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Conceptualization of function as a covariational relationship between two quantities through modeling tasks

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ABSTRACT

In this paper we use learning trajectories to study 11th grade students' conceptualization of function as a covariational relationship between two quantities while they engaged in modeling tasks to support their experimentation and conceptualizations. Pairs of students used digital tools that offer integrated representations of functions while working on an instructional sequence of modeling tasks in their mathematics classrooms. The analysis shows students' progressive conceptualization of functional relationships starting from quantitative and covariational relationships using learning trajectories. The findings indicate the potential of upper secondary students to conceptualize function as a covariational relationship involving the rate of change, as well as the role of the available tools and the role of models and their connections in students' conceptualizations.

1. Introduction

Covariational reasoning involves "coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002; Thompson, 1994). Existing literature indicates that covariational reasoning is foundational for the development of functional thinking (e.g., Confrey & Smith, 1994; Blanton et al., 2015) and many studies have focused on the importance of students' reasoning about a covariational relationship between quantities that results in a functional relationship (e.g., Carlson et al., 2002; Castillo-Garsow, 2012; Johnson et al., 2017; Thompson & Carlson, 2017; Ellis et al., 2020). Another strand of research highlights an increasing need for coupling mathematical modeling and quantitative, covariational and functional relationships (Basu & Panorkou, 2019). For instance, Moore and Carlson (2012) study showed that students' quantitative and covariational reasoning can facilitate their understanding of functional reasoning under a modeling approach. Other researchers have indicated that modeling of dynamic situations has the potential to help students understand functions and make connections between mathematics, daily life, and sciences (Kaiser et al., 2011; Lagrange, 2018). Functional relationships appear in different settings and corresponding models such as a physical device, a dynamic figure, the covarying magnitudes, and the algebraic functions (Lagrange, 2014). Recent studies (e.g., Psycharis et al., 2021) that connected quantitative, covariational, and functional relationships to mathematical modeling have shown the potential of using a modeling process to study students' conceptualization of function as a covariational relationship between two quantities while modeling dynamic situations. However, there are still a number of open issues in the related literature. These include the diversity of mathematical models involved in the modeling process, the models' descriptions, the connections between them, and the kinds of involved covarying quantities (e.g., magnitudes, variables). Our study aims to contribute to this field of

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research.

The general goal of our study is to investigate students' evolving conceptualization of quantitative and covariational relationships to functional relationships at the upper secondary level while the students engage with digital tools and tasks inspired by authentic situations in a modeling context. The students used a digital environment that offers interconnected representations of functions that allow the manipulation of covarying quantities and the treatment of the corresponding functions. Our goal here is to highlight students' trajectory of this progression. We focus on the ways that they treated the covarying quantities in the different models involved in the modeling process and pay special attention to the connections between these models. We expect this study to inform other research that focuses on students' developmental progression through modeling and also teaching approaches that target the coupling of covariation and modeling through the use of digital tools.

2. Theoretical background

2.1. Quantitative reasoning, covariational reasoning and functions

Over the years, many approaches have emphasized the covariation aspect of function (e.g., Carlson et al., 2002; Thompson, 2011; Lagrange & Psycharis, 2014). Function, when seen covariationally, has recently been defined as "the conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other" (Thompson & Carlson, 2017, p. 444). In this vein and in order to reason covariationally, it is crucial for a person to conceive the quantities involved and the relationships among them. Quantitative reasoning offers the means for thinking quantitatively (Thompson, 2011). It refers to the reasoning about quantities' relationships and serves as a prerequisite for the conceptualization of measurable attributes of objects (Thompson, 2011). To reason about quantities, a person engages in a quantification process that has been defined as "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bilinear, or multi-linear) with its unit" (ibid, p.37). Thus, starting from quantities, a person can reason quantitatively about quantities and their relationships, coordinate the changes of two covarying quantities, and reason covariationally about function.

In our study, we adopt Thompson and Carlson (2017) definition of the term quantity as "someone's conceptualization of an object such that it has an attribute that could be measured" (e.g., attribute of quantity: length, area or volume considering a unit of measure). We define a priori some more terms to characterize the work with quantities in all steps of students' experimentation. We define magnitudes as quantities' measures (Lagrange, 2014; Saldanha & Thompson, 1998) indicating attributes of quantities considered independently of the unit of measure (e.g., the value of a length, an area, or a volume). Thus, magnitudes connect the work with attributes' dependencies and the work with algebraic functions (Psycharis et al., 2021). This transition from attributes of quantities to magnitudes and then to mathematical functions indicates the crucial role played by quantitative and covariational reasoning in students' development of functional thinking. In our study, we classify a priori students' reasoning about covarying quantities from attributes of quantities (as dependencies between objects) to magnitudes (as measures of dependent quantities) and to variables (as elements of mathematical functions)(Fig. 1).

Research literature offers insights about the meaningful transition from quantitative and covariational relationships to functional relationships between covarying quantities. For example, Confrey and Smith (1994) studied the exponential functions using tables of values and differences between values and reported that continuous covariational reasoning is foundational for thinking about modeling natural phenomena. Moore (2014) argued that quantitative and covariational reasoning are central to students' understanding of functions. The 2015 study from Blanton et al. indicated generalization of covarying quantities, their justification using various representations and reasoning on these representations as a prerequisite to functional thinking. Moore and Carlson (2012) reported that robust quantitative and covariational reasoning are critical for conceptualizing functional relationships in a modeling approach based on applied problems (e.g., the box problem).

Another strand of research on covariational reasoning has focused on identifying different levels of students' conceptualizations of functional relationships. For instance, studies on elementary level learners have categorized students' thinking about generalizing functional relationships (e.g., Blanton et al., 2015; Panorkou & Maloney, 2016) and showed elementary school students' potential to reason covariationally. Researchers have also focused on undergraduate students' thinking about the covariation of quantities in dynamic situations (e.g., the bottle task) resulting in a framework of five levels of covariation, which were described through corresponding mental processes (Carlson et al., 2002). These levels and processes are: (1) *dependence* (observation of changes in the two variables); (2) *direction* (increase or decrease in one variable with changes to the other one); (3) *quantitative correlation* (coordination of the amount of change of a variable with changes of the other one); (4) *average rate* (correlation of the average rate of change with uniform increases of the independent variable); and (5) *instantaneous rate of change* (correlation of the instantaneous rate with continuous increases of the independent variable). In this framework the first level refers to the work with dependencies, the middle levels (2) and (3) include additional details about covariation (e.g., direction, amount of change), and the highest levels (4) and (5)

Dependencies between objects(Attributes of quantities) →
Measures of dependent quantities (Magnitudes) →
Functions (Variables)

Fig. 1. Sequence of constructs from attributes of quantities to variables.

refer to the rate of change, which is part of our research interest. As regards the rate of change, it constitutes a rather demanding concept that requires "conceptualizations of ratio, quotient, accumulation, and proportionality" (Thompson & Carlson, 2017, p. 441). Our study identifies a trajectory of students' conceptualization of function as a covariational relationship between two quantities involving rate of change in increasingly sophisticated ways through the use of digital tools and modeling tasks.

2.2. Functions and modeling

"Modeling is considered as work on various models of a reality, belonging to different scientific fields, with varied mathematizations" (Lagrange, 2018, p.1). In recent years, the use of modeling to study covariational and functional relationships has gained increased research interest. For example, there are studies focusing on how covariational reasoning offers new opportunities for students' understanding of functions through modeling dynamic situations (i.e., Castillo-Garsow, 2010; Moore & Carlson, 2012). Recently the quantitative and covariational relationships have been studied under a modeling perspective (e.g., Paoletti et al., 2018). In 2019, Czocher and Hardison focused on the model evolution in modeling tasks. Their results indicate a fruitful coordination of quantitative and covariational reasoning. In some studies, the use of digital tools and special task design were used to promote students' conceptions between quantities intertwining the modeling perspective and the quantitative approach (e.g., Ellis et al., 2015; Johnson et al., 2020). For example, Johnson et al. (2020) indicated the importance of graphs as representing relationships between quantities when students work with quantitative and covariational relationships in modeling tasks.

Modeling tasks have been indicated as providing rich opportunities for students to engage in functional thinking and therefore, to interpret function as a covariational relationship between quantities (Lagrange & Psycharis, 2014). Existing research shows that in many upper secondary level mathematics curricula, teaching approaches to functions through modeling tasks based on real world situations overemphasize the use of algebraic methods to support translation into the mathematical language, but overlook work that needs to be carried out earlier in other spaces (e.g., geometrical). This constrains students' passage to the algebraic space and their meaningful engagement in functional thinking (Minh & Lagrange, 2016). Lagrange (2018) proposed modeling situations as a basis for tasks that facilitate students' meaningful learning of mathematics through models that have different relationships to reality and mathematics. The focus was on helping students to make connections between these models. These models include mathematical objects, their relationships, expressions, and related mental images. In literature, modeling problems are characterized as authentic (the noun "authenticity") when being genuine (Vos, 2011). According to Kaiser et al. (2011), designed tasks about authentic modeling situations are somewhat simplified from industry so that they can be embedded into reality. Existing research indicates students'

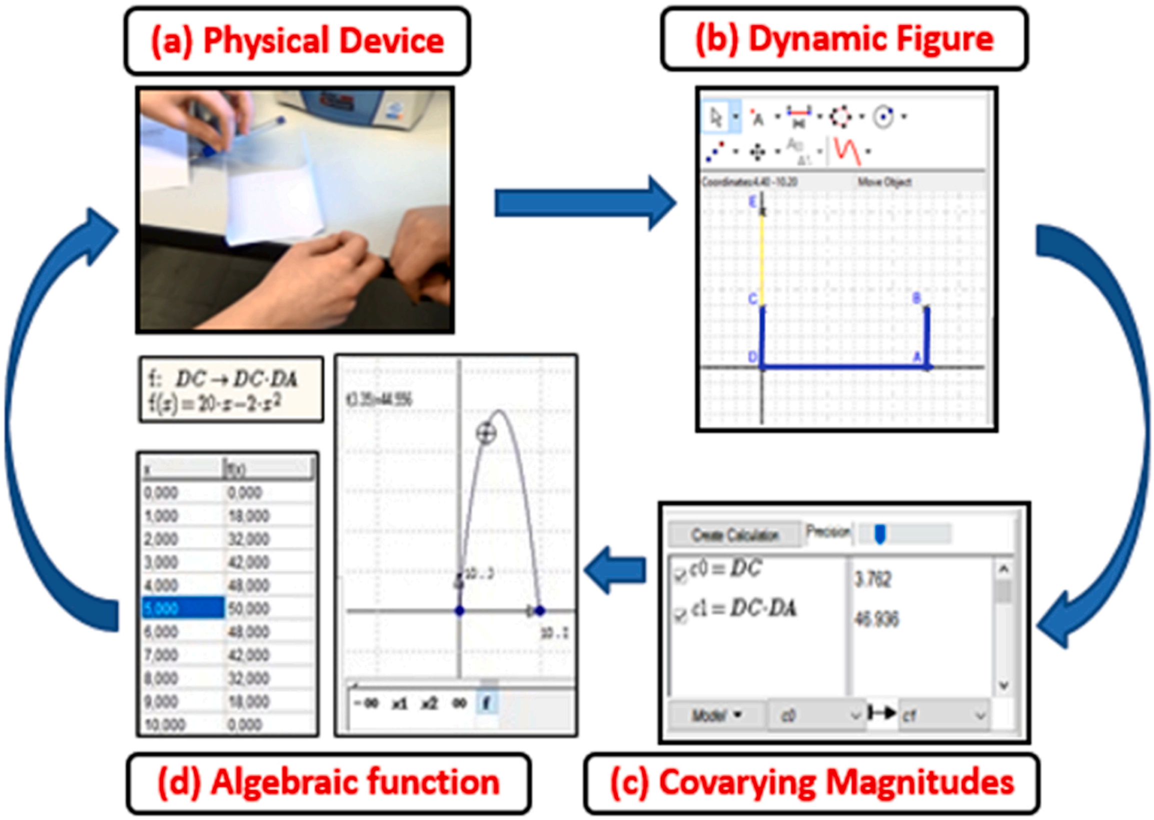


Fig. 2. Problem solving across the different settings of the modeling cycle.

benefits on dealing with tasks based on authentic situations as such tasks can facilitate students' engagement with school mathematics and representations and help them appreciate the utility of mathematical ideas (Palm, 2009). However, "educationalization" of authentic situations to be accessible by students is rather complex. One way of educationalization proposed in the literature is to design tasks inspired by authentic situations that orient diverse distances between school mathematics and authenticity (Dierdorp et al., 2011). The modeling tasks in our study are inspired by authentic situations, have varied connections to authenticity and involve the use of digital tools to facilitate students' conceptualization of function as a covariational relationship between two quantities.

As regards modeling situations targeting students' learning of function through real-world problems and digital tools, Lagrange (2014) described the *modeling cycle* using four settings based on the use of the digital environment Casyopée. This learning environment was created to facilitate students' treatment of functional relationships while students work with different models. The four settings of the modeling cycle are: (a) a physical device (e.g., a simulation model), allowing students to experiment; (b) the dynamic figure resulting by modeling the dependencies in a digital tool (e.g., a dynamic figure in a dynamic geometry window); (c) the co-varying magnitudes (e.g., the measures of the dynamic figure); and (d) the algebraic functions that model problem (Fig. 2).

Lagrange (2014, 2018) distinguished four successive models within each one of the different settings of the modeling cycle (e.g., students' work with material objects/devices can define a model called manipulatives) and paid special attention to students' passages and linkages between these models. We consider that this transition across different settings and the corresponding models (i.e., the passage from experimentation with the dependencies in the physical device to algebraic functions) is complex as it is mediated by different kinds of covarying quantities (e.g., magnitudes, variables). In our study, we examine students' trajectory from quantitative and covarying relationships to functional relationships in the complex path from physical context to variables. In this path, a function emerges first as a dependency between attributes of quantities (e.g., physical or geometrical objects), then between magnitudes (i.e., quantities' measures), and finally between variables (i.e., mathematical function). We pay special attention to students' transition to the different models and the corresponding covarying quantities involved in the trajectory.

2.3. Learning trajectories

We adopt learning trajectories (Clements & Sarama, 2009) as a framework to create and analyze students' learning paths. A learning trajectory includes three essential elements: (a) a mathematical goal, (b) educational activities to achieve the goal, and (c) a description of the development of students' thinking while they are engaged with the activities. The idea of hypothetical learning trajectories (Gravemeijer, 2004; Simon, 1995) provides a guideline in designing tasks and results to an instructional sequence. A hypothetical learning trajectory is a useful initial hypothesis of teacher's planning, which fosters intended learning, but in some cases might deviate from the actual learning trajectory (Simon, 1995). Especially in the digital technologies context and considering the task's complexity, the actual learning trajectory might be different due to the plurality of the events that the digital tools bring to the fore (Sacristán et al., 2010).

In this study, we adopt learning trajectories as a framework both at the level of design and analysis. Inspired by the framework of Carlson et al. (2002) about covariational reasoning, which addresses explicitly the idea of rate of change, we develop a hypothetical learning trajectory to highlight a potential development of students' thinking about function as a covariational relationship between two quantities. At the level of analysis, the comparison between the hypothetical learning trajectory and students' actual trajectories

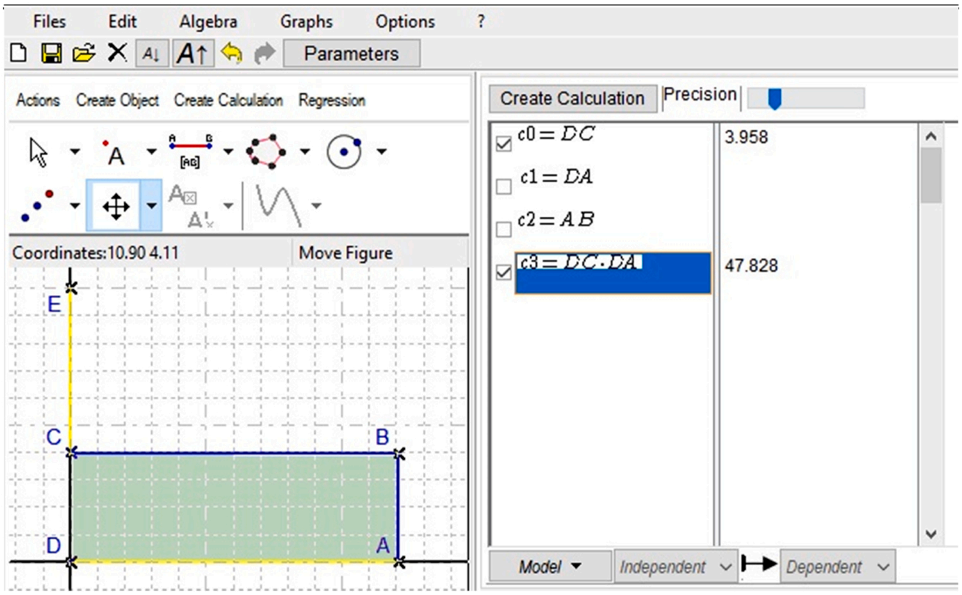


Fig. 3. DG, geometric calculations, and automatic modeling.

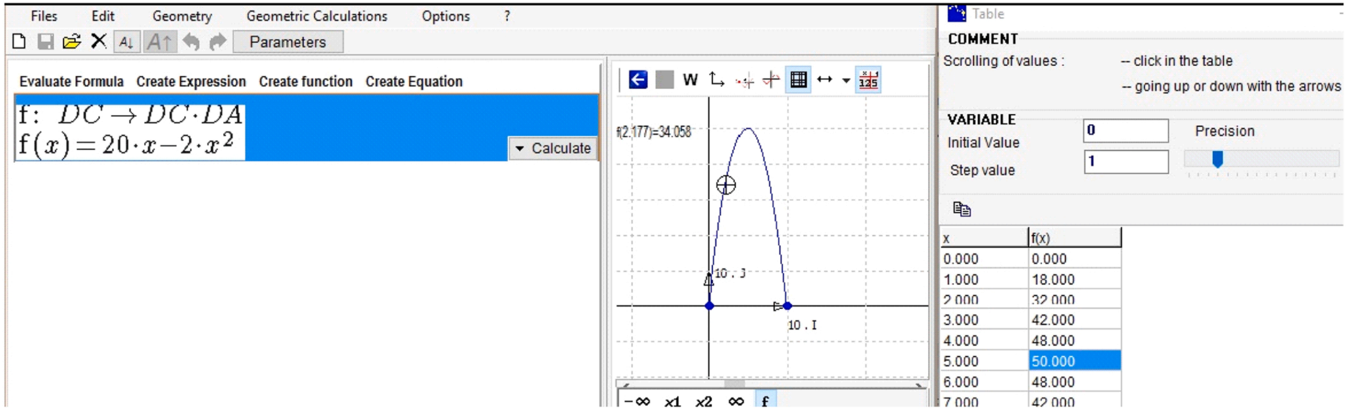


Fig. 4. Algebra window, graph, and table of values.

allows us to describe students' progression in relation to their expected conceptualizations.

Our research aim is to explore the potential of 11th grade students to conceptualize function as a covariational relationship between two quantities in the context of modeling tasks involving the use of different models, manipulatives, and digital tools. Also, we aim to highlight the connections between the models and their role in students' conceptualizations.

3. Tools and tasks

3.1. Casyopée

Casyopée is a digital environment that was developed to enhance the teaching and learning of functions through multiple representations (Lagrange, 2010). It combines a dynamic geometry (DG) window and an algebra window that are interconnected and allow students to deal with geometrical dependencies and algebraic settings at the same time. While modeling a problem, students can create free or fixed geometric objects (e.g., points, segments) in DG, define independent and dependent objects in DG and consider them as attributes of quantities, create the correspondent measures in a geometric calculations window (e.g., lengths and areas symbolized as c0, c1, c2, etc.), and investigate their covariation through the "automatic modeling" functionality.

The students can check whether a function can be defined using two covarying magnitudes (e.g., c0 =DC and c1 =DC*DA, Fig. 3) through the "automatic modeling" functionality by selecting the first one as the independent variable and the second one as the dependent one. If a function can be defined from the selected magnitudes as variables, its algebraic formula is automatically generated in the algebra window (Fig. 4); otherwise Casyopée provides appropriate feedback that it cannot create the function from this pair of covarying magnitudes. Finally, a function can be interpreted using multiple integrated representations, such as the algebraic formula, the table of variables' values, and the graph for further exploration.

3.2. Task sequence

The designed tasks are optimization problems (see Appendix) inspired by authentic situations. We collaborated with professionals to become familiar with authentic practices from specific fields (e.g., industrial design, architecture, fuel vending). Following Lagrange (2014) cycle of modeling, we designed tasks with the aim to promote increasingly sophisticated conceptualizations of a covariational relationship by students. Their work with the tasks involved experimentation with manipulatives or relevant material, modeling of situations with the use of digital tools or work with given models, definition of a function and use of its available representations so as to provide a solution. The *Gutter Design* task refers to the sensual experience of the covariation with manipulatives and tools for the optimal design of a rain gutter. The *Front of a Store* task relates with the covariation between areas and the transition to the rate of change taking into account the restrictions of an architecture design. The *Fuel Tank* task refers to an extensive experimentation with covariation and rate of change in the fuel station context.

The *Gutter Design* task requires maximizing water capacity of the designed gutters. The task was inspired by the authentic practice of experts searching for the optimal design of gutters. The aim is to engage students in conceptualizing function as a covariational relationship by experimenting sensually with the manipulatives and Casyopée's functionalities. The design includes students' engagement in: experimenting using a rectangular piece of paper (10 by 20 cm) as a prototype of the metal plate (Fig. 5) used in professionals' authentic practice; observing the covariations and expressing the algebraic relationship using a variable; designing and exploring a dynamic rectangle in Casyopée and keeping track of the calculations, so as to examine each different folding (e.g., folding the 10 cm side as 3–4–3: E = 12 cm^2); experimenting with covarying magnitudes and creating the function $f(x) = 20x - 2x^2$ that models the problem [$x \to x(20-2x)$, where x is the side length]; and reaching a solution using function representations.

The *Front of a Store* requires optimal design based on authentic construction site restrictions provided by an architect. The authentic context lies in the background and priority is given to students' focus on the construction elements and their restrictions/relations. The first part of the task includes a discussion of critical construction elements by showing relevant photos from an architects' work to help students to concentrate on relations that underlie construction elements that appear in the front of the store (e.g., the area of the supporting beam changes along with the distance between the vertical columns). The second part of the task relates to a pre-made dynamic front in Casyopée (Fig. 6). The width of the supporting beam (DA) is dependent on the distance between the vertical columns (OE) and works as an important parameter of the construction. The aim is to let students experiment with covariation changing the dependent variable and to introduce them to the rate of change. The task requires the maximization of the area of the upper shop

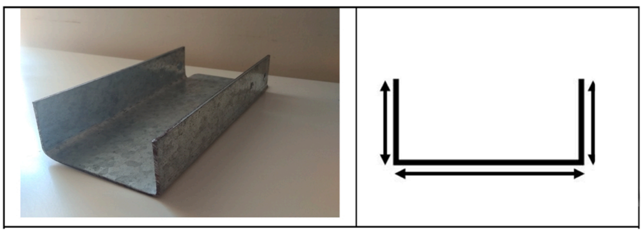


Fig. 5. Gutter.

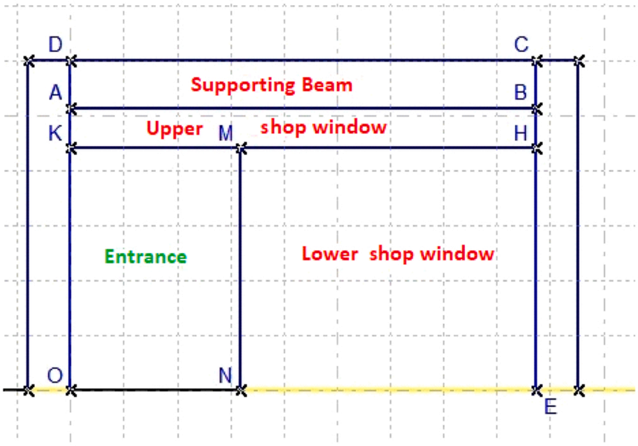


Fig. 6. Front of a Store.

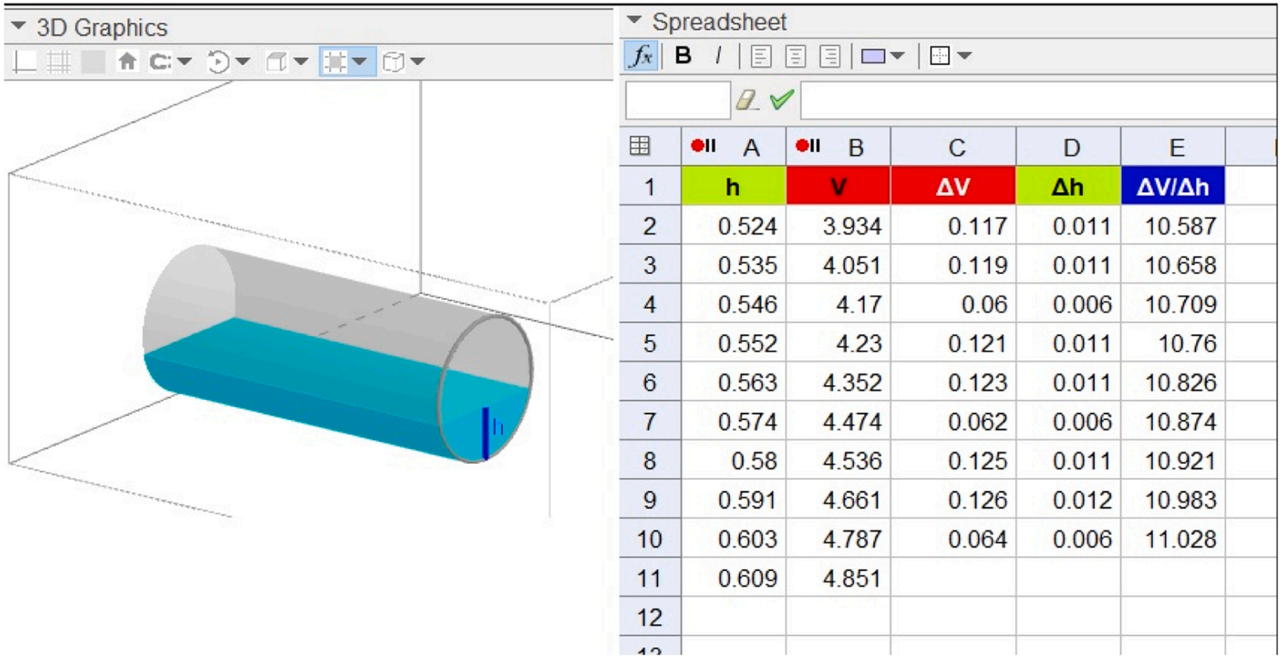
window (ABHK) and it is enriched with the exploration of the summary of two quantities and the relation with the covariation and the rate of change. The covariation appears from the experimentation of students with areas in DG to their work with functions in the algebra window and the related representations. The corresponding functions are: (a) the area of the upper shop window (OE→ABHK), (b) the area of the lower shop window (OE→MHEN), and (c) their summary (OE→ABHK+MHEN). The students are asked to identify the covarying quantities in the DG figure; identify the distance between the columns, in which the upper shop window is maximized; create the total area, and experiment with the rate of change of the areas.

The *Fuel Tank* is inspired by the authentic fuel vendors' practice of measuring the fuel in the cylindrical tanks placements (vertically and horizontally) using a measuring rod. Again, the authentic context remains in the background and the task requires students to explore the way that the values taken by a measuring rod change in relation with the volume of the fuel inside the tank for the two tank placements. The first part includes a classroom discussion of a fuel vendor's practice of measuring the fuel using a measuring rod. In second part of the task a GeoGebra file was given to the students including a dynamically-filled, horizontally positioned tank and it was interconnected with a dynamically-filled table of values (Fig. 7). It required students to describe the refilling process, provide the algebraic solution of the problem using the measuring rod and to make a height-volume graph. The instruction goal was students' engagement in working further with the graph and the relation to the rate of change, according to the main question: "Explore if a change of ¼ of the fuel height corresponds to ¼ of the whole volume of the tank." This question requires thinking covariationally and comparing different instances of the horizontally positioned tank. The function in the vertical case was $f(h) = h\pi r^2$, where h is the height of the fuel volume in the tank as shown by the measuring rod and the r is the base radius. In the horizontal case, the algebraic solution refers to a series of steps for the calculation of the volume of the fuel. The covariation appears in the exploration of the vertical placement of a tank and then it is present in the horizontal placement of a tank mainly during the creation of the graph.

3.3. A priori analysis of tasks

The tasks follow the same structure and the mathematical goal in terms of learning trajectories is to conceptualize function as a covariational relationship between two quantities. The hypothetical learning trajectories from simple to more advanced conceptualizations are: (I) intuitive approach to covariation through manipulatives; (II) covariation between attributes of quantities; (III) covariation of magnitudes; (IV) covariation of variables; and (V) rate of change. Thus, a priori we expected students to experience covariation intuitively (Gutter task), then to experiment with the quantities and the magnitudes in Casyopée (Gutter and Store tasks) and finally to focus on the rate of change (Store and Tank tasks). We defined a priori the following models related within the different settings of the modeling cycle: (1) manipulatives, (2) dynamic geometry, (3) measures, and (4) functions.

The manipulatives model is the first model during the modeling approach and refers to the experimentation with physical objects. In this model students' work resembles in a way to experts' authentic practice. The covariation appears between attributes of quantities, however there is poor dynamicity of the model due to physical constraints. The dynamic geometry model is the expression of the physical device in a dynamic geometry environment. In this model there are two characteristics: (a) the constraints of the construction play a critical role for the modeling of the problem and (b) the dynamic construction will offer interactivity of the situation. The covariation appears between attributes of objects or quantities' measures. The measures model is a result of the quantification process as the covariation appears between quantities' measures rather than geometrical entities. In the functions model, a function is exported as a result of a selected pair of two covarying quantities (e.g., a length of a rectangle and an area) and the students work with the different function representations. This step is not obvious since it requires a robust understanding of the situation and a strong mathematical background (i.e., the selection of independent and dependent variables). Table 1 shows a characterization of the tasks in relation to learning trajectories and models.



	A	B	C	D	E
1	h	V	ΔV	Δh	ΔV/Δh
2	0.524	3.934	0.117	0.011	10.587
3	0.535	4.051	0.119	0.011	10.658
4	0.546	4.17	0.06	0.006	10.709
5	0.552	4.23	0.121	0.011	10.76
6	0.563	4.352	0.123	0.011	10.826
7	0.574	4.474	0.062	0.006	10.874
8	0.58	4.536	0.125	0.011	10.921
9	0.591	4.661	0.126	0.012	10.983
10	0.603	4.787	0.064	0.006	11.028
11	0.609	4.851			
12					

Fig. 7. *Fuel Tank* in GeoGebra.

Table 1
A priori analysis.

	Gutter Design	*Front of a Store*	*Fuel Tank*
Learning trajectory	(I)-(IV)	(II)-(V)	(II)-(V)
Models (modeling cycle)	(1)-(4)	(2)-(4)	(2)-(4)

4. Methodology

4.1. The context of the study

Under a design research perspective (Prediger et al., 2015), we initially designed the three modeling tasks by collaborating with professionals and we implemented them in three different schools in Athens during two years (design-redesign cycles). In this paper we analyze data from the second year of implementation in one public school (4 two-hour sessions over one month) that corresponds to the redesign cycle and consists of richer data as the tasks have been modified to facilitate students' engagement in mathematical inquiry. The class teacher and a researcher had three, five-hour redesign sessions regarding the implementation of the tasks and the use of digital tools in the classroom so as to let the researcher act independently in the classroom. The changes from the initial tasks were to include an initial discussion for the engagement of students in the modeling task and to enhance the inquiry dimension of the tasks by making them more open. All the tasks followed the same implementation: (a) task launching by showing students a short video (1–2 min) or photo presentation related to the authentic situation to help students link the given task to the situation; (b) an initial whole class discussion to bring to the fore aspects of the authentic practice and let students recognize the covarying quantities working mainly on the manipulatives or the DG models; (c) cycles of autonomous student work in groups followed by whole class sessions to discuss students' work; and (d) a final whole class session devoted to the presentation and discussion of students' solutions. The 25 students (grade 11) worked mainly in pairs. The students were taught about function according to the Greek mathematics curriculum, which favors the correspondence aspect of function and reference to covariation is limited. The instructional sequence was implemented by taking into account the key features of inquiry pedagogy, such as problem-solving culture, exploration, autonomous learning, student collaboration, and working in a scientific manner (Artigue & Baptist, 2012). The class teacher implemented the tasks and acted as facilitator, challenging the students with specific reference to the authentic context. A researcher was a participant observer who also triggered the students to explain their reasoning within the group. A camera and four recording devices were used for data collection and specifically captured two focus groups and the whole classroom experimentation. We selected the focus groups according to their experimentation with Casyopée in the familiarization phase.

4.2. Data collection

The data collected consists of video recordings of two focus groups and audio recordings of the classroom implementation. In this paper, we analyze video recordings of two focus groups (group 1: S1, S2 - group 2: S3, S4) that were studied in their autonomous work and whole-class discussions (e.g., during the presentation of their solutions) throughout the implementation of all the tasks. The data were fully transcribed for the analysis.

4.3. Method of analysis

The analysis focused on progressively identifying more sophisticated ways of thinking in students' activity of conceptualizing function as a covariational relationship between two quantities to highlight the developmental progression in a learning trajectory. In the first analysis phase, the transcripts were complemented by a separate column in which we inserted comments related to students' actions and gestures from the video and audio recordings. In this phase, we adopted a grounded method (Charmaz, 2014) to identify and code (open coding) incidents of students' actions and utterances in the data that indicated ways students referred to the core dimension of covariation, that is, covarying quantities. Two researchers carried out the coding independently based on three criteria: (a) the kind of dependencies in terms of quantities, magnitudes, and variables, the model in which the students worked (manipulatives,

Table 2
Coding of incidents.

Task number	Phrase / Utterance from incidents	Model	Tool	Code
1	As this length [*of the cross-section side*] changes, the area changes too	Manipulatives	Paper	Interdependence of covarying quantities
2	The upper shop window becomes bigger and bigger. Also, the width of the supporting beam decreases, too	Measures	Geometric calculations (Casyopée)	Direction of covarying magnitudes in the geometric calculations window
2	The g(x) increases as the x increases. [*g(x) is the function of OE→MHEN*]	Functions	Function representations (Casyopée)	Covariation of variables using algebraic symbolism
3	The rate of change increases constantly for any interval	Functions	Function representations (GeoGebra)	Instantaneous rate of change

DG, measures, and functions), and the tools they used; (b) how the students conceived the covariation of the dependent quantities/magnitudes/variables using both codes from the existing literature with reference to the rate of change that was part of our research interest (e.g., dependency, direction, covariation, rate of change, for example, Carlson et al., 2002) to highlight expected learning trajectories and emerging codes stemming from the specificities of the current context (e.g., use of available tools in Casyopée); and (c) if and how students expressed the aforementioned conception of covariation (e.g., through the use of algebraic symbolism, gestures, verbal descriptions - see Table 2).

Using constant comparisons, two researchers cross-checked similar incidents in the data, dismissed non-relevant codes, and reached an agreement in the final coding. The above codes were then grouped according to their level of mathematical sophistication and the approximate emergence in students' thinking across the teaching sequence (Battista, 2004). From this grouping, a preliminary progression capturing students' trajectory of conceptualization of covariation was checked, resulting in the initial characterization of the different ways in a list. This characterization indicates how students' thinking shaped the actual learning trajectory (Clements & Sarama, 2004).

In the second analysis phase, representative conversational episodes (i.e., a speaker's turn or a sequence of speakers' turns) involving any student activity of covariation were identified and subsequently coded in relation with the model that the students were working with. An additional coding has been implemented to indicate identified influences of other models. These episodes allowed us to illustrate the characteristics of each conceptualization, the current model, the kind of the involved covarying quantities, and the presence of influences by other models. Again, two researchers checked similar episodes in all the data and refined the progression until no new codes emerged in the data. This led to the final characterization of the developmental progression resulting in seven conceptualizations of function as a covariational relationship. Finally, by tracing the sequence of codes in the different models we highlighted the connections between them.

5. Results

As a result of our analysis, we present the actual learning trajectory from dependencies between attributes of quantities and covariation between magnitudes to the covariation of variables and the rate of change (Table 3).

In the next sections we show the occurrence of students' learning trajectory in the focus groups during the instructional sequence while working with the different models and we highlight students' transitions between them.

5.1. Manipulatives

In the manipulatives model the students could identify dependencies between attributes of quantities (interdependence, e.g., the cross-section area is dependent from the side length, Figs. 8–10). Students could also relate changes in one attribute in relation with another and frame the initial part of the trajectory. In addition, in this model the students could extend the interdependence of quantitative relationships to the correlation of a covariational relationship by recognizing that their variations are connected, (correlation, e.g., S1: "As long as one attribute changes, the other changes too" referring to the side length and a corresponding area).

For example, the following episode from the *Gutter Design* implementation took place during the initial whole class discussion following students' experimentation with the paper model to find ways to maximize the quantity of water passing through the gutter. In the excerpt, the four students from the focus groups (Group 1, S1 and S2; Group 2, S3 and S4) are discussing with the teacher (T) about the core elements of the problem. S3 illustrates their experimentation by folding the paper model. Specifically, S3 shows the interdependence of the two gutter sides in order to maximize the amount of water passing through the cross-sectional area.

76 S1: [*Showing the paper model folded] I think that the maximum water volume depends on the maximum volume of the metal sheet (*Fig. 8).
77 S3: [*Showing the cross section of the gutter] It depends on the area of this figure.*
78 T: Why?

Table 3
Students' learning trajectory.

Developmental progression	Description	Models
Interdependence	Identifying mutual dependencies of covarying quantities (e.g. physical/geometrical objects)	Manipulatives, DG
Correlation	Relating changes in one magnitude in relation with the other (without indication of increase or decrease)	Manipulatives, DG, Measures
Direction	Relating the direction of change of one magnitude with changes in the other (i.e., increase or decrease)	DG, Measures
Variable selection	Identifying the pair of magnitudes as a pair of dependent and independent variables	DG, Measures, Functions
Coordination	Coordinating changes in the independent variable values with changes in the dependent one in different function representations	Functions
Amount of change	Coordinating the amount of change in the independent variable values with changes in the dependent one in different function representations	Functions
Difference quotient (Uniform and instantaneous rate of change)	Coordinating the rate of change of the function values with uniform or indications of continuous changes in the independent variable through the difference quotient	Functions

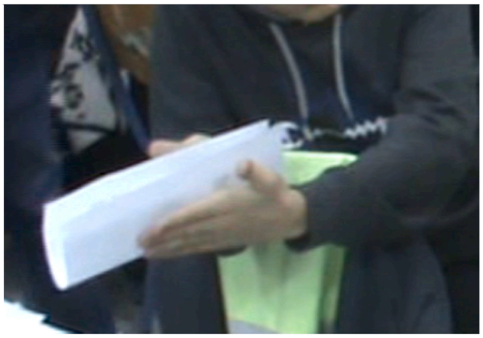


Fig. 8. Piece of paper model.

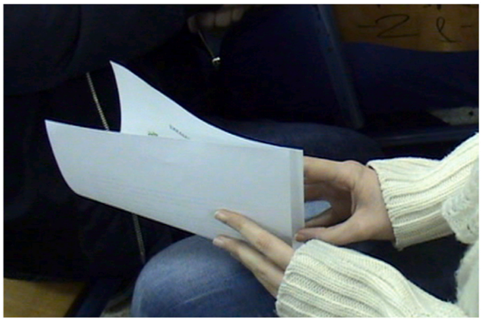


Fig. 9. Identifying the restrictions of the gutter.

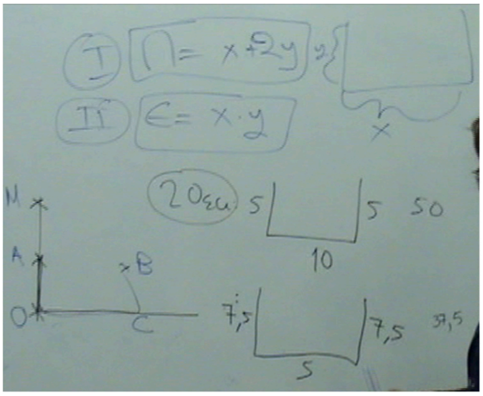


Fig. 10. Paper folding as a basis of DG.

79 S3: Because the volume would be this [*showing the cross-section] multiplied by the length of this whole thing [gutter].*
80 S4: I think that it [*the maximum water volume] is a combination of both, the lower part [shows the bottom of the folded sheet] and the sides [shows the height] since these retain the water.*
81 S2: Therefore, this product [*of the bottom and gutter side] should be as large as possible.*

In this excerpt, in which the students have been working exclusively in the manipulatives model, they are able to quantify the problem and identify the interdependence between the changes of the sides and the cross-sectional area. At the beginning of the episode, S1 suggests that the maximum amount of water is related to the volume given by the folding of the metal sheet – and thus, tries to identify the two quantities. Then, S3 conceives that maximizing the area of the cross section was sufficient for solving the problem (lines 77, 79) and S4 combines the changes of both sides as they fold the sides of the paper model (line 80). We note that in their responses, the students directly link the interdependence of the dimensions of the rectangular cross section of the gutter to the water volume and the way that water is supposed to pass through the gutter. Finally, S2 replaces the quantity of volume using the cross-sectional area and realizes that the product of the two dimensions should be maximized (line 81). This model is characterized by a

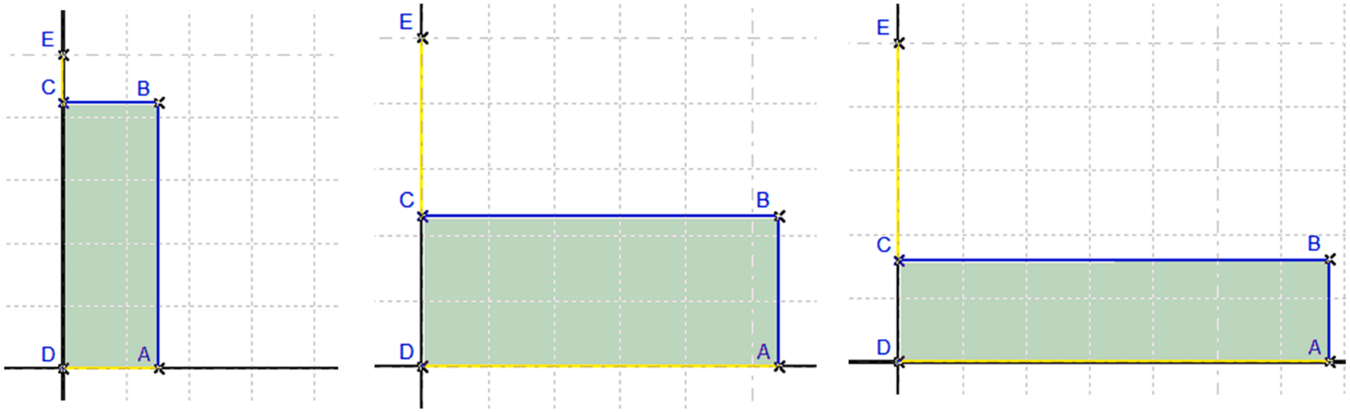


Fig. 11. Instances of the dynamic rectangle in DG.

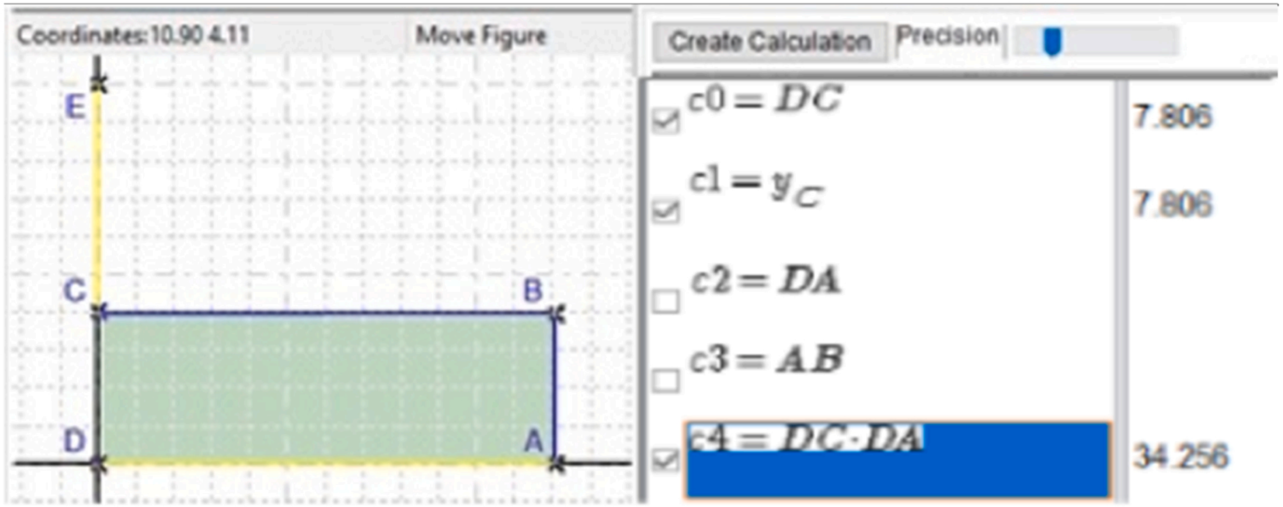


Fig. 12. DG and geometric calculations windows.

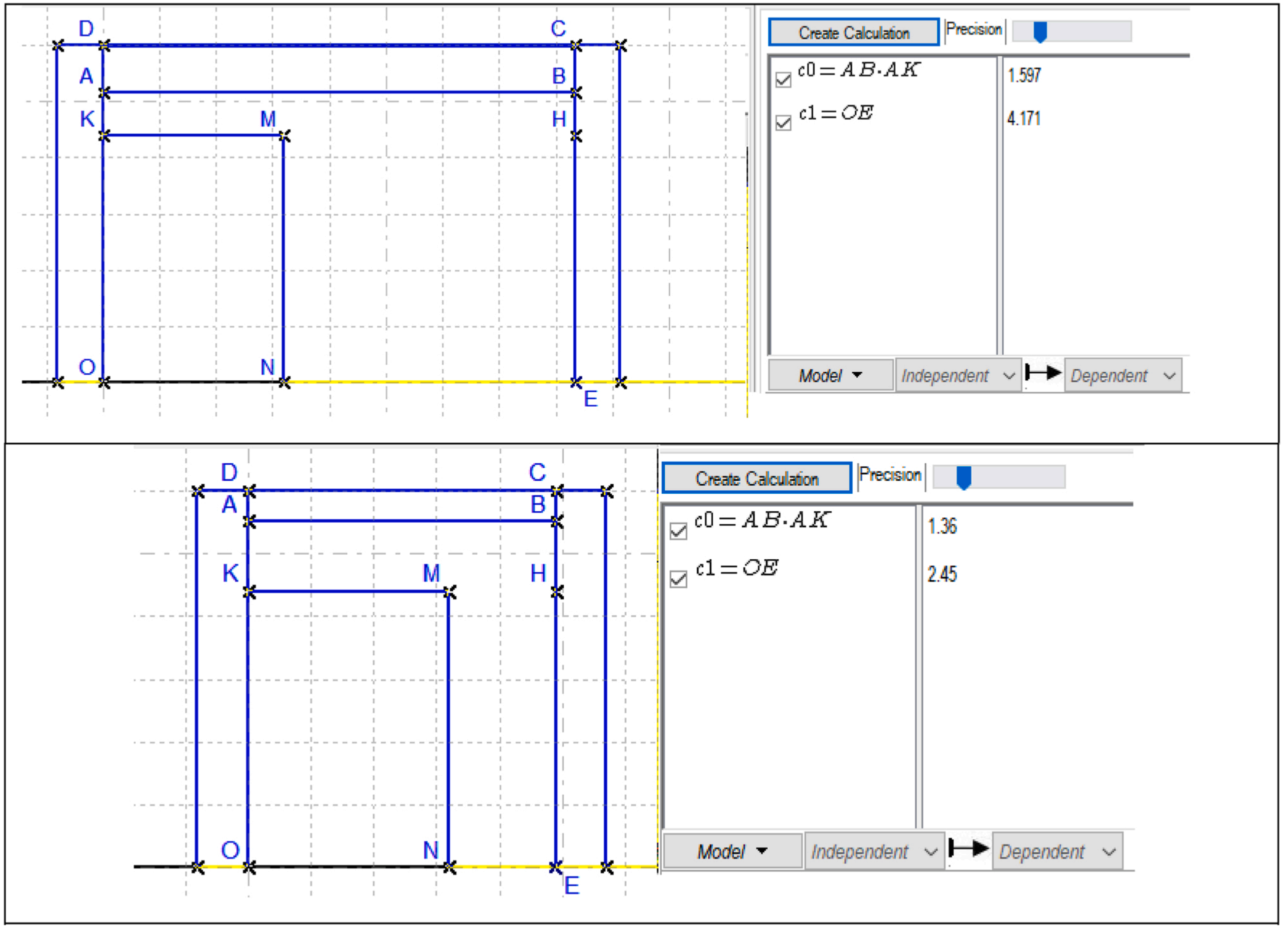


Fig. 13. Instances in DG and geometric calculations windows.

vivid interplay between the mathematical world and the task in students' references to dependencies between attributes of quantities. For example, in the above episode, the students initially talk about the volume of water, then about the volume of the gutter, and then about a combination of sides. They finally quantify their product, which represents the cross-sectional area.

5.2. Dynamic geometry

In this model the students experimented with the DG window of Casyopée. Building on their recent conceptualizations (interdependence and correlation) they could relate the direction of change of one magnitude with changes in the other (i.e., increase or decrease, S2: "As the one magnitude increases the other increases, too", referring to the side length and a corresponding area). A crucial difference from the previous part is that the students referred mainly to magnitudes (Figs. 11–12). Often the coordination of dragging to the corresponding changes of covarying magnitudes' values in the geometric calculations window seemed to facilitate students' attention to the correlation and the direction of change.

For example, during a whole class discussion in the implementation phase of *Gutter Design*, the teacher talked with some students from different groups about the elements and restrictions of the construction (see Fig. 12, the construction in DG). The students of group 1 started to model the problem in DG by proposing the free point C and the dependent points A and B of the gutter.

172 T: You will need one point for the lower part of the gutter and one point which describes the maximum folding. Then we need another point between these two points to describe every time the different folding, but first of all, we need to find the restriction of the construction.
173 S1: I propose to put point D on (0,0).
174 S2: We have to create a point E as (0,10) in case we fold the metal plate in the middle so that we get a segment for positioning the free point C.

After creating C, the students observed the folding in order to find an expression for the x-coordinate of A. Most of the groups in the classroom attempted to find it through solving the equation $x + x + y = 20$ for y (e.g., S3: "I tried to create A, as (20–2 *x,0) but it did not work!", S4: "We created A as (20–2 *yC,0) and it worked! [*using y-coordinate of point C*]", S1: "As for us, we created A as (20–2 *DC,0) and it worked also, because as we move point C, point A is moving too!"). Here, the students work in the DG model as a result of their prior work with the manipulatives model and make sense of the connection of the covarying magnitudes (the connection of covarying lengths based on points A and C) and later on they could give more details about the direction of change. In this episode, S2 identifies that the restriction of the gutter must be indicated by point E and then most of the group works further to identify the coordinates of point A using a simple equation. Then, using a single free point and connecting it to the other points, S1 indicates the correct coordinates for the dependent point A and conceptualizes the correlation of magnitudes. In this whole-class interaction, S1 indicates the covariation, by saying: "as we move point C, point A is moving too" and it emerged from the experimentation with Casyopée in the DG model.

5.3. Measures

In this model the students could extend their previous conceptualizations (correlation and direction) and engage in exploring the changes in covarying magnitudes. More specifically, they could identify a pair of magnitudes as a pair of dependent and independent variables. In this model there are two main steps: (a) the creation of calculations and (b) the conceptualization of covarying magnitudes as variables. However, the students appear to be sensitive as regards the selection of the independent and dependent variables. In most cases the students exploited their preceding experimentation with the dynamic figure in the DG to support their selection (Fig. 13).

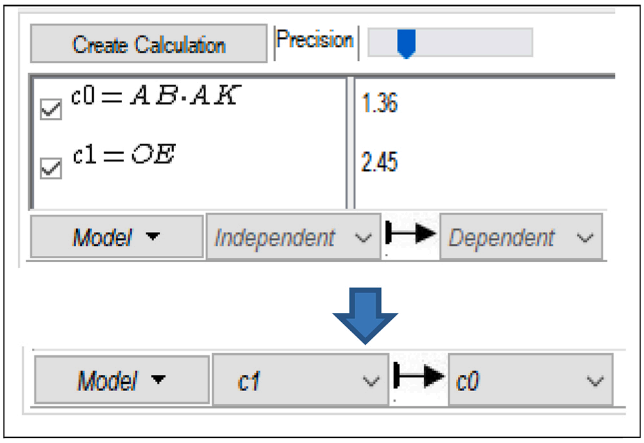


Fig. 14. Selection of magnitudes as variables.

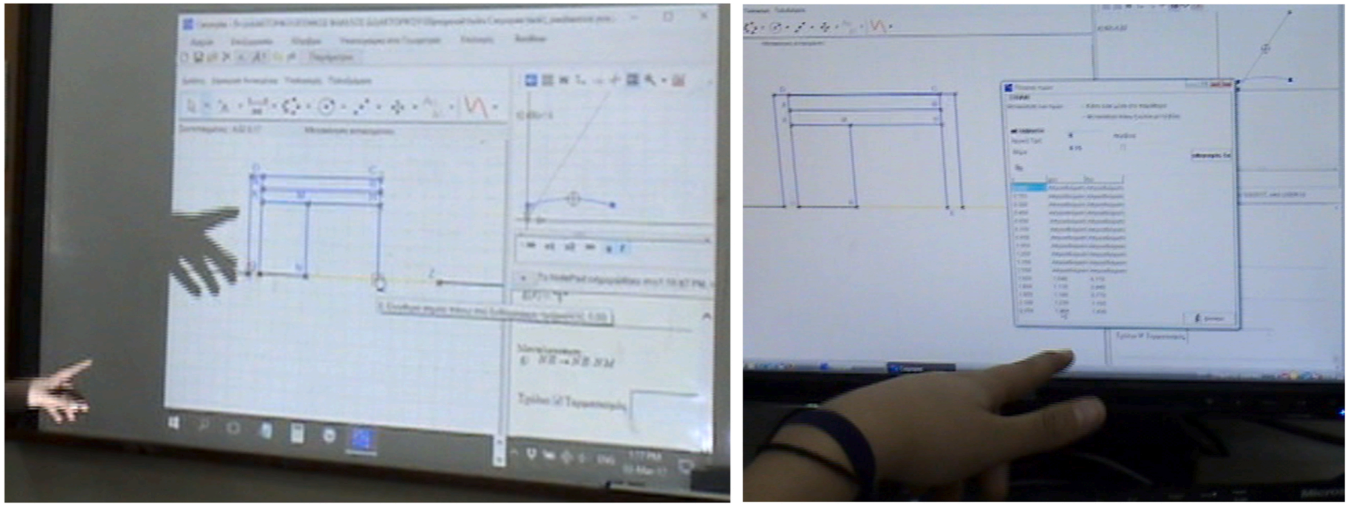


Fig. 15. Students showing and coordinating functions' representations.

The following episode from the *Front of a Store* shows how during their autonomous work, focus group 1 students' work refers to the appropriate selection of geometric calculations to create a function. After experimenting with erroneous pairs of dependent and independent variables, S1 proposed creating the area of the upper shop window as a dependent variable of the appropriate function and look for the independent variable.

652 S1: In the geometric calculations window we must create only the area of the upper shop window, which will change in relation to another independent variable. (R).
653 S2: So, here is the calculation of the area of the upper shop window [S2 *creates a geometric calculation]*.
654 S1: OK, select this area as a dependent variable.
655 S2: Wait. We must also calculate OE, because we move point E. I think this will be the independent variable.

Here, the students identify the covariation of magnitudes, propose the creation of other geometric calculations in Casyopée, and talk about a pair of variables instead of magnitudes (line 652). Next, they select the area of the upper shop window as a dependent variable (line 654). Finally, S2 proposes the creation of OE as a geometric calculation and suggests it as the appropriate independent variable (line 655) based on the observation of how the dynamic area changes through moving the free point E in the DG window (Figs. 13–14).

5.4. *Functions*

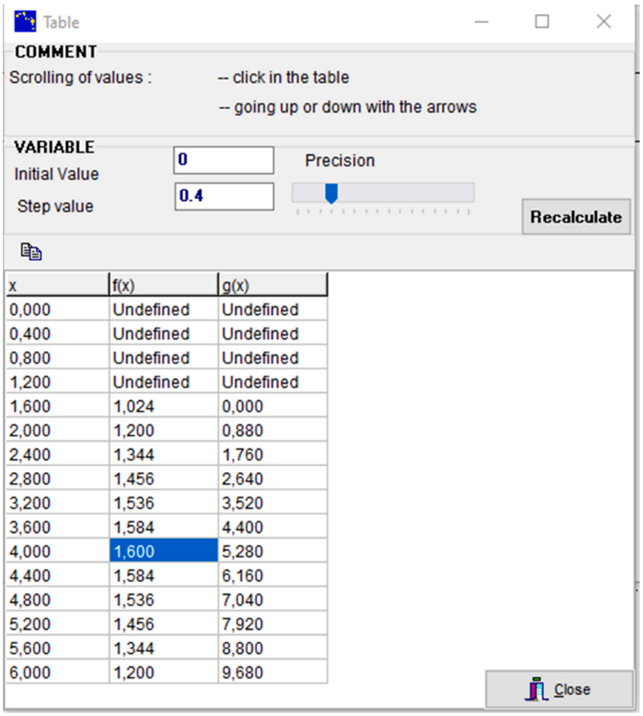
The main characteristic of this model is that the students refer exclusively to variables sometimes using algebraic symbolism and speak about function as a result of two covarying quantities by: (a) linking the newly offered representations of function (algebraic formula, table of values, and graph), (b) identifying the amount of change, and (c) conceptualizing the uniform and instantaneous rate of change through the difference quotient.

5.4.1. *Linking representations*

Initially the students created a function and identified the coordination of changes in the independent variable values with changes in the dependent one of different representations using Casyopée's multiple representations (e.g., S3: "The g(x) increases as the x increases and we can also see it from the table of values and the graph!" - Fig. 15).

5.4.2. *Amount of change*

In the last two tasks, students' focus on the differences between values in successive cells on the Casyopée's function table provided a basis for the conceptualization of the rate of change. This focus was initially characterized by a more systematic examination of the



x	f(x)	g(x)
0,000	Undefined	Undefined
0,400	Undefined	Undefined
0,800	Undefined	Undefined
1,200	Undefined	Undefined
1,600	1,024	0,000
2,000	1,200	0,880
2,400	1,344	1,760
2,800	1,456	2,640
3,200	1,536	3,520
3,600	1,584	4,400
4,000	1,600	5,280
4,400	1,584	6,160
4,800	1,536	7,040
5,200	1,456	7,920
5,600	1,344	8,800
6,000	1,200	9,680

Fig. 16. Table of values for upper and lower shop window: f(x) and g(x).

way that the values change in the function table and in the graph.

For instance, some students focused on equal differences in the different pairs of function values, e.g., around the pair (4, 1.6) in the columns x and f(x) as shown in the table in Fig. 16, such as (3.6, 1.584) and (4.4, 1.584) where similar values appear. However, in the evolution of students' exploration of the values they took a comparative look at the differences of values between different pairs in different function values. For instance, the students' identification that a constant increase of 0.4 in the values of x leads to constant differences of 0.88 in the values of g(x) (see columns of x and g(x) in Fig. 16) challenged them to compare the amount of change between different functions (e.g., f and g).

For example, in the episode below the students' focus on the function values shifts to a focus on their differences favoring a comparative view to the amount of change of different functions and a further elaboration of these. The episode takes place during a cycle of students' autonomous work. Here, they had created the required functions f, with g and h standing for the area of the upper and lower shop windows and their sum. Students also had replied to all questions by proposing the most suitable construction using the table of values and the corresponding graphs (Fig. 17). Before the episode during a whole class discussion, a student wondered how it was possible for h(x)=f(x)+g(x) to be increasing while f(x) was decreasing in a specific interval. In mathematical terms, this question is related to the comparison of the corresponding rates of change. For example, in the following episode S1 and S2 work with the table of values (Fig. 16) and thus, make sense of the amount of change.

860 S2: I see that the differences of the values in the column f(x) are not the same.
861 S1: In contrast, look at the column g(x). Here there is a constant rate in the changes. The amount of change as regards the lower shop window from one value to the next is the same [i.e., 0.88] [*S1 points to the consecutive cells on the table of values. The teacher opens a class discussion and the researcher discusses more with the students.*].
902 S1: The rate of increase of the lower shop window is bigger than the rate of decrease of the upper shop window. The lower shop window is increasing with a specific rate of change. The upper shop window is decreasing with another rate of change. The rate of increase of the lower shop window is bigger than the rate of decrease and thus their sum, which is the total area, is increasing.
903 S2: We found that the rate of increase of the lower shop window is much bigger than the rate of decrease of the upper shop window.
904 R: What do you mean when you refer to the rate of change?
905 S2: The lower shop window increases by 0.88.
906 S1: It always increases.

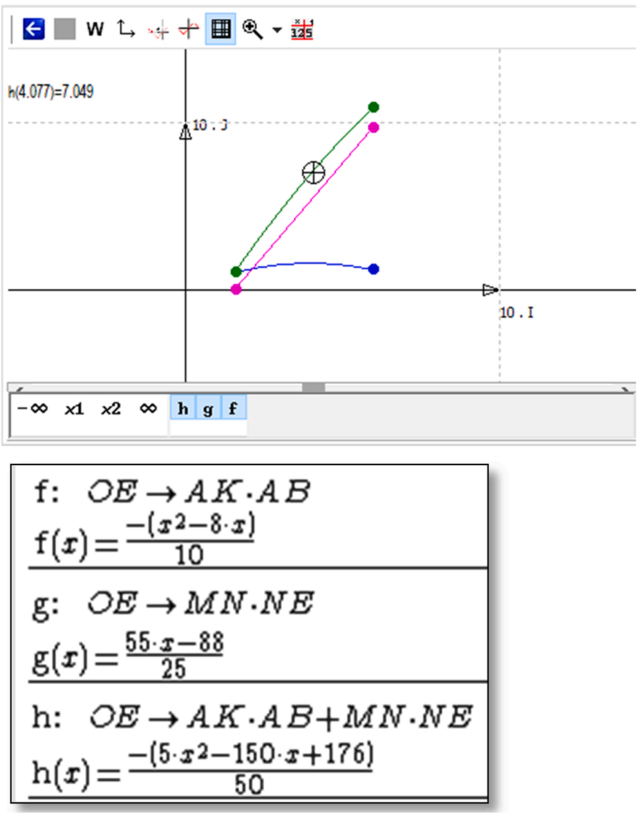


Fig. 17. Graphs and algebraic formulas of f (blue), g (pink) and h (green) functions.

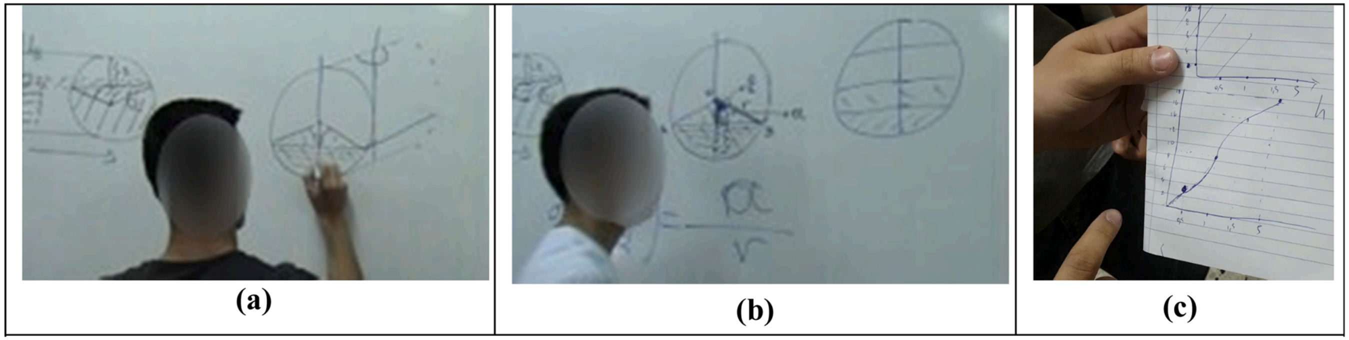


Fig. 18. Students' exploration in *Fuel Tank* (horizontal position).

907 S2: It increases with this constant rate of change and we observe that the upper shop window decreases here for example by around 0.02 then.
908 S1: Here it decreases by around 0.14.
909 S2: It never reaches 0.88.
910 S1: Thus, it makes sense that the total shop window will continue to increase.

Here, students' focus on the function values is followed by an initial focus on the differences of values (line 861). The students try to describe verbally the amount of change through the differences of function values (line 902) to explain the changes of values in the consecutive cells of the table (Fig. 16). S2 compares the two amounts of change verbally (line 903) and identifies that the one is "much bigger" than the other. Then, students see the differences of the function values (f, g, h) and compare them to support this statement (lines 905–909). They then conclude that function h(x) will continue to increase (line 910). The students seem to take a global view of the co-varying values combining the shape of the graph and the table of values, which seems to provide a scaffold that facilitates students to identify the function values differences and to conceptualize the amount of change while dealing with different representations of function.

5.4.3. *Difference quotient*

In functions model some students could conceptualize the average rate of change using the difference quotient. Also, in some cases these students referred to the difference quotient without defining a specific interval or step for the independent variable. These are indications of an early conceptualization of the instantaneous rate of change. Our analysis shows that students' conceptualizations were mainly facilitated by their experimentation with the graph.

For example, the following episode shows how the focus groups' students conceptualized the rate of change as a difference quotient during the final whole class presentation of their work with the *Fuel Tank* task. The students had already found an analytical way to calculate the fuel volume using the measuring rod, and they had created the graph of the covariation of the relevant variables (height-volume) on their worksheet (Fig. 18c). The other groups in the class, had constructed on the board their own solution as regards the height – a volume graph resulting in five different graphs (e.g., straight lines, curved lines) and a debate about which was the correct one. In the realm of the whole-class discussion, students from group 1 wanted to support their own suggestion (i.e., the correct graph involving an inflection point in the middle, Fig. 18c).

To this end, they constructed a graph based on the table of values in the column E of the GeoGebra file (Figs. 19, 20a). The table seemed to facilitate further students' focus on the difference quotient. This was evident in the whole class session near the end of the lesson, when the focus groups' students (S1-S4) were invited to present their solution to the last question of the task (i.e., exploring if a change of ¼ of the height of a tank corresponds to the ¼ of tank's whole volume). During the presentation, the students use the graph created by group 1 in order to support their exploration. In the following extract, S2 recognizes that there is at least one point on the graph that indicates a change of the function curvature.

1127 S2: I think that the critical point is in the middle or in the quarters. (Fig. 20b).
1128 T: OK. Let's see what is changing in this graph.

Spreadsheet

	A	B	C	D	E	F
1	h	V	ΔV	Δh	ΔV/Δh	
2	0.524	3.934	0.117	0.011	10.587	
3	0.535	4.051	0.119	0.011	10.658	
4	0.546	4.17	0.06	0.006	10.709	
5	0.552	4.23	0.121	0.011	10.76	
6	0.563	4.352	0.123	0.011	10.826	
7	0.574	4.474	0.062	0.006	10.874	
8	0.58	4.536	0.125	0.011	10.921	
9	0.591	4.661	0.126	0.012	10.983	
10	0.603	4.787	0.064	0.006	11.028	
11	0.609	4.851				
12						

Fig. 19. Table of values (GeoGebra).

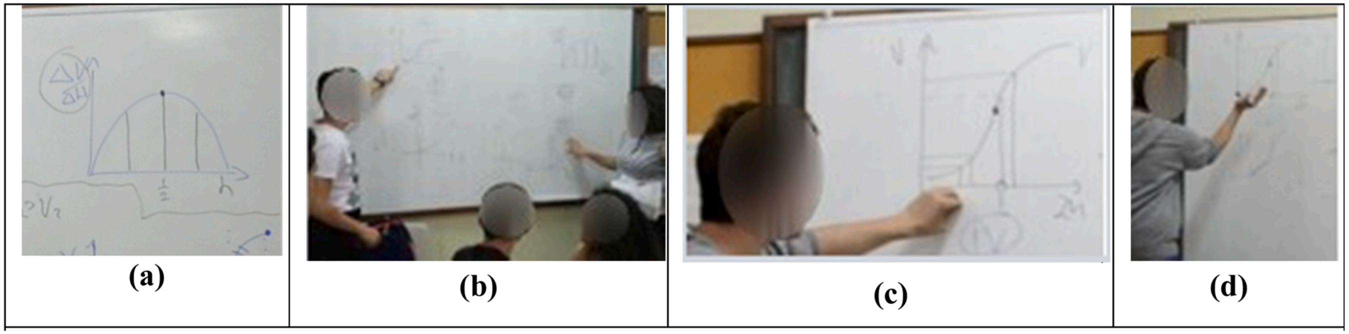


Fig. 20. Students' progress in the graph creation and interpretation.

1129 S4: The rate of change is changing.
1130 T: What do you mean?
1131 S2: We calculated here V/h and….
1132 T: Thus, this is the number in which the volume changes in relation with the height, but here there is a maximum point where the change of the volume in relation with the height is maximum.
1133 S1: It changes in every quarter of the height.
1134 S3: No, it does not change in every quarter.
1135 S2: Using the rate of change we mean how quickly the volume changes in relation with the height.
1136 T: OK. How quickly does it change: fast or slow? Where can you see how quickly the volume changes in relation to the height?
1137 S4: If we take ΔV/Δh here [lower part of the graph] and ΔV/Δh here [near the middle of the graph], then this [ΔV/Δh in the lower part of the graph] is smaller than this [ΔV/Δh near the middle of the graph] (Fig. 20c).
1265 T: *[After some time the teacher asks all students]* Can you tell me at last what is changing in this situation?
1266 S4: The velocity of the fuel volume changes. The fuel that we get for every Δh. This changes, the rate of change. (C).

Here, the students conceptualize the rate of change for different values of height differences (Δh) focusing on the changes in different intervals appearing in the graph. S1 and S3 identify the "critical" point of the rate of change in the middle of the graph (lines 1133–1134) and S2 describes verbally the rate of change (line 1135) without reference to a specific step for the variation of variable h. Then, S4 conceptualizes the rate of change as a difference quotient by comparing the rate of change algebraically in two different instances of the tank filling (Fig. 20c). Also, S4 uses hand gestures to show the changes in the graph curvature and has in mind the way that the quantities change in relation to each other. Finally, S4′s words indicate his potential to conceptualize the instantaneous rate of change concept (constructing, line 1266). At the level of tools, the focus group students experimented with the dynamic manipulation of the tank height in the provided GeoGebra model and the resulting filling of the table values. This experimentation facilitated students' attention to the changes in the function values for different height intervals. The rate of change graph offered by another group provided a scaffold that triggered these students' thinking beyond the differences between values.

5.5. Model connections

The experimentation with the paper model triggered students to search different folds of the piece of paper, which was considered by the students as a simulation of bending a real metal sheet in the professional context and started to examine different cases of folds using specific numbers. The interdependence and the correlation identified in the manipulatives model were critical for a robust functional thinking developed in the other models. For instance, students' experimentation in the manipulatives model facilitated the identification of restrictions in the DG model (e.g., S2: "point E in case we fold the metal plate in the middle") and the correct creation of point A using the equation $x + x + y = 20$ (e.g., "We created A as (20–2 *yC,0) and it worked!"). Also, the use of DG tools (dragging) reinforced the dynamicity of the interaction with the geometrical representation of the covariational relationships. Students' work with DG (e.g., creation of restrictions for the points' coordinates) facilitated them in distinguishing the dependent and independent features of the construction and linking these to specific geometrical entities. This transition from the geometrical objects to quantities' measures was a critical step to the function model. Here, students had to select one magnitude as an independent variable and another one as a dependent in the "automatic modeling" functionality. The automatically exported function provided students the opportunity to work further with the covarying quantities in the functions model through linking multiple representations.

6. Discussion and conclusion

The analysis of episodes reveals the learning trajectory of the conceptualization of function as a covariational relationship between two quantities and indicates students' potential to proceed from quantitative and covariational to functional thinking. More specifically, while working within the different models in the modeling tasks, the students identified the interdependence of the covarying quantities sensually through their experimentation with manipulatives. Additionally, they correlated the changes of one covarying magnitude in relation to the other. Students also made sense of the direction of change and focused on the covarying values. Furthermore, they selected a pair of dependent and independent variables and coordinated changes in the independent variable values with changes in the dependent one in different function representations. Finally, they conceptualized the amount of change and the difference quotient (uniform and the instantaneous rate of change) through their engagement with the digital tools and their comparative view in function values.

In this trajectory, the connections between the models show the gradual progression in students' thinking. The sensual experimentation of covariation, related with the identification of interdependence in the manipulatives model (see manipulatives episode) was a basis for connecting magnitudes in DG. In the DG model, a variety of conceptualizations appears from interdependence (identification of free and stable points and their relation in the dynamic figure) to the variable selection supporting the formalization of covarying magnitudes in the measures model (see Section 5.3). The measures model provided a link between DG and functions models. Specifically, students' engagement with covarying magnitudes in the measures model offered an opportunity for them to link geometrical entities with covarying variables. Additionally, this model allowed students to create a function and to revisit the DG model to justify the selection of independent and dependent variables. Finally, the students worked with a pair of covarying variables in the functions model. This resulted in abstract conceptualizations, such as the coordination of variables linking representations and the rate of change. These findings highlight students' mathematical work within and between the different settings of the modeling cycle that lead to the transition from attributes of quantities to variables.

The results allow a discussion of the role of tools in students' learning trajectory through the corresponding models. In the manipulatives model, students' conceptualization was based on the sensual experimentation of covarying quantities (e.g., with a piece of paper). In the DG model, students' conceptualizations ranged from interdependence to the conceptualization of magnitudes as variables. The digital tools triggered students to identify relationships between coordinates of points, which can be interpreted as an early engagement with algebraic symbolism. In the measures model, the students referred to abstract conceptualizations of magnitudes conceptualizing covariation, direction, and the crucial variable selection that was facilitated by the use of the geometric calculations window and the "automatic modeling" functionality. Finally, in the functions model, students' justifications using various representations of function and the reasoning on these representations facilitated their further conceptualizations of functional relationships, which is in line with existing research findings (i.e., Blanton et al., 2015). By using the available representations, students compared the differences of function values and working with covarying variables made sense of the amount of change. Also, they conceptualized the difference quotient (the uniform and instantaneous rate of change) by using a specific step or by continuously changing it.

The developmental progression of students' trajectory contributes to the existing literature by indicating potential levels characterizing the transition from quantitative and covariational reasoning to functional thinking (e.g., Thompson & Carlson, 2017; Ellis et al., 2020). Specifically, the developmental character of the emerging trajectory allows considering the categorization of the episodes as a basis for level characterization from simple conceptualizations to more abstract ones. Another contribution of the study is that it brings to the fore the diversity of models that facilitate conceptualization of function as a covariational relationship and the diversity of covarying quantities within and between the different models. In particular, the present study informs existing research on the evolution of students' models and their conceptualization of a covariational relationship (e.g., Czocher & Hardison, 2019) by providing a detailed account of students' mathematical work within each model and their interconnections. This allows us to better understand students' passage from dependencies between attributes of quantities to mathematical function, the kinds of quantities involved in the passage and the means by which it was mediated.

As regards the adopted theoretical perspective, the analysis shows that the combination of learning trajectories and modeling appeared productive in addressing our research focus. A priori, this combination was useful to (a) describe the potential progression of students' mental path in relation with the task sequence from an intuitive approach to covariation to a focus on the rate of change and (b) define the models involved in the settings of the modeling cycle (physical objects, DG, covarying magnitudes, function representations). A posteriori this combination allowed a productive description of students' evolving conceptualizations of covarying quantities within and between the different models by taking into account the diversity of covarying quantities in the developmental progression.

This paper is informative in terms of the requirements of task design and implementation relevant to support the conceptualization of function as a covariational relationship through modeling tasks inspired by authentic situations. For example, the analysis may inform task designers, teachers, and researchers about orienting students' activity during classroom implementation by identifying a priori critical points regarding students' conceptualizations. These include (a) the critical role that different models play in students' activity, especially those exploiting the dynamicity of digital tools, (b) the diversity of covarying quantities in the models, and (c) the requirements of task design that integrates authentic situations and modeling. A similar point is also raised by Lagrange et al. (2022). They argue for the need to design tasks that pay specific attention to transitions between different working spaces that can be defined by an authentic situation and the diversity of models related to it. Our study contributes to this needed and ongoing discussion on the process of mathematical modeling concerning the nature of the involved models and their relation to students' conceptualizations. Finally, the study creates new terrains for further research as regards students' conceptualizations when different types of manipulatives and digital tools are involved.

CRediT authorship contribution statement

Kafetzopoulos Georgios-Ignatios: Investigation, Conceptualization, Methodology, Data curation, Resources, Formal analysis, Visualization, Writing – original draft. **Psycharis Giorgos:** Investigation, Data curation, Resources, Formal analysis, Validation, Writing – review & editing, Supervision, Project administration.

Declarations of interest

None.

Appendix A. Supporting information

Supplementary data associated with this article can be found in the online version at doi:10.1016/j.jmathb.2022.100993.

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