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Combining theoretical frameworks to investigate the potential of computer environments offering integrated geometrical and algebraic representations

Jean-Baptiste Lagrange[1], Giorgos Psycharis[2]

[1]*University Paris Diderot and University of Reims, France,* [2]*University of Athens, Greece*

In this paper we aim to address the problem of fragmentation of theoretical frameworks within the field of mathematics education with technology while exploring the potential of computer environments offering integrated geometrical and algebraic representations. We follow a 'cross analysis' method to analyse an experiment taking place in the Greek context under a constructionist theoretical perspective through the lens of the theory of didactical situations and of an epistemological model of activity with functions from the French research context. The analysis indicates that the aforementioned method enhances our efficiency to capture the potential of computer environments linking geometrical and algebraic representations.

Introduction

Several computer environments now offer integrated geometrical and algebraic representations and functionalities. Researchers and innovators stress their potential to support especially the teaching and learning of functions at various levels. However, it seems difficult to really appreciate this potential, since every author usually bases his/her findings on a vision provided by his/her specific framework and because of "the fragmented character of the theoretical frames which have been developed in order to approach learning and teaching processes in such environments" (Artigue et al. 2009, p. 5). Our aim in this paper is to combat fragmentation, trying to connect visions based on different theoretical perspectives. We have chosen the metaphor of networking theoretical frameworks and the idea of combining and coordinating frameworks "for the sake of a practical problem" (Prediger et al., 2008 p.172). We also chose to analyse concrete teaching experiments taking place in real classroom settings. Consequently, we adopt "cross analysis" as a method, according to which researchers analyse jointly an experiment taking place in a given national and didactic context under different theoretical frames and research approaches.

Each of us used a software environment that has been developed under specific theoretical frameworks in specific research and national contexts. The first one (called *Turtleworlds*) is a piece of geometrical construction software which combines symbolic notation, through a programming language (Logo), with dynamic manipulation of variable values (Kynigos, 2004). The design and the research on the use of Turtleworlds is inspired by constructionism (Harel & Papert, 1991). The second software (called *Casyopée*) offers a dynamic geometry window connected to a symbolic environment specifically designed to help students to work on functions. Casyopée's design and experimentations occurred in a French context shaped by didactical theoretical frameworks and epistemological considerations. We focus here on a framework preeminent in the French context –the theory of didactical situations (TDS, Brousseau, 1998), and on an epistemological model of activity with functions built to make sense of the potential of computational environments with interconnected algebraic and geometrical representations, especially of *Casyopée* (Lagrange & Artigue, 2009).

We separately conducted research about the potential of computer environments offering integrated geometrical and algebraic representations, each using his own software in his specific context and with his specific frameworks. We are interested in exploring deeper this potential and the question we aim to tackle is: what new insight about this potential might be gained from analysing an experiment carried out by researcher A involving his own environment within his specific framework, when combining this framework with researcher B's framework? We consider here a teaching experiment designed and implemented with Turtleworlds in the Greek context. As a way to combine A and B's frameworks we cross a constructionist analysis of this experiment with analyses carried out by way of TDS and of the epistemological model mentioned above as distinctive features of the French research context. We introduce briefly situated abstraction as the main idea of constructionism for mathematical learning since it was the framework that informed the design of Turtleworlds and the implementation of the teaching experiment and then we report on the teaching experiment. In the cross-analysis we first highlight the elements of an analysis from the constructionist perspective before we demonstrate how key notions of TDS and of the epistemological model shed light upon these elements.

Constructionism and situated abstraction

Constructionism is a theory of learning that incorporates and builds upon constructivism's connotation of learning as "building knowledge structures" through progressive internalization of actions, in a context where learners are consciously engaged in constructing (or de/re-constructing) something on the computer. Constructionism attributes special emphasis on students' activity when using mathematics to construct a model on the computer: the notion of construction refers both to the 'external' product of this activity as well as to the theories constructed in students' minds (Papert, 1980). Aiming to elaborate a theoretical account of how mathematical meanings can be situated –in terms their genesis, means of expression and use - and yet abstract in that they extend beyond immediate concerns to more formal conceptions of mathematical knowledge, Noss and Hoyles (1996) introduced the notion of *situated abstraction* to describe how learners construct mathematical ideas by drawing on the linguistic and conceptual resources available for expressing them in a particular computational setting. A basic tenet of situated abstraction is that the computational tools, in turn, shape the ways the ideas are expressed and thus the respective computational environment can be considered as a system through which mathematics can be expressed.

Turtleworlds

Turtleworlds is a microworld designed to integrate formal mathematical notation with dynamic manipulation of variable values (Kynigos, 2004). In Turtleworlds, the elements of a geometrical construction can be expressed in a Logo procedure. After a procedure depending on variables is defined and executed with a specific value for each variable, clicking the mouse on the turtle trace activates the "variation tool", which provides a slider for each variable (see at the bottom of Figure 1). The dragging of a slider results in a continual reshaping of the figure according to the corresponding variable value. Thus, the user is able to use Logo formalism (a) for describing geometrical figures algebraically by using variables and/or relations to represent segments and/or angles and (b) for dynamically manipulating the geometrical objects embedded in the construction of these figures for controlling their shape according to geometrical properties. The software thus behaves like a hybrid between programming tools (e.g. Logo-like microworlds) and expressive tools (e.g. Dynamic Geometry Systems) proposing formalism as a means of representing mathematical ideas in a geometrical construction context. The user is able to write, run and edit Logo procedures to 'drive' the construction of geometrical figures by the turtle providing also a way to construct relationships that render the construction visible.

The experiment

The experiment with Turtleworlds took place in a secondary school in Athens with two classes of 26 pupils aged 13 years old and two mathematics teachers. All teaching sessions were video-recorded by a team of two researchers who acted as participant observers in the classroom. The constructionist theoretical framework that underlies the study suggests designing a task that engages students in producing a meaningful outcome, and simultaneously in appreciating the utility of the respective mathematical ideas, the why and how these ideas are useful (Ainley et al., 2006). The task – called *Dynamic Alphabet* – engaged each class in constructing enlarging-shrinking models of all the capital letters (i.e. of variable sizes) with one variable corresponding to the height of the respective letter. Moving the slider of the variation tool in this case would result in the visualisation of the letter as an enlarging-shrinking geometrical figure. In formal mathematical terms this means that each letter procedure had to contain only one variable, so all of its varying lengths would be expressed with appropriate multiplicative functional relationships. We stress that these relationships were not initially explicit to the students, the aim being that they experience visually-based cognitive conflict, particularly when using additive strategies.

Early in their work most of the students constructed a model of their letter – which we refer to as the *original pattern* – sometimes without using any variables. In subsequent phases of their exploration, students experimented with the use of variables for all of its segments, to change it proportionally, until they built their final model with one variable. We focus on a pair of students - Christina and Alexia- exploring the construction of an enlarging-shrinking model of the letter N that they completed in the following successive six phases during four classroom sessions:

Phase 1: Construction of the original patterns of two models of N: N (35°) (vertical segments=200, slanted segment=240, internal angle=35°) and N (45°) (vertical segments=100, slanted segment=145, internal angle=45°).

Phase 2: N (35°) construction with two variables for the vertical segments and the slanted segment respectively. Recognition of the interdependence of variables. Exploration of the construction of similar N (35°) models of different sizes.

Phase 3: N (35°) construction with one variable and specification of an additive functional relation between the vertical segment and the slanted segment. Experimentation with changes to the constant value of the additive functional relation used to represent the slanted segment.

Phase 4: N (45°) construction with one variable and specification of a multiplicative functional relation between the vertical segment and the slanted segment (not appropriate function operator). Experimentation with changes to the constant value of the function operator used to represent the slanted segment.

Phase 5: N (45°) construction with one variable and appropriate multiplicative functional relation between the vertical segment and the slanted segment.

Phase 6: Exploration of the construction of different models of N (25°, 30°, 35°, 45°) and specification of appropriate multiplicative functional relations between the vertical and the slanted segment.

During phase 3, in order to challenge these students to consider the co-variation of the two variables (:r and :t) for constructing similar models of N (35°) in different sizes, the researcher asked them "how many times the one segment is the other". The students translated the relation of the two values as "200 plus forward 40" and substituted variable :t with the functional expression (:r+40). Dragging the only slider :r, students realised that the figure was obviously distorted for most of the values of variable :r. However, they observed that when :r took values 'near 200' -which was the value of the vertical lengths in the original pattern- figure distortion seemed to be minimised. Thus, they tried to identify a sub-domain of the functional relation to control figure distortion and validate the effectiveness of their method for enlarging-shrinking the geometrical figure. According to the graphical outcome they concluded that the figure was less distorted when variable :r took values between 195 to 205. In the stream of their subsequent exploration the students experimented with new additive functional relations (i.e. :r+45 and :r+50 respectively) trying again to identify new sub-domains of these relations for controlling the graphical distortion on the figure caused by different values of the only variable :r. In this vein, they drew a line at the letter base so as to precisely evaluate the accuracy of their method. Again, testing dragging on the variation tool confirmed that the use of an additive algebraic expression constituted an erroneous strategy for constructing an enlarging-shrinking model of N holding for 'all values of :r'. From this point students started to rethink the correlation between the two varying magnitudes in the Logo code.

Researcher: Since the one [i.e. the vertical segment] is 100 and the other one (i.e. the slanted segment). How many times of 100 is 145?

Alexia: One and …

Christina: One and ... Yes, one and a half.

Researcher: Well, how one and a half can be expressed in a relation?

Alexia: Maybe, one plus :r divided by two. Let's try it [i.e. on the computer].

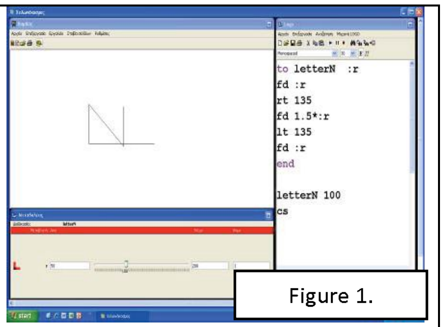

Figure 1.

In the above excerpt (Phase 4) the students had already started to experiment with the identification of an appropriate functional relation for the slanted length so as to construct an enlarging-shrinking model of N (45°) (vertical segments=100, slanted segment=145 in the original pattern). The researcher took the opportunity to intervene so as to challenge students move the focus of their attention on the multiplicative dependence between the two varying lengths by comparing the numerical values of them in the original pattern.

Analysis from a constructionist perspective

The above episode brings to the foreground two critical aspects of the construction of geometrical figures according to proportionality: first, how the students appreciated the inappropriateness of additive strategies and, second, how they identified and expressed multiplicative functional relations in formal notation. In our view, a critical step in this direction was the translation of the dependency between the vertical and the slanted side in symbolic notation through a process in which values, variables and everyday language were simultaneously interlinked. Although the correlation between the two magnitudes was initially perceived by the students as additive, the computational environment provided a structure which they used to express the corresponding function, tools for experimenting with it and further elaborating its formula. At that time we see that meanings were reshaped as the students moved the focus of their attention onto a relation which was a new object within the setting. Thus, the students moved from identifying dependence and co-variation between magnitudes to identifying and expressing co-variation between magnitudes represented through variables. The chain of functional meanings here involved (a) the idea of variable as representing a general entity that can assume any value and symbolise general rules, (b) the specification of domains of validity for an additive functional relation and the development of methods to take control of the distortion of the figure (e.g. design of straight line in the letter base), (c) experimentation with the symbolic form of the additive functional relation and (d) implications of the potential emergence of the multiplicative correlation between the two variables representing the vertical and the slanted side. The chain of functional meanings was reshaped through the use of the variation tool. The dragging on the one and only slider of the variation tool refuted the proportional enlarging-shrinking of the geometrical figure for all values of the respective variable, thus providing a link between the graphical distortion and the symbolic aspect of the functional relation.

We consider this excerpt as an illustrative example of the dynamic nature of the functional meanings developed by the students. The researcher's remark about the correlation between two specific numerical values triggered students' focus on the functional relation between the two corresponding lengths. Christina describes the emergent relation by words while Alexia seems to be able to articulate the dependent length as situated abstraction with direct reference to the independent variable :r. Although it is not clear what is the symbolic form of the functional relation suggested by Alexia– i.e. (:r + :r/2) or (1 + :r/2) or (1+:r)/2- the available symbolic component of the environment allowed students to test these relations, thus providing a basis for further elaboration based on the use of the variation tool and the graphical feedback. Here situated abstraction emerged facilitating students' reshaping of functional meanings towards the identification and expression of the multiplicative co-variation between the two variables.

Analysis from a TDS perspective

Brousseau (1998) presents TDS as a way to model mathematical situations in a learning context. In this model, a central notion is the "milieu", a device which justifies the use of knowledge objectively to solve a given problem. He establishes a list of conditions to satisfy including: (a) the mathematical knowledge aimed at should be the only good method of solving the problem (b) students should be able to start working with inadequate "basic knowledge", (c) students should be able to tell for themselves whether their attempts are successful or not, (d) the feedbacks should not indicate the solution, but they should be suggestive of ways to improve strategies, (e) students should be able to make a rapid series of attempts, but anticipation should be favoured.

Considering the example with Turtleworlds we can see that these conditions are important for the success of the situation. Students could start with inadequate conceptualisation and the multiplicative functional relation appeared to be the only good method for completing adequately the construction. Dragging on the slider of the variation tool allowed them to realize by themselves whether their attempts succeeded or failed. The graphical distortion produced by additive relation brought suggestive information about the inappropriateness of this solution. Expressing the relation into the Logo procedure favoured anticipation. It seems to us that the constructionist researchers implicitly build their analyses upon a model of mathematical situations consistent with Brousseau's TDS. However, the episode when the researcher intervenes so as to move students' attention on the multiplicative dependence would be interpreted by TDS as a too direct indication towards a solution (an "effet Topaze"). TDS researchers would note that the students do not follow the intervention, but rather continue to explore addition based relationships. The exploration with the software is productive because it helps to distinguish between multiplicative expressions (like :r + :r/2) and semi-additive expressions like (1 + :r/2).

Another difference is that constructionism takes care of "meanings" rather than directly of "mathematical knowledge". The effect of students' interaction with the "milieu" is described in terms of development of functional meanings. It helps to incorporate in the analysis the symbolic components brought by the "milieu" and their effects onto the identification and expression by the students of the multiplicative co-variation. Furthermore, as shown in the presentation of the task constructionism favours situations where students can appreciate the utility of mathematical ideas. This means that the "milieu" should not only favor the emergence of meaning and of symbolic components, but also help students appreciate their utility beyond the boundaries of school mathematics. In contrast, TDS privileges in the analysis of the situation the knowledge to be taught and stresses the connection between knowledge built by interacting with the "milieu" and the standard mathematical knowledge at stake. This means that in a TDS perspective a process of institutionalization should be organized by the teacher in order that students interacting with Turtleworlds access some standard knowledge on multiplicative functions, whereas constructionism insist on the emergence of situated abstraction in the interaction itself.

For us adding a TDS analysis of the experiment helps (a) to better understand the conditions that make the interaction with Turtleworlds work in a productive way, (b) to focus on another aspect of learning: in addition to the development of functional meanings useful beyond school, the Turtleworlds situation can help students reach standard knowledge about multiplicative functions.

Analysis based on the model of algebraic activity

In order to makes sense of the potentialities of new computational environments for learning about functions, Lagrange and Artigue (2009) classified the various activities about functions into a model of algebraic activity. We limit here the model to a classification based on the epistemological assumption that the notion of function is connected to the idea of dependency in physical systems where one can observe mutual variations of objects. The model distinguishes three levels: (1) activity in a physical system where dependencies are "sensually" experienced (Radford, 2005); (2) activity on magnitudes, expected to provide a fruitful domain that enhances the consideration of functions as models of physical dependencies; (3) activity on mathematical functions, with formulas, graphs, tables and other possible algebraic representations.

The model helps to analyse the students' activity in the Turtleworlds experiment. The physical system is a path of the turtle in three segments with a given angle between them. In phase 1, the path is fixed with given length of the segments. In phase 2 it depends on two variables and in the next phases, the challenge is to program the path in order that it depends on one variable while conforming to the goal that it represents the letter N. At the level of magnitudes, angles and lengths are involved. What is at stake is formulating a dependency between the length of the vertical segments and the length of the slanted segment in a functional form allowing its expression in a Logo procedure. At the level of mathematical functions, the dependency between the "vertical" and the "slanted" length is mathematically a multiplicative function, whose coefficient depends on the given angle.

In phases 1 to 5, while activating the sliders and working on the Logo procedure, the students consider together the physical system and the dependency between magnitudes. Their task is actually to understand the constraints of the physical system as a dependency linking two magnitudes and to find an expression for this dependency in order to write the procedure using a single variable. This implies to choose one length as an independent variable and the other as a dependent variable before building a suitable algebraic expression. In phase 6, the students move to mathematical functions. Taking the functional expression of the dependency for one angle into account, they understand that the same multiplicative model holds for other angles. Their task is then to find the multiplicative coefficient for each angle. Further tasks could deal with comparing the functions for different angles by using tables or graphs, allowing students more activity at this level of mathematics functions.

The model of algebraic activity is based on an epistemology of functions. It has been built to make sense of the design of geometrical and symbolic environments different from Turtleworlds and of their use by older students (aged 16 and more) to learn about functions. The distinction between the three levels of activity helps here to analyse the progression of functional meanings, showing the importance of working with magnitudes as a bridge between sensual experience and mathematical functions and suggesting further tasks.

Conclusion

Our concern was the fragmentation of theoretical frameworks within the field of mathematics education and its negative consequences in terms of appreciating research results and validating the constructed knowledge. Being interested in investigating the potential of computer environments offering integrated geometrical and algebraic representations we considered one environment and an experiment with students aged 13. The constructionist perspective brought elements of analysis: a chain of functional meanings was observed involving the idea of dependency between variables, a functional relation and the symbolic form of this relation. Thanks to the activity with the software the students recognised the inappropriateness of spontaneous models and identified and expressed appropriate multiplicative relations in formal notation. The TDS offered an appropriate framework to analyse this activity in terms of interaction with a "milieu", specially adequate to justify the use of the appropriate knowledge objectively to solve the problem. The model of algebraic activity helped here to analyse the development of functional meanings, highlighting the importance of working with magnitudes as a bridge between sensual experience and mathematical functions. This brings evidence that by combining constructionism with TDS and the model of algebraic activity, the cross analysis captures more efficiently the potential of Turtleworlds as compared to the use of a single framework specific to the experiment. This is clearly a first step towards coordinating these approaches in order to get an integrated framework to analyse the potential of computer environments offering integrated geometrical and algebraic representations.

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