own pace, to return multiple times and to provide more information by posting, commenting and reacting. The results highlight the potential of informal online spaces in providing diverse collaborative opportunities to teachers and participation in professional development.

The above three examples indicate that tools of this category offer affordances that facilitate sharing of materials and enriched forms of interaction between trainers and teachers and between teachers (e.g. cyclic interaction in the form of 'feedback-responses'). The resources that shape the design of collaboration include material resources (e.g. shared computer drives) and human and socio-cultural ones (e.g. verbalisation, communication at own pace, long collaboration time), while the available records of interaction (e.g. written communication through social media) allow addressing the evolution of collaborative talk and the quality of collaboration. The status of mathematics in teachers' work is related to the everyday teaching practice while feedback and redesigning indicate improvement of designed activities to support student learning.

5.5.2.3 Tools and Resources Designed for Representing Teaching in New Ways

In this category, we refer to three examples of studies in Italy, USA and Israel. These studies have also been discussed in previous sub-sections, with a focus on how the tools representing teaching may help in linking PD context and school context (see Sect. 5.3.3) and how they could support teachers' reflection on teaching (see Sect. 5.4.3).

In the first one, we refer to Cusi et al.'s (2020) study that engages teachers to design, reflect and redesign scenarios through ordered sets of scenes in Lesson Sketch. The tool allows teachers to focus on and discuss the various possibilities in which an interaction might evolve during the classroom activity. The resources shaping the teacher collaboration include scenarios represented as stories with cartoons through the use of the depicted tool and the character set of Lesson Sketch. The final product of the teachers' collaborative work is a net of comic strips. The results indicate that its development is facilitated by two affordances of the tool: (a) 'tree of possibilities' that allows representing in different ways the evolution of a classroom interaction as 'ramifications'; (b) 'thought balloons' that allow to make explicit the reasons behind teacher reactions/interventions. Mathematics in teachers' collaborative work appear interrelated to different aspects of the teaching-learning processes (e.g. teaching practice, teachers' justifications of didactical choices).

The second example concerns Herbst and Milewski's (2020) Story*Circles* (see Sect. 5.5.1), another approach based on the use of Lesson Sketch. Story*Circles* deploy upon two kinds of infrastructure of teacher collaboration: social infrastructure that supports conversations about teaching and representational infrastructure which is used in making teaching an object of negotiation in such conversations. The reported study from the USA involves teacher participants using some resources to script a lesson (e.g. records of students work), visualise classroom interactions by putting together various script moves and offer justifications for alternatives to what is represented. The status of mathematics in teachers' work is dynamic, since Story*Circles* enable viewing a lesson as a multiverse that could be composed of many related but divergent stories.

As regards the third example, Weingarden and Heyd-Metzuyanim (2020; see Sect. 5.5.1) explore the potential of the RTA tool to facilitate prospective teachers' collaborative discussions on explorative teaching by altering the use of its available affordances from assessing teaching to represent teaching. These affordances include mathematical objects (e.g. linear function), their various realisations in classroom teaching (e.g. visual, verbal, algebraic) and the links between them around a common mathematical idea. The tool affords teachers opportunities to focus on the mathematical objects and their emergence in teaching as well as to discuss opportunities for student meaning-making. The results indicate that through these challenges, the status of mathematics becomes more prominent in teacher discourse and it is explicitly linked to the teaching practices that afford students' explorative participation in the lesson.

The above examples show that the main affordances offered by tools for representing teaching are related to two main kinds of infrastructures for teacher collaboration: (1) representational infrastructures, i.e. materialising the diversity of classroom interactions and their underlying dynamics, as well as the emerging mathematical objects (e.g. trees of possibilities, reasons of teachers' decisions); (2) social infrastructures, i.e. enabling teachers' collaborative work/reflection on classroom teaching and students' mathematical understanding (e.g. paths of classroom interactions). Links between mathematics in teachers' collaborative work and classroom practice seem to be enhanced.

In summary, in this sub-section we address the issue of design for teacher collaboration as regards the three categories of tools and resources presented in Sect. 5.4.1. The provided examples of studies bring to the fore the following findings:

- tools of category A offer affordances facilitating joint preparation of materials and self-facilitated collaborative inquiry in PD settings;
- tools of category B1 are used for quite similar activities with the aim to facilitate rich interactions between teachers and trainers (e.g. around a task), as well as storing and sharing of resources in PD settings—with the exception of social media that support non-formal ways of collaboration outside PD settings;
- tools of category B2 offer affordances to visualise aspects of teaching in innovative ways while teachers can be engaged in activities such as scenario design and joint creation of representations of lessons.

5.5.3 Theoretical Tools and Professional Practices Towards Teacher Collaboration

A number of theoretical tools and professional practices have been used to support mathematics teacher collaboration and the communities in which they work. In this sub-section, we refer to such tools by distinguishing two broad categories: those that are shared with teachers and, in this way, become tools to support collaboration, and those that are used only by researchers to frame the design of PD settings or to interpret interactions within these settings. In terms of the categorisation introduced in Sect. 5.5.1, theoretical tools of the first category can be considered as tools 'designed' for teacher collaboration—as they are used operationally by researchers to facilitate teacher collaboration—while theoretical tools of the second category can be considered as tools 'adapted' by researchers to design and study teacher collaboration in PD communities in different (national, institutional, etc.) contexts.

The theoretical tools of the first category are used by researchers/teacher educators in relation to appropriate professional practices and methods (e.g. teacher noticing, Lesson Study) explicitly to orient teacher collaboration and facilitate teacher collaborative work. For instance, *teachers' professional noticing* (i.e. making sense of students' mathematical thinking during instruction and

deciding how to respond to that thinking) (Jacobs et al., 2010) is a practice that has recently attracted research interest in professional development contexts where groups of teachers work together. Pynes et al. (2020) use the Collaborative Inquiry Tool designed to support upper-elementary teachers in self-facilitated collaborative inquiry to explore teachers' collective noticing of children's mathematical thinking. In a PD context, three teachers participated in 12 collaborative inquiry sessions to examine and discuss student work for a common story problem they each posed to their own students with a focus on children's thinking of key mathematical relationships. The tool provides access to descriptions of the mathematical thinking of students that are not familiar to teachers, as well as to artifacts from the teachers' classrooms. Teachers could consider the different perspectives and may confirm or extend their own noticing. The results indicate the critical role of the tool in supporting teacher collective noticing by allowing multiple perspectives around the same piece of student thinking to be shared and discussed.

Another example concerns the *Lesson Study* (LS) (Huang et al., 2020) that has been a popular teacher-directed professional development approach in many countries to improve mathematics teaching and learning and strengthen connections between research and practice. Recently teacher collaboration has emerged as a promising research area in studies combining LS with different theoretical and methodological perspectives. Díez-Palomar et al.'s (2020) study taking place in a High Education Spanish institution explores how LS and the *Didactical Suitability Criteria* (DSK) (Font et al., 2010) can complement each other when pre-service teachers collaborate in designing interdisciplinary (mathematics and science) lessons for pre-K and K students. While LS is adopted as a context for engaging teachers in the cycle design-implementation–reflection as regards mathematics teaching, DSK provides a set of observable indicators for different types of criteria/suitability (i.e. epistemic, cognitive, interactional, mediational, emotional, ecological) that may help teachers to design and assess their teaching in terms of different sets of 'mathematics teachers' competencies'. The results indicate that the combination of LS and DSK enriches the available professional tools to support teachers' collaboration and further develop teachers' competencies, such as assessing epistemological aspects of mathematical concepts, addressing their teaching and learning and using appropriate resources.

A third example concerns the qualitative study of Bağdat and Yanik (2020) who investigate changes in question types of two novice mathematics teachers participating in a collaborative PD program in Turkey, focused on designing and implementing cognitively demanding tasks. The program focused on identifying collaboratively factors associated with the decline or maintenance of cognitive demand, modifying mathematical tasks to increase cognitive demand and using the theoretical tool of five practices (*anticipating, monitoring, selecting, sequencing, connecting*) to orchestrate whole-class discussions while maintaining the cognitive demand of the tasks at a high level (Smith & Stein, 2011).

The results indicate that, due to their collaborative PD experience, the teachers after the program maintained the cognitive demand of the task at high level and improved in their questioning and discussion techniques. Thus, the approach of five

practices supported the design and actualisation of PD, and allowed describing the evolution of teachers' practices. As regards the use of theoretical tools as resources for teachers to prepare and implement lessons in their classrooms, there are studies using *learning trajectory* (Simon, 2014) (e.g. Albano et al., 2020; Huang, 2020), *scenario design* (Cusi et al., 2020), etc.

As regards the theoretical tools of the second category, our focus here is on how researchers/teacher educators design their PD settings, so as to support teacher collaboration and make sense of the interactions taking place within these settings, teacher knowledge and learning. For instance, Huang (2020) combines LS to *boundary objects* (Akkerman & Bakker, 2011) to study how a mathematics teacher educator and a group of 12 primary and secondary teachers in a Chinese PD setting collaboratively worked to design a research-informed exemplary lesson. He provides an integrated framework to support teacher–researcher negotiation of meanings of effective teaching and learning of mathematics in PD initiatives, where members of the research and teaching communities come together.

Ferretti et al. (2020; see Sects. 5.4.2 and 5.5.1) develop a model to design activities for mathematics teachers' PD by networking Jaworski's (2006) notion of *community of inquiry* and the *Mathematics Teacher's Specialised Knowledge (MTSK)* model (Carrillo-Yañez et al., 2018), that is based on Shulman's (1986) notion of *Pedagogical Content Knowledge* and Ball et al.'s (2008) notion of *Mathematical Knowledge for Teaching*. The model is also based on the affordances of Gestinv database providing information about standardised assessment and mathematics tests in Italy. Teacher collaboration in the model involves interaction with Gestinv's resources and critical reflection on the complexity of standardised assessment in mathematics. Both community of inquiry and MSKT allow describing possible changes in teachers' beliefs about mathematics and mathematics teaching pursued through the inquiry attitude and addressing the formation of mathematics teacher identity.

Other researchers based their approaches to teacher collaboration in PD settings to broader theories of education and learning. As an example, we refer again to Herbst and Milewski's (2018) Story*Circles* in which groups of teachers work together to create a representation of a lesson using a web-based storyboarding tool and cartoon characters collectively. The goal of engaging teachers in making a collective product is inspired from Papert's (1991) *constructionism*, an educational theory of design and learning according to which learning happens best through designing external and shareable artifacts valuing engagement, exposure, bricolage, ownership and discourse.

Summarising, theoretical tools that are shared with teachers to support their collaboration (e.g. DSK, five practices) seem to be used to bring to the fore the complexity of mathematics teaching and the diversity of practices related to it, while theoretical tools that are used by researchers to design their PD activities concentrate on interactions within these settings and how these influence practice and promote teacher learning.

5.5.4 Transversal Issues and Perspectives Around Fostering Teacher Collaboration

A global look at the tools and resources for fostering teachers' collaboration shows a diversity of categorisations in relation to their nature, design purposes, theoretical perspectives and professional practices. As regards the nature of different tools and resources shaping the design of collaboration, our analysis shows that the current landscape in the field is oriented by the following categories of technological tools: (A) designed for teacher collaboration; (B) adapted to operate as formal or informal environments for teacher collaboration, including tools and resources for distance working, storing/sharing and social communication (category B1) and tools promoting new ways of representing teaching (category B2). These tools and resources are based on different kinds of technologies, provide diverse affordances and allow the design of a range of activities for teachers.

Category A includes specially designed digital platforms and online tools (e.g. Collaborative Inquiry Tool, RAMZOR) that allow teachers to develop, share, and jointly improve teaching materials in PD settings. Category B1 includes e-learning platforms (Moodle) and specific tools (Assignment, Workshop), cloud-based storage facilities (Google Drive), interactive databases (Gestinv) and social media (e.g. Facebook). The tools and resources of this type are adapted to be used for quite similar activities with tools of category A involving cyclic interaction (i.e. 'feedback-responses') between trainers and teachers as well as between teachers (i.e. reviewing other teachers' submissions) around an assigned task (Moodle) and sharing online resources (Google Drive) in PD settings.

An additional feature of social media is that they allow teachers to use them to establish groups and collaborate asynchronously outside formal settings (e.g. PD) by posting, commenting and reacting on members' artifacts and practices. Category B2 concerns those technological tools that share the affordance of representing teaching in innovative ways allowing activities, such as designing scenario collaboratively, representing classroom interactions through cartoon stories (e.g. Story*Circles*) and triggering teachers' attention to representations of mathematical objects emerging in a lesson (e.g. the RTA tool).

As regards the theoretical tools oriented towards teacher collaboration, the quoted studies reveal that under broader professional development approaches (e.g. teacher noticing, LS) teacher collaboration is targeted through: (a) theoretical constructs shared with teachers to address the complexity of teaching and the practices related to it (e.g. DSK, five practices); (b) theoretical constructs used by researchers to design PD settings and study the collective part of teachers' work (e.g. community of inquiry), as well as teacher knowledge and learning (e.g. MTSK, constructionism, boundary objects).

Taking a broader look at the research in the field, we can draw some main conclusions. Digital tools and resources seem to have a protagonist role in studies addressing teacher collaboration due to their wide/flexible range of uses such as supporting synchronous/asynchronous interactions around teaching resources,

acting as platform and repository for supporting joint work of teachers, and allowing representations and analysis of the finer nuances of teaching practice through digital representations. Online spaces, such as Facebook, Twitter, WhatsApp and Global Math Department Virtual Meetings, constitute an emerging category of tools and resources mediating contextually relevant teacher collaboration outside formal PD settings.

These places have provided teachers with opportunities not only to exchange resources, but also to build learning communities (Larsen & Parrish, 2019) and address individual problems of practice (Risser et al., 2019). Finally, tools and resources providing innovative representations of teaching allow a new look at the social and representational infrastructure of teacher collaboration and which elements of them support teachers in building a broader professional knowledge for teaching (Milewski et al., 2018).

5.6 Tools and Resources for Studying Teachers' Collaboration

The previous sections have addressed: tools and resources collaboratively designed for teaching (Sect. 5.3); tools and resources for collaboratively inquiring about teaching and fostering teacher learning (Sect. 5.4); tools and resources for fostering collaboration (Sect. 5.5). As we consider the tools and resources developed to foster teacher collaboration or evolved from collaboration, we should also consider how these tools and resources can be used to examine the form and the purpose of teacher collaboration. For this section, we consider the methods and theories that are used to examine in what ways, and for whom, the tools and resources developed for and within teacher collaboration are effective, and determine the tools and resources to be developed that will support teachers and teacher educators. This section addresses the tools and resources that are currently available to examine: the impact that teacher collaboration may have on the actors themselves (Sect. 5.6.1); the theoretical and methodological tools that researchers use to examine structure of or interactions within the collaborations (Sect. 5.6.2); the suggestion of potential development for infrastructures to study teacher collaboration (Sect. 5.6.3).

5.6.1 Reflecting on the Impact of Collaboration Tools on the Actors

Researchers and facilitators may examine the tools and resources used in teacher collaboration as a source of data to analyse the impact of the collaboration either directly or indirectly. In using resources directly, researchers may analyse observation notes or recordings of the collaboration, documents for and created from

collaboration, observation notes or recordings from the classroom and documents from the classroom (e.g. student work and teacher recordings). Researchers may also use these resources indirectly, examining teachers' reflections on these documents and how their teaching practices have evolved as a result of collaborating with peers. In either level of use, tools and resources can support both teachers and researchers in considering how ideas from teacher collaboration are connected to classroom practice, and viewed as generative and productive for teachers' professionalism.

Albano et al. (2020; see Sect. 5.4.1) analysed a direct resource that was created as part of a professional development activity for secondary teachers. This resource was an instructional plan that the teachers created and revised over a period of time and with the support of professional development. Albano et al. were interested in tracing the impact of the collaboration from both the mathematics teacher experts and the peer teachers in the revisions of the instructional plan. The researchers analysed each iteration of the instructional plan submitted, in order to identify the revisions that the teacher made as a result of the professional development and, in particular, the interactions that may have supported this revision. The final analysis identified a scale of three levels (p. 577):

- Level 0: the teacher made no changes in their instructional plan, or any suggested changes were not properly integrated;
- Level 1: the teacher modified the instructional plan or integrated design details, but there was no evidence these changes were made as a result of the professional development;
- Level 2: the teacher's modifications demonstrated evidence of interactions with content experts and peer teachers.

The researchers found most teachers improved their instructional plans as a result both of the targeted professional development and of peer feedback. Moreover, evidence suggested that, for almost half of the teachers, the feedback from their peers had a greater influence on their task design than the professional development alone. This evidence was supported by analysing several revisions of one instructional plan, therefore future researchers may consider collecting more than two iterations of a document to identify how the collaboration may or may not have supported an individual or set of teachers.

Direct resources may be created for the purpose of supporting collaboration, co-constructed during collaboration or collected from individual teachers before or after collaboration. Through the examination of direct resources, researchers use their own perspectives and theoretical frames to examine the impact of teacher collaboration. However, what could be missing in this analysis is the teachers' voice, or how the teacher identifies the impact of collaboration.

We now turn to indirect resources that researchers use to consider how teachers communicate the impact collaboration has on their practice or beliefs about teaching. Indirect resources include teachers' reflections on the collaboration and can be used to identify the tools and resources that teachers believe are supportive when collaborating with peers. In Sect. 5.4.4, we highlighted how reflexive mediation may impact teachers' learning in collaboration. The following studies demonstrate the

indirect resources researchers use to elicit teachers' perceptions of how the tools and resources teachers use in collaboration contribute to their learning about teaching. Some researchers, such as Hollingsworth and Clarke (2017) utilise semi-structured teacher interviews, while others, such as Albano et al. (2020) and Segal et al. (2020), may elicit teachers' reflections on how specific collaborations influenced either their perspectives or beliefs about teaching with follow-up questionnaires.

One form of a semi-structured interview is a stimulated teacher reflection. Hollingsworth and Clarke (2017) created a tool supporting teachers in examining and obtaining feedback about their own teaching practice. The primary purpose of the study was to examine opportunities for teacher learning within a structured stimulation prompting teacher reflection. When developing the observation protocol, the researchers intended the feedback from colleagues to be a conversation about teaching, rather than opportunities to make a critique.

To elicit teachers' perceptions of how the tool supported efforts in improving mathematics instruction, the researchers invited two Australian teachers to participate in the study. These teachers were asked to select the dimensions they were most interested in developing, to video-record one mathematics lesson and to analyse this lesson prior to a video-stimulated feedback conversation with the researchers. After the conversation, teachers reported that the protocol encouraged focused feedback, rather than the more generic feedback they may typically receive outside of structured conversations. The teachers also suggested that the opportunity to observe specific dimensions in their own practice through video was more generative for promoting reflection and informing areas of improvement. Although this particular collaboration was between teacher and researcher, this tool can be used to inform future teacher collaboration protocols that promote teacher agency and self-reflection.

In addition to eliciting teachers' perceptions of collaboration through semi-structured focus groups or interviews, many researchers also elicit teachers' views through written questionnaires or surveys. In this sub-section, we described the analysis tool Albano et al. (2020) created to identify how the comments of both the teacher educators and teacher peers influenced revisions in the teachers' instructional plans. Albano et al. also posed a questionnaire to the teacher participants at the end of the study. The focus question for the questionnaire translated to "What advantages for your teaching profession can you identify in the peer review activity?" (p. 579).

In another study, Segal et al. (2020) discussed how they encouraged collaboration in a digital environment, RAMZOR (see Sects. 5.4.1 and 5.4.4), to support teachers in planning and implementing complex tasks. The researchers developed a questionnaire to determine, via a Likert-type scale, the degree to which the participants believed collaborating in this space with other teachers contributed to both their mathematical and didactic knowledge for teaching and contributed to a sense of belonging to the community (see Sect. 5.2.4). In addition to the scale, the questionnaire included an open-response prompt for teachers to elaborate on the level of agreement selected. Both studies reported teachers appreciated the opportunity to receive constructive feedback from peers. Future researchers may create items to

identify teacher perceptions on the use of particular tools and resources used in collaboration.

Indirect resources can inform both researchers and facilitators with evidence for how teachers perceive a collaborative community to support the improvement of teaching practice in mathematics. Eliciting regular feedback from teachers can provide opportunities to revise tools or support structures for teacher collaborations, or create teacher agency to adapt tools and resources to fit better their needs and purposes for collaboration that is generative for their practice.

Both direct and indirect resources can provide researchers and facilitators with a sense of how teachers take up issues from collaboration. The evidence to examine this take up may come from observations of how protocols are implemented or are modified over time; the discourse patterns among the teachers with or without the presence of a facilitator; and observations of classroom practice. Moreover, when evaluating the impact collaborations have on teachers, it is important to include the perceptions of the actors involved in the collaboration.

As we consider the type of resources available to identify or evaluate the impact of the collaboration on the actors themselves, we also find the need to examine the resources available to study the structures in which collaboration takes place. In the next sub-section, we consider methodological and theoretical tools available to researchers to examine the interactions within teacher collaborations at a variety of levels (e.g. school-based, region-based, web-based) and provide examples of how current tools and resources could be used for this purpose.

5.6.2 Studying the Interactions in and Frameworks of Teacher Collaboration

In Sect. 5.2.4, we introduced our shared glossary on tools and resources, and referenced the theoretical frameworks that inform our work as researchers in mathematics education. In this sub-section, we leverage the theoretical frameworks and methodological tools that have been introduced and expanded on: (1) to demonstrate the tools and resources researchers have recently used to study the learning opportunities teachers have in a variety of contexts and structures for teacher collaboration; (2) to introduce new frameworks for studying these opportunities; (3) to consider the learning opportunities that representations of practice afford teacher communities.

5.6.2.1 Learning Opportunities for Teachers in Collaboration

Regarding the various structures of teachers' collaborative work and the learning opportunities that exist within these structures, we examine the work from Chang et al. (2020) and Anderson (2020). From Chang et al., we consider the opportunities

one teacher had to revise an instructional task as she moved through three different work groups, or different communities, and from Anderson we consider the learning opportunities for teachers as they discussed problems of practice on a social media platform.

Chang et al. (2020; see Sect. 5.4.1) reported on a lesson-design model that provided teachers the opportunity to create and revise a mathematical task based on feedback. The researchers presented these opportunities through a case study of one teacher, and shared interactions within three distinct communities: the whole group; a small group; a group of mathematics teacher educators. Data sources to capture how the groups supported teachers in learning included revisions to the instructional task, video records of the interactions, and video and written records of the teacher reflecting on the revision process.

Researchers employed a meta-didactical transposition model (Arzarello et al., 2014) to consider how brokers (Wenger, 1998), or the different group settings, supported the mathematics teacher to learn through task revision. Initially, the teacher was resistant to criticise in the whole-group setting. However, as the teacher continued to discuss and receive criticism within the small group and professional group setting, the teacher was more receptive to suggestions in subsequent whole-group discussions. The small group also anticipated how the students might take-up the model, which led to more pedagogical problems that were then discussed with the professional group. These discussions provided the teacher with key questions to consider when revising the task and for what purpose.

Another structure to consider when examining collaboration is the form social media plays in creating spontaneous communities that allow teachers an opportunity to crowd-source for specific ideas. As teachers post problems of interest, they can receive ideas from members with varying experiences and locations. Anderson (2020; see Sect. 5.5.1) examined the discourse structures within one Facebook community to understand better how these collaborative environments could be a generative space for teachers' professional learning and noted four structures of interaction.

Within these discourse structures, made available through social media, teachers not only have access to other's ideas, but also an opportunity to collaborate through exchanges that build on one another's ideas. Therefore, Anderson highlights the ways in which social media platforms can be a generative space for teachers to grow in their professional learning in a more immediate way that not all collaborative communities can provide, and these spaces can provide researchers with a means to identify the current needs communicated by teachers. Next, we review the theoretical tools researchers utilise to identify learning opportunities within teacher collaboration.

5.6.2.2 Emerging Frameworks to Theorise Learning Opportunities for Teachers in Collaboration

A variety of theoretical tools are used for studying teacher collaboration. In Sect. 5.2.4, we introduced a subset of theoretical frames that underlie our understanding of

learning opportunities for teachers in collaboration. For the purposes of this chapter, we highlight new research that utilised boundary crossing and boundary objects (Robutti et al., 2020) to theorise how learning is transferred across settings, referencing studies described in previous sections, and also introduce new taxonomies researchers recently created to characterise the learning opportunities teachers have in a variety of collaborative settings.

To theorise how different communities interact to exchange knowledge, some researchers employ boundary crossing (Akkerman & Bakker, 2011; see Sect. 5.2.4). Within these interactions, or boundary encounters, teachers have an opportunity to identify and negotiate new understandings. For example, Psycharis et al. (2020; see Sect. 5.3.1) and Huang (2020; see Sect. 5.5.3) introduced a collaboration that included both primary and secondary teachers to discuss and develop resources that would support students in the development of algebraic thinking. Both frames provided an opportunity for teachers from different schools to collaborate around boundary objects (e.g. lesson plans, student tasks and materials).

To inform their analysis, the researchers employed boundary crossing to consider how the primary and secondary teachers collaborated with one another, given their different instructional contexts and pedagogical and mathematical knowledge. For Psycharis et al., the primary teacher was able to discuss how to promote algebraic thinking and to consider how to support the transition from early algebra to algebra with the secondary teacher. Through analysis, they identified boundaries for teachers including mathematical knowledge (i.e. how the teachers viewed algebraic concepts for their grade level) and pedagogical approaches (e.g. contextualised problems, open-ended tasks, development of generalisations) that were discussed and negotiated as the teachers began to share a view of how to characterise and foster algebraic thinking across the grade levels.

Another theoretical frame comes from the work of Horn et al. (2017) developing a taxonomy to characterise teachers' learning opportunities. When analysing the workgroups, the researchers considered three main questions: (1) the purpose and result of the meeting; (2) the focus of the facilitator; (3) when teachers engaged in dialogue, what was their focus? Through analysis, Horn et al. created six categories of workgroups. The researchers characterised four of the categories as low-depth meetings, suggesting teachers' opportunities for learning within these workgroups were limited (e.g. focusing on pacing and logistics). Horn et al. note these types of workgroups often resulted in one teacher sharing, limiting the opportunities for discussion and/or collaboration. The researchers found that teachers had richer conversations, and thus a greater opportunity to learn, when the workgroup centred on a collective interpretation as they investigated problems of practice.

Similar to Horn et al., Brodie and Chimhande (2020) recently introduced a framework for considering the quality of the content and depth discussed within teacher collaborations. Using this framework, the researchers analysed six activities that the collaboration is centred upon (e.g. analysing assessments, discussions around readings, lesson planning), the content of the talk (e.g. focus on the learner,

the mathematics, instructional practice, identified priorities) and the depth in which the teachers engaged with the content. For the analysis of depth, they characterised four levels ranging from no or little engagement with the content (Level 1), to generalising the content or coming to new understandings (Level 3 plus). Similar to Horn et al., Brodie and Chimhande determined that different collaboration activities provided different learning opportunities for the teachers to engage in the content, and the depth of the teachers' conversations did not necessarily shift over time. This analysis suggests researchers still need to develop resources to support the quality of teachers' engagement with the content.

Researchers continue to advance the frameworks we use to examine and characterise the learning opportunities available to teachers to improve their practice, and more research is being done to consider how collaborations are a generative space for teacher improvement. In addition to the theoretical frameworks that conceptualise the mechanisms that facilitate teacher improvement within collaboration, researchers also consider the development and use of representations in collaborative spaces.

5.6.2.3 Theorising How Representations of Practice Provide Learning Opportunities for Teachers in Collaboration

In the previous sections, we discussed the use of representations and the learning opportunities afforded to teachers in two distinct manners. In Sects. 5.3.1 and 5.3.2, representations were developed and/or discussed by the teachers to demonstrate conceptual ideas (such as integer operations or functions), and the section argued how selecting and discussing these representations in collaboration provided generative opportunities for teachers' mathematical knowledge for teaching (Ball et al., 2008). The previous sections also discussed how tools and resources are used to represent teaching practices from two distinct perspectives: the use of representations to support the teaching of mathematics (see Sect. 5.3.3) and encourage reflection (see Sect. 5.4.3); how representations are considered in the design of tools and resources to support mathematics teachers in collaboration (see Sect. 5.5.2). In this sub-section, we consider the theoretical tools or frameworks researchers consider when studying how the learning opportunities' representations of practice afford teacher communities.

To examine records produced through collaboration, Trouche et al. (2019) analysed the collaborative work of two French middle-school teachers planning instruction for a new topic using three theoretical perspectives: Documentational Approach to Didactics (DAD), Anthropological Theory of the Didactic (ATD) and Cultural–Historical Activity Theory (CHAT). Introduced in Sect. 5.2.4, DAD informs the analysis of both how the teachers use resources to create their lesson planning document and how the teachers create this document as a collective. CHAT informs the analysis through the frame of organisational learning, broader than the

mathematics education. Trouche et al. argue that, through CHAT, researchers can interpret representation of practice from "rules, artifacts, and division of labour, as well as from community feedback" (p. 55) to theorise how the organisational structure contributed to learning opportunities.

Lastly, the researchers state that, although ATD is a known theoretical frame for mathematics education research, it was not developed to analyse the collective work of teachers. However, they argue that ATD allows researchers to analyse representation of practice through the structure of knowledge and practices, the dynamic work of designing and implementing tasks, and identifying the conditions that afford and/or constrain this work through an ecological perspective.

Many theoretical frames underlie the learning opportunities representations of practice that both afford and constrain teachers' collective work. Teacher communities are often organised around representation of practice, either in the form of considering the classroom practices that already happened (e.g. Pynes et al., 2020; Uzuriaga et al., 2020), but they can also provide teachers with an opportunity to consider decisions from a multitude of perspectives through the work of anticipating student thinking and scripting lessons (e.g. Cusi et al., 2020; Díez-Palomar et al., 2020; Weingarden & Heyd-Metzuyanim, 2020). The theoretical frames that inform our understanding of how teacher communities are generative are not only essential for determining which representations to organise teachers' work, but also in identifying the impact of resources in teacher and student learning.

The theoretical tools and representations of practice described are examples of how researchers are examining the structures of and interactions within teacher collaborations. These tools help researchers to identify the learning opportunities teachers have within these groups and resources to be developed to support teachers in future collaborations. In addition to analysis tools that reflect on the impact of teacher collaborations on mathematics teaching, the Theme D participants also argued the importance of creating analysis tools that consider the affective development of teacher collaborations, that is, how do teachers learn to collaborate and, in particular, what are the differences when this work is either voluntary or obligatory.

In the Theme D Plenary, Brodie (2020) argued that, "safety and trust are important to be able to learn with others" (p. 40), and therefore collaborative tools should also provide space for teachers to build a community of trust and, as researchers, we should also analyse the development of this trust to support teachers in collaboration. Developing these relationships within collaboratives is important, so that teachers are comfortable sharing perspectives that may not have been introduced to the group and assuming positive intent as differences are discussed. We also wonder how researchers could make more apparent their own role as they study teacher communities. As we could consider the relationship between the researcher and the teacher to be a form of collaboration, we assume the researcher takes on the role of participant in some form.

The next sub-section considers resources that still need to be developed to support researchers developing and studying larger infrastructures that organise teacher collaboration at a variety of levels, including international communities.

5.6.3 *Developing and Studying Mathematics Teachers' Collaboration Across Instructional Settings*

In this sub-section, we question the tools needed for researchers to study teacher collaborations across instructional settings, including collaboration between: grade levels; locations (e.g. urban, suburban, rural, country), funding sources and space (e.g. in-person, virtual, hybrid). We address: (1) the design considerations for platforms that support collaborative settings in the context of mathematics education; (2) the need for infrastructures to analyse collaborative settings; (3) the importance of supporting collaboration at the international level.

The design of technology for researchers' study of teacher collaboration needs to attend simultaneously to (1) enabling teachers to collaborate on the practices of mathematics instruction and (2) enabling researchers to set up, observe and facilitate such collaboration. The first activity calls for technologies that allow teachers to communicate with colleagues in a context where language is not sufficient. The practice of teaching mathematics in classrooms, just like any teaching, lacks a common technical language for practitioners to communicate. While some have proposed designing such language (Grossman, 2020), others have noted the reductive nature of such project (Horn & Kane, 2019) and yet further others (e.g. Herbst & Kosko, 2014) insisted that practice relies on collective tacit knowledge (e.g. knowledge of instructional norms) that cannot be represented in language. The use of videos, animations, storyboards and classroom artifacts has been useful for teachers to demonstrate what they know.

Particular processes of teacher collaboration through technology, like Story*Circles* (Herbst & Milewski, 2018; see Sect. 5.4.3), have created contexts for teachers to collaborate about practice that accommodate teachers' tacit knowledge in the context of scripting lessons (see also Zazkis & Herbst, 2018). For these technologically mediated collaborations to support the transaction of tacit knowledge of practice, the technology needs to be capable of handling multimodal representations of practice. The design of this technology thus requires not only technological specification (e.g. the capacity to handle rich media) but also semiotic considerations (e.g. the systemic capacity to enable the reading and writing practice-related meanings through the manipulation of multimodal expressive tokens).

For the scripting of lessons in Story*Circles*, the existence of a set of graphic characters and a storyboarding software has been essential. These resources permit the storyboarding multimodality to achieve the same flexibility as writing in language and similar capacity as video for the expression of tacit meanings. The design of this representational infrastructure, in ways that it permits it to be an open system for meaning-making, is an important task for researchers to dedicate time to.

The work of teacher collaboration also requires a social infrastructure for teachers to discuss or exchange representations of practice. Technologies that support the capacity to edit storyboards collaboratively or share them in forums, or that support the collective annotation of media (e.g. Anotemos; www.anotemos.org) are

therefore important as well. In particular, considering that instructional practice relies on tacit knowledge, it is important to conceive of this social infrastructure as enabling transactions that are multimodal in nature. Thus, the notion of annotation behind Anotemos is a multimodal one, one can annotate a piece of media by interacting with it graphically (selecting regions or making marks on the screen, attaching images), aurally (attaching an audio file), or in writing (by adding comments to moments in the timeline).

Whereas software that allows for some of those functionalities can be obtained off-the-shelf, their integration in the service of long-term research agendas is an important consideration that researchers need to make. The Lesson Sketch platform (which was operational 2011–2020; see Herbst et al., 2013, 2016) contained annotation and storyboarding tools, and included them in the context of a larger infrastructure. In the platform, researchers could organise experiences for practitioners to interact with practice, and could collect the data that practitioners would generate. The Lesson Sketch platform allowed researchers to create such experiences, assign them to prospective participants and obtain reports. The software would record log data of practitioners' perusal of videos and storyboards, as well as report comments made in annotations. To that end, it is key to enable not only the practitioner collaboration, but also the collection of data that can support research on teacher collaboration about practice.

It leads naturally to the issue of developing specific infrastructures for analysing teachers' collective work. Globally, we must admit that research, regarding teachers' collective work, typically puts more energy for collecting data than for analysing it. And yet, this analysis is complex, due to the amount of data at stake when we consider the nature of teachers' collaborative work, such as: the resources each participant brings to the collective setting; the resources produced by the collective; the variety of contexts collaborations occur within and across; boundary objects (Akkerman & Baker 2011). As part of a national research project (ReVEA, https://www.anr-revea.fr/), the AnA.doc platform (Alturkmani et al., 2019; Trouche, 2019), a prototype developed in France, demonstrates the interest, and the difficulties, of such an infrastructure of analysis. AnA.doc is a platform structured on three levels: data collection; data analysis; a shared glossary of concepts.

At the level of data collection, the platform allows the storage of data related to a variety of *situations* of teachers' individual as well as collective documentation work (e.g. preparing a progression, or a lesson, reflecting on his/her practices). Each situation is described following the same model (i.e. history of the actors; context of the school; context of the curriculum; intentions of the researchers guiding the data collection strategy). Each data related to this situation (e.g. resources used versus produced; videos of teachers' work; self-representations of teachers' resource system; questionnaires) is associated with meta-data facilitating their use.

At the level of data analysis, documents created by the teachers on the Ana.doc platform are utilised, composed of a situation or a set of situations (e.g. two teachers co-constructing a lesson and then implementing it in their own class). As teachers

can upload a variety of media, the Ana.doc platform provides an opportunity to analyse a portion of this data (e.g. extracting an excerpt of video). Through the platform, members can conduct initial analyses with a small data set (e.g. the role of textbooks in lesson planning) and communicate the findings on the platform, which could be considered as a draft of a final product. Members of the community are then encouraged to comment on this analysis that can support future revisions, or propose an alternative analysis leading to the generation of a new document.

At the level of glossary, each member of the community had an opportunity to define the concepts that were used in the analysis. Community members may reference the same term or concept, but perhaps have different understandings or perspectives. These instances encouraged members of the community to discuss and come to a shared understanding to define and articulate clearly the different possible meanings of this concept, with respect to different theoretical frameworks.

Ana.doc, as a component of the project ReVEA (2014–2018), provides researchers with a prototype that could feed further projects at an international level. Of course, such a project is quite ambitious and needs important human and technological means to develop. We could imagine, as it was suggested during the Theme D sessions, to use platforms such as RAMZOR (see Sect. 5.4.1) as a single repository of lesson plans around the world. These platforms could include data about individual users for covariate analysis (e.g. location; title or position; years of experience; teaching interests). Repositories could also be dynamic, providing an environment that allows for a variety of actions (e.g. commenting on uploaded documents; creating and attaching supplemental documents; suggesting revisions or modifications) and meta-data could be collected by the platform.

As these platforms collect data, researchers could collaborate to analyse samples from these databases and generate claims around products of teacher collaboration, for example comparing lessons across a set of countries. As we expand our communities and boundaries, we must reconsider our ethical obligations to the communities we work with and learn from to ensure respectful collaborations that meet the needs of each community. Theme D participants also discussed the importance of considering the accessibility, adoptability, adaptability, and sustainability of tools developed to support teacher collaboration within varying contexts and levels.

As we reflect on past research and consider the future of teacher collaboration, we argue the need to create more formalised infrastructures that could allow researchers to draw from the same set of data and provide opportunities for analysis in both novel and collaborative manners. These infrastructures could also open the boundaries and provide accessible opportunities for teachers to collaborate with teachers who work in other countries. In this sense, these structures provide for collaboration at the international level for both teachers and researchers. This sub-section considered potential next steps for creating infrastructures that support mathematics teachers in collaboration and argued for the development of tools that support collaborations among international communities.

5.6.4 Transversal Issues and Perspectives Around Robust Analysis

As teacher collaboration, and the tools and resources that support teacher collaboration, continues to evolve, the theoretical frameworks and analysis tools researchers use to characterise these collaborations continue to develop as well. In our current time, geography does not necessarily restrain collaboration and the COVID-19 pandemic demonstrated the potential of web resources to facilitate collaboration. This section of the chapter surveyed the theoretical frames and tools researchers are currently using to study the impact of teacher collaboration (see Sects. 5.6.1 and 5.6.2), and provided suggestions for future development (see Sect. 5.6.3).

While many researchers are studying teachers in collaboration, we thought it was important to highlight the methods and analytical tools researchers have used to study these collaborations and suggest papers that focus on particular methods could be fruitful for the field. The studies presented in the previous three sections highlight the analysis techniques or theoretical frameworks researchers are currently using to study the interactions and opportunities afforded to teachers in these settings. From these studies, we gain new perspectives as we consider collaborations as a part of professional development, a bridge between professional development and the teacher' practice, or the collaborations teachers create for themselves, and identify both direct and indirect resources that are used to analyse the impact of collaborations. Many researchers have provided evidence to demonstrate how teachers take up ideas from professional development into their collaborations or practice, but we also encourage future analysis to consider the impact of collaboration on the learning opportunities for students.

As teacher collaborations require teachers to devote a portion of their time, both teachers and those who support teacher collaboration aim to ensure this investment is productive and generative for teachers. Therefore, it is imperative that as researchers analyse the impact of teacher collaboration, teachers' voices and perspectives are included in this analysis. Especially as we consider the many spaces in which teachers self-organise and collaborate that may not be visible to researchers. This leads to the potential in developing tools that provide the following: a repository for collecting resources developed by teachers both *for* and *from* collaboration to be shared with teachers and researchers and provide opportunity for collaboration; the proposal of analysis tools that can be applied across collaboration contexts; tools for ensuring teacher perspectives are included and valued in the final analysis.

We propose repositories should be dynamic and provide an opportunity for researchers to make connections across the media users (e.g. teachers) upload and conduct both qualitative and/or quantitative analysis. To support this endeavour, researchers should make more explicit the types of data that should be collected, in order to perform robust analysis, and how to identify and make best use of meta-data that digital platforms can generate. In collecting this data from teachers, researchers can monitor the needs of mathematics teachers, the types of resources that they

request and share, and the different solutions that are generated in collaboration in relation to the specific areas of mathematics.

5.7 Weaving Threads, Perspectives for Research and Development

This final section weaves together the themes discussed throughout the chapter to highlight the main research questions that Theme D aimed to address (Sect. 5.7.1). The section also discusses to what extent the COVID-19 pandemic affected the accessibility of resources and collaboration (Sect. 5.7.2). For this reason, we address the issue of equity as a transversal issue (Sect. 5.7.3). And, finally, we end this chapter by suggesting necessary perspectives of research to be developed (Sect. 5.7.4).

5.7.1 Weaving Threads, Enlightening Initial Questions of Research

In this sub-section, we summarise the findings provided by Sect. 5.5.3 through 5.6, identify themes that cross these sections and discuss main issues that remain to be addressed.

From the previous sections, we retain some main results, in terms of power and necessity of teacher collaboration.

- From Sect. 5.5.3, we retain the power of collaboration for supporting teachers in developing resources for addressing complex issues: implementing a new curriculum, new topics to teach and new practices to develop. Although the resources *for* and *from* collaboration are presented as distinct categories, both influence each other as anticipating practice, sharing experiences of practice and reflecting on practice co-occur within interactions in each collaborative setting.
- From Sect. 5.5.4, we retain the power of (certain) tools and resources for fostering teachers' collaborative inquiry: dynamic settings rather than static ones; open rather than closed format; oriented towards redesigning rather than transmitting; giving room for reflective analysis. This section demonstrates the fundamental role played by *reflexive mediation* when teachers collaboratively interact with different tools and resources to inquire into teaching.
- From Sect. 5.5, we retain the power of (certain) tools and resources for fostering teacher collaboration. Tools and resources are sorted into two main categories: (A) tools and resources *designed for* teacher collaboration; (B) tools and resources *adapted by* teachers or teacher educators to operate as environments for teacher collaboration. The latter category includes resources designed for distance working and social communication, and resources promoting new

ways of representing mathematics teaching. The critical feature of each of these resources is not in their original purpose, but rather their affordance in allowing teachers to develop teaching materials through feedback, discussion and reflection.

- From Sect. 5.6, we retain the necessity of (certain) tools and resources for studying teachers' collaboration, its forms and effects. Among them: repositories for collecting resources developed by teachers both *for* and *from* collaboration to be shared by teachers and researchers; analysis tools that can be applied across collaborative contexts; tools for ensuring teacher perspectives are included and valued in the final analysis. Some of these tools emerge throughout the developmental projects, but we must still develop more formal research infrastructures that could be shared at an international level.

When we examine the themes across the sections, the following results seems to be critical:

- the necessity of tools and resources explicitly designed to support teachers in collaboration and achieve the aims of collaboration in the specified educational context;
- the power of instrumentalization processes to support teachers in adopting and adapting tools and resources designed: for collaborating: for collaboration outside of educational settings; or not initially designed for collaboration;
- the double aspect of resources as *supports* for achieving a given goal, and as *objects* needing an effort to be appropriated. Adopting a resource leads always to adapting it, and that is particularly the case in the context of teacher collaboration, consisting in several stages: discussing classroom issues; designing for addressing these issues; adapting for his/her own classroom, sharing experiences; revising after a process of negotiation. Using and designing are then to be considered as two intertwined processes;
- the dialectic relationship between the nature of resources and the nature of collaboration: resources shape the collaboration and resources are shaped by the collaboration. The living character of digital resources leads to living interactions between teacher educators and teachers, as well as among teachers (e.g. reviewing each other's work). Reciprocally the quality of collaboration conditions the quality of the resources that are developed;
- The sensitivity of teacher professional development to the resources and the interactions developed within the collaborative settings. We imagine the consideration of resources as a collaborative triangle: developing collaboration, developing resources, and developing teachers' knowledge.

Across the sections the main issues still to be deepened appear to be:

- the issue of *quality* and *coherence* of resources and tools collaboratively designed (Pepin et al., 2015). In some cases (see Psycharis et al., 2020), researchers take care of these essential features. In other cases, the design process itself guarantees quality and coherence due to the continuous improvement of resources used by a large number of teachers (the case of Sesamath, see Pepin et al., 2015);

- the issue of *sustainability* of tools. For example, projects may sponsor a web-based tool and lose future funding for hosting, or the tools themselves may become outdated (i.e. either the content or the program code is no longer supported);
- the issue of *scaling-up*. Resources are often designed for and from teachers involved in small collaborative settings. Under which conditions these resources could benefit teachers beyond these settings? These conditions may be related to the forms of collaboration, the nature of resources, the agents involved in the collaboration or institutions;
- lastly, the issue of *digitalisation* of teaching and learning environments. Under which conditions digitalisation could benefit teacher collaboration and support the improvement of collaboration structures.

5.7.2 Rethinking Resources for/from Collaboration Over the Epidemic Period

As we prepared for the ICMI conference in Lisbon, and as we began to write this chapter after the conference, COVID-19 was recognised as a pandemic. This led to the closing of schools and sheltering-in-place for many teachers and students across the world. Bakker and Wagner (2020) and Engelbrecht et al. (2020) provide evidence of the challenges emerging in such a situation. Under these circumstances, we wondered to what extent could teachers' collaboration constitute a necessary counterpoint against the isolation many experienced. We also wondered which resources and tools teachers used or developed for, as a result of these collaborations. For these reasons, in May 2020, we asked Theme D participants to share their own experiences of teaching mathematics in a time of pandemic.

We received nine responses from: Algeria (Sayah), China (Huang), Colombia (Castro), India (Kumar), Israel (Segal and Movshovitz-Hadar), Italy (Faggiano and Robutti), South Africa (Brodie) and an international team (Aldon et al., 2021). These contributions underlined the enormous amount of work that teachers had to accomplish, in a very short time, when asked to move traditional face-to-face classrooms to a virtual environment. Castro identified the following major issues: curricular changes and adjustments; contextualised activities with less or greater complexity; adaptation of evaluation schemes—formative versus summative; the technological infrastructure of teachers; the technological infrastructure of students; platforms, applications, mobile devices, free software; 'orchestration' between training, evaluation and technology programs; changes in schedules and forms of interaction—synchronous and asynchronous—changes in knowledge' beliefs of teachers and students; parental involvement; institutional support.

What emerges from these contributions is the critical aspect of resources both for and from teachers' collaboration for facing these issues during the pandemic. These issues include: adapting existing resources and/or their uses; designing new

resources; and identifying missing resources. These categories are not mutually exclusive and, in our email exchange, our colleagues highlighted a variety of techniques developed by teachers and the discourses that supported their choices.

5.7.2.1 Adapting Existing Resources and/or Their Usages

From Israel, the Mathematics News Snapshots Research & Development project (MNS) was described by Segal et al.:

> According to the project policy, the MNSs have been made available only to teachers who participated in a professional development program [. . .]. As soon as the emergency remote teaching/learning began, the project team made a decision quickly to make the 24 MNSs that were prepared up until that point, openly accessible through RAMZOR [Sect. 5.4.1] a designated Hebrew website, [...]. The openly accessible MNSs enabled teachers who wished to present the MNSs, to do it from the newly created website through Zoom, or even dare to let the students access the website and go through any MNS on their own, then run a flipped-classroom style of discussion.

From Italy, Faggiano described moving a course for the prospective teacher education program online:

> This has offered an unexpected opportunity: an online teaching experiment would have been conducted with students at lower secondary school (Grade 7). Hence, some lessons of the course have been devoted to designing collaboratively the activity to be experimented. The whole group of prospective teachers took part in the online teaching activity. Finally, they collaboratively reflected on it, not only during the lessons, but also in further group meetings that they have autonomously organised alongside the course. An online shared folder became the learning environment by means of which the university students built their storyboard. They annotated every comment to the collaborative design; they uploaded the videos of the teaching activity and their transcriptions; finally, they wrote a collaborative text containing the analysis of the results and their self-reflection on-action. In particular they have been interested in the unexpected changes in the activity and in its analysis that were required by the distance teaching–learning mode.

From Colombia, Castro shared with us that:

> Teachers feel alone facing the challenges imposed by the pandemic, neither the Ministry of Education nor the officials seem to comprehend the harsh time teachers have to face to continue teaching mathematics and complying with the academic standards upon which officials assess teachers' work. Teachers turned to the most experienced colleagues and attended online meetings to share documents and tips to use apps, technology and resources. Once in possession of the resources, the teachers dedicated themselves to sharing suggestions for use and didactic adaptation of the documents, videos and free applications. They also made changes in the management of the courses: for example, they proposed projects that involved the participation of several teachers. In this case, the pandemic leads to new forms of collaboration for sharing resources.

Lastly, we heard from the University of Turin in Italy, regarding the decisions of a mathematics education laboratory for secondary school prospective teachers, directed to designing mathematics activities inside the national curriculum of secondary school, and to deepening didactical concepts. The examination of the laboratory—for these Master's students—usually consists in carrying out the

designed activities with a selected group of high-scored secondary school students, engaged in a project called 'Stage di matematica'. Usually, Master's students designed the tasks individually. Due to the pandemic, the professor of the laboratory took new decisions: to hold virtual lessons synchronously via a platform; to ask Master's students to work collaboratively on a common task. All the students, collaborating online as a virtual community of practice (Dubé et al., 2005), were asked to design—for the secondary school students—a mathematical 'escape room', to be implemented experimentally the following school year either remotely or in-person.

This working modality was revealed to be successful, as the students—working together collaboratively at the same task—developed a real community of practice with common tasks, vision, aims and practices. The platform, used previously as a repository, became a virtual room for synchronous classes and included a virtual space for asynchronous interaction. The final product came from the interactions among students, developed along the semester and consisted of a set of virtual escape boxes. Each box contained one or more very challenging mathematical problems, quiz or activities for secondary students. To make a pilot experiment of this product, three teams were remotely involved: secondary teachers; university students; secondary school students. From these three trials, feedback and notes were collected to support the development of the final version of the escape boxes.

In all cases shared, the resources and/or their usages were adapted for a purpose that was not the original intent of the resource. This adaptation may lead teachers to develop new ways of collaborating and reflecting. This was the case for Story*Circles* (Sect. 5.4.3), described by Milewski et al. (2020) for a special issue of a journal dedicated to the pandemic. But, as Sayah emphasised, for a case in Algeria, in the total absence of face-to-face collective work, the possibility and effects of using a variety of ICT resources remain important issues: how has the collective work through ICT contributed to the richness of interactions between teachers during this crisis? How did these interactions contribute to developing other resources for supporting teachers' activities during this crisis?

5.7.2.2 Designing New Resources

We learned that a group of expert teachers in China (Huang et al., 2023) were asked by their institution to develop, with their school colleagues, new resources such as online videos that appear fundamentally important in helping teachers adapt to online teaching, although there were some difficulties in the adoption. The school-based Teaching Research Group played a critical role in helping teachers to develop complementary materials for addressing different student learning needs and various technological constraints. This case demonstrates that the process of appropriation of new resources, even those dedicated to this time of pandemics, fosters teachers' collaboration, and leads them to adjust these resources to their own needs.

In South Africa, Brodie describes a case co-written with a group of mathematics teachers. They collaborated to develop resources to teach mathematics through

COVID-19 in their classrooms. The collaboration took place in the context of a Master's course on pedagogy, which looked at how learners' experiences out of school might be drawn on to teach mathematics and the strengths and limitations of doing this. The case of COVID-19 gives an especially useful example because: (a) it combines everyday and scientific resources for understanding the pandemic, its causes and how we deal with it; (b) it supports integration across contexts differently from how this is usually understood; (c) it shows the strengths and limitations of mathematical knowledge in understanding our environment and experience. The teachers present a newspaper article with data from the first 100 days of COVID-19 in South Africa, together with a set of questions which help learners to understand the mathematics involved and relate it to their on-going, lived experience of the pandemic. The teacher collaboration is not analysed, but this presents a first example of developing a resource through collaboration for the COVID-19 experience.

5.7.2.3 . . . and Being Aware of Missing Resources

The pandemic reveals that using existing resources, and developing new resources based on teacher collaboration is not enough. Castro shares:

> The collaboration between the teachers allowed them to respond to the challenge of teaching entirely online, but according to some teachers now is time for a new form of collaboration among education stakeholders—teachers, students, parents and Ministry of Education officials—in order to redefine the school dynamics and to respond to the challenges bring about by the pandemic. Unfortunately, teachers feel that this type of collaboration is out of their reach.

All the issues linked to the pandemic exacerbated, and were exacerbated by, the inequities between students, particularly regarding social issues (see also the following section), in many countries. For example, in the context of India, Kumar shares:

> In the present context of pandemic, the inequities have become exacerbated because of the issues of access and excessive focus on the digitalised interventions for addressing educational needs, while leaving students without the access of devices and connectivity completely in the lurch [. . .] most of the interventions focus on online teaching using video conferencing apps or a mix of chatting apps with the synchronous interactions and assessments, while few have focused on providing most basic resources like textbooks to students who may not have access to device or internet and may be economically disadvantaged to get this kind of access.

Two strands of literature provide good reasons to be highly concerned about the detrimental effects of the school closure on learning outcomes, in particular to students with a disadvantaged background. There is evidence to suggest that: (a) the time spent in school reduces the learning gap with respect to privileged students, particularly in mathematics (Bovini et al., 2016) and in disadvantaged areas (Battistin & Meroni, 2016); (b) long summer breaks have a negative short-run and long-run effect on educational outcomes and are perhaps one of the major sources of learning inequalities (Alexandre et al., 2007).

In the University of Turin, a team of academics from mathematics education and economics and social statistics joined together to develop a project designed to measure the effects of the COVID-19 school-building closures on the mathematics skills and mathematics learning inequalities in primary school children. This evaluation will be the first done in Italy and will provide a timely assessment of educational impacts (Contini et al., 2021).

The United Nations Development Programme (UNDP, 2020) estimates a steep reduction in human development in 2020 as a consequence of COVID-19, due to the educational and economic losses. These considerations lead us to dedicate a special sub-section to the issues of collaboration, resources and (in)equity.

5.7.3 Rethinking Resources and Collaboration Under the Light of Inequities

In this sub-section, we discuss issues related to equity and a few examples of tools and resources developed collaboratively to address this issue. The word 'equity' appears ten times in the ICMI 25 proceedings, indicating that very few collaborations have directly focused on this issue. We focus here on: collaboration for designing resources to face issues of inequity; resources for supporting teachers' collaboration for facing inequities; collaboration for addressing the issue of *accessing* digital tools; collaboration for addressing the issue of equity when *designing* digital resources.

5.7.3.1 Collaboration for Designing Resources Addressing Inequities Issues

Realising that not only schools, but community and family spaces need to be engaged mathematically, South African Numeracy Chair Project worked in collaboration with teachers to use classrooms after school meetings and community math clubs to promote meaningful and fun-filled engagement with mathematics using take-home resources (Graven & Venkat, 2017). Alternatively, recognising that addressing equity involves not only access to school mathematics, but also taking into account marginalised students, studies have built culturally responsive tools in collaboration with teachers belonging to marginalised communities, such as studying patterns in the Maori art and adopting Maori culture in building relationships (Hāwera & Taylor, 2014). In the context of schools located in slums, for the teaching of proportions Bose and Subramaniam (2019) have highlighted the interest of taking into account the resources resulting from the children's own experience.

5.7.3.2 Resources for Developing Collaboration Among Educational Agents to Address Inequities

Teachers may come together in different forms of collaboration to address their own issues, in rejecting the professional learning communities (Brodie, 2021) due to their working conditions and constraints, for systemic or personal reasons. This may occur in collaborative groups that reflect contrived collegiality or those which come together due to voluntary, but isolated action. Louie (2016) illustrated how even equity-oriented mathematics teachers experienced tensions between inclusive and restrictive discourses about mathematical competence during the collaborative interactions.

This finding highlights the challenge of maintaining the focus on equity- and reform-oriented learning in professional learning communities. Professional learning communities could develop in the classroom themselves, as described by Eden (2020) where co-teaching in the classroom setting proved as a resource for noticing student thinking and "thus expanded access to resources for practice" (p. 300). Professional communities at large may also be a resource: according to Nieman et al. (2020), not only teachers, but school leaders, principals and facilitators can also play a big role in establishing equity-focused collaborations.

However, creating teacher collaborations to address equity issues is not simple, as illustrated by Bottia et al. (2016) stating that, "teacher collaboration will only be effective for Latino/a students who are English language learners if the collaboration is accompanied by both adequate pedagogical and cultural understandings of these students" (p. 527), which includes the lived experiences of Latinos who speak Spanish at home.

Another study conducted by Kokka (2018) focused on social justice in STEM education. This study shares how four STEM teacher activists became involved in grassroots organising, sparked by their own experiences of marginalisation and structural oppression, and how the organisation became a vehicle for their own healing, as well as a means toward addressing the inequities they witness and experience in their communities. Among interventions focusing on developing teacher leadership for developing community of learners (Harris et al., 2017), some involved formation of a series of professional learning communities at different levels ranging from whole staff in a school to particular division to "teacher math buddies", with students across grades to study understanding of one topic of measurement across grades (Lieberman et al., 2016).

5.7.3.3 Collaboration for Addressing the Issue of Accessing Digital Tools

Access to and use of digital technology varies indeed across the countries, depending on socio-economic, cultural and gender factors (Forgaz et al., 2010). This gap is usually referred to as the *digital divide* between the people who have access to and

knowledge of using technology and the people who have no access to technology. The issue of access to digital resources can be considered at the following levels:

- issues of access to computing devices for teachers as well as students. Initiatives, like *one laptop per child*, have tried to provide low-cost devices to students aiming to establish a collaborative and constructivist environment in the classroom (Buchele & Owusu-Aning, 2007; Kraemer et al., 2009). However, teachers were often not involved as participants and therefore not provided with adequate resources to support students in making the best use of this technology. Studies have identified the need for adequate teacher education (Pischetola, 2014), because access to digital media is not sufficient to bring about transformational change. In addition, under the present circumstance, the access to devices for teachers at home to support virtual learning environments may depend on either the number of devices at home in relation to the number of users, and/or the gender dynamics in the family for determining who may have priority access to the device for their work;
- the issue of 'where' and 'when' the access to digital devices is provided—at school, at home, as shared community resources or as personal devices. Each type has its implications for access, as well as for collaboration among students, between students and teachers, teachers and parents, and even across schools and regions;
- the issue of lack of connectivity, or internet, required to access the digital resource. The lack of connectivity limits the users access to online resources and prevents devices from receiving the latest updates, which may be critical for its use;
- licensing issues related to the use of proprietary digital resources which may be mitigated by the provision of open education resources (e.g. Sesamath in France https://www.sesamath.net, or Connected Learning Initiative India, see below);
- issues of designing for accessibility using Universal Design Principles, so that the resources are accessible to all;
- issues of cultural norms which constrain access to digital resources (e.g. gender or marginalised students in the classroom) as others fail to share the resources equally.

Over the years there has been a shift from interventions focusing on providing low-cost hardware to developing software that supports teacher collaboration across schools and geographies using hand-held personal devices, such as smartphones which are becoming more and more ubiquitous. The software developed is similar to the ones discussed in the chapter, which might be used as a representational aid for mathematics, a pedagogical aid, or a collaborative aid. However, the issue of access to digital resources for teachers and students, and the quality of experiences of students, remain a complex issue to be addressed, at least in developing countries.

5.7.3.4 Collaboration for Addressing the Issue of Designing Digital Resources

While collaborative research for developing content and pedagogical knowledge among teachers exists in large numbers, there are few studies which explicitly focus on the equity aspects while designing tools and resources. When designing for equity, which needs should we focus on? Here, we discuss a few possibilities.

An important consideration for equity is the language used in the resources designed for teacher collaboration. With the availability of digital resources, how do designers address the language concerns of minorities and make the resource available to those who speak other languages? In many developing countries, there exists a tension between considering English as the preferred language of instruction and utilising the learner's home language to develop conceptual knowledge. The politics of the stakeholders who are often in the position to make decisions about what language is used and by whom underlie this issue of language and equity.

In India, a large-scale intervention, called Connected Learning Initiative (https://clixoer.tiss.edu/home/e-library), released its resources in three different languages (English, Hindi and Telugu). This addition allows students to toggle between and change the displayed language of the resource. Revisions to improve the quality of translations and suggestions of more familiar words, for both teachers and students, were made based on feedback from teachers. Through this addition, the resource is more accessible to learners.

Multiple languages thus become a resource for learning by making resources more accessible. When considering translation from one language into another, an additional issue is to be taken in consideration as a great value: the culture. Language is not only a tool for communication, but also the substantial ground for sharing history, culture and values of a population. So, even if the translation is well done, what we have to save with the translation is this substantial ground, to be effective in supporting learning.

Adler (2017) argues that, in the developing world, providing access to education does not ensure that learning takes place, as there is restricted access to valued knowledge. Giving resources to teachers for their classrooms does not really make sense, if they are not provided with the knowledge of why the specific resource is used or how the resource is best utilised. In this manner, the embedded pedagogy becomes opaque to teachers and fails to empower them to develop the knowledge of the content or pedagogy the resource is meant to support. For this reason, it is imperative that designers include teachers' voices describing how they have adopted and adapted resources and document the modifications teachers have made. The collaborative modes of engaging in research 'with' rather than 'on' teachers to develop tools and resources for classroom use can specifically address this inequity by developing teachers' knowledge of content and pedagogy as discussed in the previous sections (Setati, 2005).

This may further ensure that the use of resources will lead to relevant learning outcomes for students, such as exploratory talk and sense-making, while shifting the

research discourse from a deficit narrative of teachers' capacities to a respectful discussion, as reflected in the work of Graven and Venkat (2017). These and other studies highlight issues of equity that need to be considered, not only in the design of the tools for collaboration with the teachers, but also in the tools designed for teaching and for reflections on teaching (see Sects. 5.3 and 5.4).

This sub-section has illustrated how issues of equity crop up in several ways which may determine the selection of and access to resources, the design of resources, the nature of professional development opportunities for teachers and teacher interactions within collaborative groups. It also illuminates a variety of complex factors contributing toward the development of a collaborative community through interactions between agents, and attempts to address the complexity of mathematics teaching and understanding of students' thinking for addressing equity while teaching.

5.7.4 *Looking into the Future: New Resources, New Collaboration, New Ways of Teaching Mathematics?*

This chapter provides a structured way to look at studies on teacher collaboration that involve the use of resources and tools, first of all distinguishing between resources and tools:

for supporting teachers' collaboration;
(coming) from teachers' collaboration.

For and *from* are two main categories that are actually interrelated (see Sect. 5.2): both types of resources influence each other as anticipating practice, sharing experience of practice and reflecting on practice; they co-occur in interactions in professional development settings. In this final sub-section, we propose a reflection in terms of interactions: interactions between tools and resources; interactions between the agents of mathematics education; interactions between theoretical frameworks.

At the beginning of the chapter (see Sect. 5.2), we have tried to differentiate the concepts of *tool* and *resource*, a *tool* being something allowing us to search for, and/or manipulate a given *resource*, grounding the teacher's work. This distinction has guided the authors in writing their respective sections. But, at the end of this writing, having a retrospective look at these sections, we have to acknowledge the fact that it is not always easy to categorise a 'thing' mediating a teacher's activity as a tool or a resource. Is a national curriculum, or an email coming from a colleague, a 'resource' or a 'tool'? It depends indeed on both the context of the teacher's activity that has to be described, and analysed, and the theoretical lens through which this analysis is performed. The meaning of words are social constructs that develop within communities of practice.

The early stages of studies on teachers were directed to investigate essentially the teachers while teaching in their classes. Recently, the research agenda has become wider and richer, exploring: teachers in communities inside or outside the

institutions using resources developed for collaboration or not, for education or not, and—more importantly—teachers working side by side with researchers, teacher educators and generally knowledgeable others. This shift in the research agenda indicates teachers' passage from being the 'object' of research to being the 'subjects' themselves, engaged in educational but also in research processes.

The complex issues of resources and tools for/from collaboration often call for networking existing theoretical frameworks (Trouche et al., 2019) and for rethinking theoretical as well as methodological frames (Arcavi, 2019). The need for analysing many people using/designing a huge number of resources calls for new approaches (e.g. linguistics, Learning Analytics, Big Data, for example). The need for considering the institutions and society grounding the use and design of resources call for socio-cultural or socio-epistemological theories for interactional learning and cognitive theories for learning mathematics.

Finally, considering the wide range of research that has been discussed, regarding resources and tools, during this ICMI conference, we would like to suggest some necessary perspectives of research:

- first of all, due to the numerous technologies involved, and their rapid evolution, their categorisations appear quite fragmented. We need a categorisation of the resources and tools for/from teacher collaboration, their affordances, potential and constraints, for inquiring, teaching and designing, as well as reflecting;
- second, due to the diversity of collaborative occasions of teacher professional development, formal or not, distant or face-to-face, the studies considering them appear often compartmentalised. We need studies taking into account the contributions of this diversity of settings.
- Third, while the effects of collaboration on professional development are often slow, and the appropriation of tools takes time, research in this area develops over relatively short periods. We need research taking into account the long time, and a variety of interactions between teachers and resources. This undoubtedly requires new research infrastructures.

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