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Meanings for Fraction as Number-measure by Exploring the Number Line

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This paper reports on a case-study design experiment in the domain of fraction as number-measure. We designed and implemented a set of exploratory tasks concerning comparison and ordering of fractions as well as operations with fractions. Two groups of 12-year-old students worked collaboratively using paper and pencil as well as a specially designed microworld which combines graphical and symbolic notation of fractions represented as points on the number line. We used the students' interactions with the available representations as a window into their conceptual understanding and struggles in making sense of fraction as number-measure. We report on the features of the available representations from an epistemological point of view, on the design of activities aiming at creating meaningful problem contexts for fractions as well as on the meanings generated by the students by some illustrative examples of their work indicating the potential of the activities and tools for expressing and reflecting on the mathematical nature of fraction as number-measure.

1 INTRODUCTION

The study reported in this paper was carried out within the framework of the integrating collaborative work developed within the European Research Team TELMA (Technology Enhanced Learning in Mathematics, http://telma.noe-kaleidoscope.org/), part of the Kaleidoscope Network of Excellence [1]. The main aim of TELMA was that of networking a selected number of teams [2] having a consolidated tradition in mathematics education with digital technologies with the aim of building a shared view of key research topics in the area, proposing joint research activities, and developing common research methodologies. The present study was designed as one part of a cross-experimentation methodology developed in the project (see Artigue, Bottino, Lagrange, Kynigos, Mariotti. and Morgan, 2006) according to which each TELMA team designed and implemented a short-term experiment with the use of a computational environment developed by another team to support learning in the area of fractions. Such experiments were conducted with the aim of gaining insight into theoretical and methodological similarities and differences. Analysis across national contexts has made more explicit the theoretical frameworks and the role of institutional and cultural contexts within which the use of ICT tools takes place.

In this paper, we report on the implementation and outcomes of the experiment carried out by the Greek team aiming to explore the mathematical meanings constructed by 12-year-old students concerning the notion of fraction as number-measure. The students worked in collaborative groups of two, using the 'Fractions Microworld' (FM) [3], a piece of software which combines graphical and symbolic notation of fractions represented as points on the number line. In the next section of the paper, we refer to the theoretical background of the study with reference to the existing research concerning fractions and with particular emphasis on the measure interpretation of them. This section ends with the main research question of the present study stemming from the need to investigate the role that specially designed computational tools may play in order to enhance students' understandings of fraction as number-measure. There is then a description of the adopted methodological approach, consisting of the description of the main characteristics of the computer environment and the ways in which they were taken into account in the present study, the design of tasks, the research setting and the participants as well as the data collection and analysis. Next, we examine students' construction of meanings for fraction as number-measure during the implementation of the tasks by providing illustrative examples of students' work. Finally, we discuss the results of the study, highlighting the potential and constraints of the particular computer environment and tasks for studying the number-measure interpretation of fractions.

2 THEORETICAL BACKGROUND

In the research literature, understanding fractions involves the coordination of many different but interconnected ideas and interpretations, such as part-whole, measure, quotient, operator and ratio (Lamon, 1999). The interpretation of a fraction within part-whole relations based on equal partitioning is the first and probably the most dominant facet of the concept presented to students at the primary level, after which the algorithms for symbolic operations are introduced (Pitkethly and Hunting, 1996). In the respective activities, students are oriented to make sense of the denominator as the number of pieces produced through the equal partitioning of a whole, and of the numerator as a number of those pieces. However, several authors (Kieren, 1980, Freudenthal, 1983, Steffe, 2002, Thompson and Saldanha, 2003) have considered that instruction should not be limited to part-whole relations but should also include situations integrating different interpretations of rational numbers. Freudenthal (1983) questioned the convenience of using the part-whole approach as a means of introducing pupils to fractions that dominates most of the existing instructional approaches, considering it too restricted since it yields only proper fractions. This may partly be an explanation for the research findings which report students' difficulties in developing a sophisticated comprehension of fractions.

Empirical studies in different countries have revealed that although students are able to develop algorithmic competencies in manipulating fractions (e.g. perform fraction computations) (Aksu, 1997), they usually face difficulties in understanding them conceptually, especially in measure situations in which a fraction has to be perceived as a number (Bright, Behr, Post and Wachsmuth, 1988, Siegal and Smith, 1997, Hannula, 2003). These situations with fractions are usually accompanied by the pictorial representation of a number line and students are expected to measure distances from one point to another by partitioning certain distances from zero in terms of some unit. The value of the resulting fraction in this case constitutes *the number* (i.e. the rational number that the fraction represents, which can also be considered as a quotient between two natural numbers a:b), while the distance on the number line is *the measure*. This dual reference of partitioning to quotients as well as to distances from zero can be seen as an example of the complexity of the situations in which the number-measure interpretation of fractional numbers occurs.

The existing research results confirm that the measure interpretation of fractions on the number line is one of the most difficult for students to acquire since it is related to the well-documented hidden discontinuities between natural and fractional numbers (Streefland, 1993) such as the uniqueness in the symbolic representation of natural numbers, which does not hold for fractions (i.e. several fractions can represent the same fractional number), or the density of fractional numbers depicted on the density of the number line (i.e. between any two fractions there is an infinite number of fractions) (Stafylidou and Vosniadou, 2004). As children's experience with numbers is usually based on the discrete integers used for counting, their theory of number may, with age, become increasingly resistant to accepting fractions as numbers that represent the continuous nature of measurable points in space. This situation is aggravated by the apparent difficulty of students to relate the number line with their real-world experiences or some kind of external realistic grounding (Marshall, 1993). In the absence of a "mental model" for filling the gaps between the integers, children are likely "to over-generalise their use of counting numbers in a way which provides a formidable obstacle to accommodating their theory of number to accept fractions" (Siegal and Smith, 1997, p. 2). Hannula (2003), for instance, identified that middle-school students' difficulties in perceiving a fraction as a number on the number line were mainly associated with their reliance on faulty equal-partitioning imagery, i.e. interpreting a fraction such as "3/4" as *three out of four* in which the denominator was not seen as representing a size (segments of a magnitude that each one is one fourth of a whole) but as a collection of four things. Another constraint concerns the students' confusion over the nature of unit on the number line. As a fraction is usually perceived by pupils through the part-whole metaphor, the unit is considered as bigger than all fractions and frequently the integer number line is treated as a unit rather than the segment from zero to one (Baturo and Cooper, 1999).

In the light of the above data, it makes no sense to perceive fraction as a mathematical notion on its own but rather it is more useful to consider it in terms of the concepts interrelated with it, the situations in which it may be used and the available representations, which constitute – in the words of Vergnaud (1990) - a *conceptual field*. For instance, a concept tightly related to fraction is that of unit, a situation in which it may be used can be the situation evoked by a given task (e.g. a measurement problem situation concerning comparison/ordering fractions, operations with fractions), while the available representations can be based on the use of paper and pencil or on the use of computational tools.

As far as the didactic as well as the research exploitation of the number line is concerned, a review of the literature reveals that there has been a recent resurgence of interest in the representational potential of number lines for studying students' ideas and interpretations for the number-measure facet of fractions based on measurement and distance (Lamon, 1999, Ni, 2001, Charalambous and Pitta-Pantazi, 2005, Yanik, Helding and Flores, 2008). Entailing "a dynamic movement among an infinite number of stopping-off places" (Lamon, 1999, p. 120), working with fractions on the number line has been considered as critical for pupils in conceptualising the different sub-constructs necessary for the deep understanding of the concept of fraction in general (Hannula, 2003). This is mainly due to the fact that the number line – tied directly to measurement and distance - integrates aspects of all the main interpretations of rational numbers that are considered important for pupils developing so-called *rational number sense*, i.e. to be comfortable performing partitions other than halving, to be able to find a number of fractions between any two given fractions, and to be able to abstract a unit interval and use it to measure any distance from the origin (Lamon, 1999). From this perspective, the number line has been regarded as a new resource for activity and interpretations (e.g. in a problem solving process), which learners can draw upon in their attempts to construct meanings. Additionally, the fact that the number line may be continuous while other models (e.g. set models) are visually discrete has further supported the consideration of it as a practical model for introducing students to the number-measure interpretation of fractions (Bright et al., 1988).

In this vein, new computational environments based on the use of the number line are designed so as to make the number-measure aspect of fractions more accessible and meaningful to children. One of the respective tools' prime affordances is the multiple linked representations designed to integrate different representational registers concerning aspects of fractions (Olive, 1996). Careful design of such media seems to facilitate the creation of learning environments where the mathematical nature and identities of fractions integrate with opportunities for experimentation, exploration and personal forms of reasoning (Olive, 1999, Proctor, Baturo and Cooper, 2001, Nabors, 2003). In this study, experimentation with fractions is considered as flexible and dynamic for the students due to the available tools in the FM that provide new kinds of graphical and symbolic fractional representations and new kinds of access to them, allowing manipulation of the provided representations. In our view, it is by acting on different types of representations, as well as exploring the connections between them, that the learner engages in developing meanings as abstractions emerging in activity (i.e. *situated abstractions*, see Noss and Hoyles, 1996 for an extended discussion on the notion of situated abstraction) structured by the use of the available tools and their specific contexts of use.

However, these new kinds of representations, the ways by which they may be integrated in the learning process, as well as the ways by which they affect students' learning activities, needs investigation. The problem that arises concerns the tension between the mathematical objects represented within computational tools and those provided by traditional modes, and how significant any differences may be in terms of the students' construction of meanings (Morgan, Mariotti and Maffei, in press). Given that there has been little research studying the nature of students' understanding of the number-measure aspect of fractions, the present study did not place emphasis on closed 'didactical goals' but, rather, attempted to highlight students' ideas and struggles for meaning when they were introduced with new visual representations of fractions on the number line within specially designed computational tools. Attention was also paid to the instrumental aspect of the activity (Artigue, 2002), the process by which the tool was appropriated and integrated into the students' practice in the evolution of their interactions with it, at the technical and the conceptual level. Thus it was of particular interest to investigate how students used the available computer-based representations, and if and how they related them to other traditional representations, such as those created with pencil and paper.

Our research question in this study concerns the ways in which students used the available representations of the FM in their attempts to conceptualise fraction as number-measure when they were engaged in specially designed tasks involving comparison and ordering of fractions as well as operations with fractions. This question was conceived at two levels. Firstly, we were interested to explore if and how, through specially designed tasks within an appropriately developed computational medium, fraction as number-measure could be approached by the students in meaningful ways while working within the respective conceptual field. Secondly, we were interested to see how the new resources available would be taken up and coordinated with other, more familiar, means of representation (e.g. pencil and paper). Thus, we took a critical stance towards the representations provided in the FM so as to identify which of those might be useful in student's understanding of fraction in measure situations and how students would put them to use.

3 METHODOLOGY

The research perspective adopts a constructionist approach to learning (Harel and Papert, 1991) focusing particularly on students' interaction with the available representations to construct mathematical meaning. An important corollary of this approach is that we prefer to study the impact of alternative representations and means of expression on meaning-making activity expected to emerge through students' active engagement in experimenting with the available mathematical objects and the relationships between them (Noss and Hoyles, 1996). Within this perspective in the present experiment we followed a design-based research method (Cobb, Confrey, diSessa, Lehrer and Schauble, 2003) which entails the 'engineering' of tools and task, as well as the study of the forms of learning that take place within the specific context defined by the means of supporting it.

3.1 The computer environment

The 'Fractions Microworld' (FM) (Bottino and Chiappini, 2002, Chiappini, Pedemonte and Robotti, 2003) is a piece of software which provides the user with the opportunity to construct fractions which are represented as points on the number line and further explore their properties concerning topics such as comparison/ordering of fractions, operations with fractions and equal fractions.

The FM consists of three main areas: the *Problem Text Space*, the *Fraction Working Space* and the *Buttons Space* (Figure 1). The *Problem Text Space* is the area in which the text of a problem at hand (e.g. given by the teacher) appears. Students cannot write in this area but can just read the text typed in it. The *Fraction Working Space* consists of two number lines: the horizontal one is called 'number line' while the slanted one is called 'partition number line' or 'multiplication number line'. In this area, the user can construct and work with fractions through the combined use of the thirteen buttons (for operations, transformations, etc.) available in the *Buttons Space.* Each fraction is represented graphically and symbolically as a point on the number line.

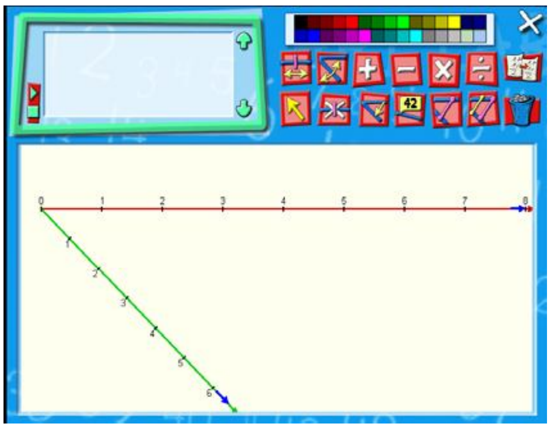
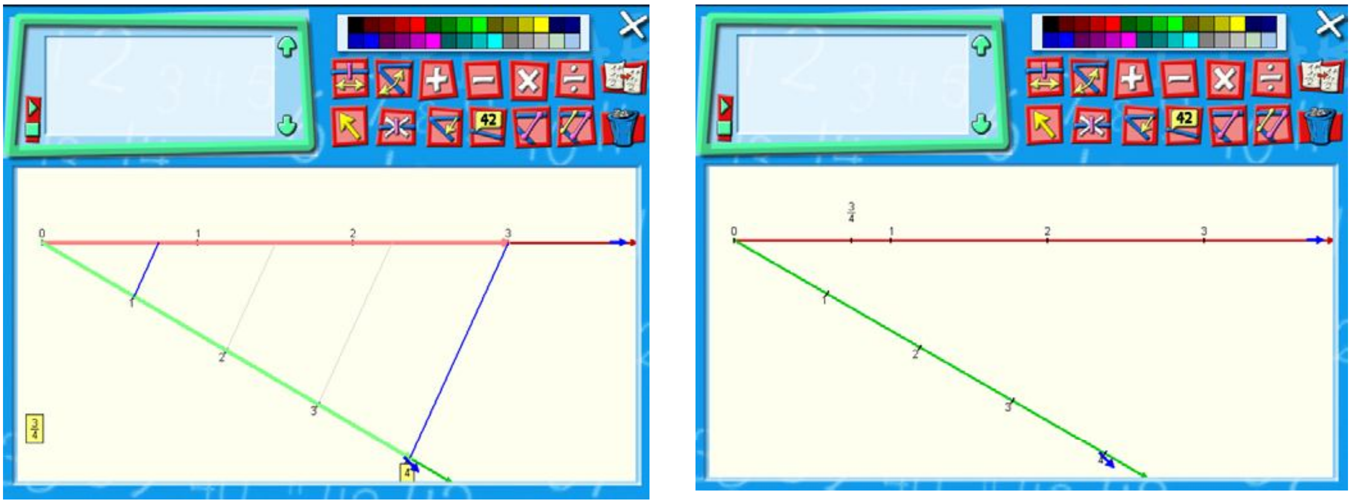


Figure 1 The main part of the Fractions Microworld

From an epistemological point of view, the construction of a fraction in the FM is based on two different types of representations: the first one is associated with Thales Theorem, while the second one with the notion of fraction as a quotient between two numbers which are selected by the user from the number line and the partition line respectively.

First, the user has to press the division button  and then select a number as a numerator (dividend) and a number as a denominator (divisor) by clicking on a number of the number line and on a number of the partition number line sequentially. The final representation of the constructed fraction is in the form of a point on the number line. The symbolic notation of each fraction is automatically given near its representing point on the number line in a "post-it" form (Figure 2). The partition number line in the construction of a fraction is thus used only for choosing the partitionings of the number selected on the number line.

After the selection of numbers from both lines in this particular way concurrently with the arithmetic representation of the constructed fraction, a geometrical representation underlying the construction technique based on the projection method involved in Thales Theorem is also provided instantly. It concerns the partition of a whole selected on the number line, into the number of parts defined by the number selected on the partition line. For example, while the construction of 3/4 in the part-whole scheme is interpreted as taking 3 times 1/4 of the numeric unit from 0 to 1, here it can also be seen as the division of 3 to 4 (divide the segment for 0 to 3 with 4) (see Figure 2). The geometrical representation is visible only when the user selects the numbers for constructing a fraction from the number line and the partition line respectively. When the construction is completed and the fraction appears – graphically and symbolically - on the number line, the parallel lines constituting the geometrical representation of the partition automatically disappear.



A B

Figure 2 Construction of ¾ (A) and visualisation of it graphically and symbolically (B) on the number line of the FM

The number-measure interpretation of fractions is thus at the core of the representational infrastructure of FM, since for each particular fraction the representation is provided as a *number*, i.e. the rational number that the fraction represents (resulting as a quotient between two specific numerical values) as well as a *measure*, i.e. the distance from 0 represented by a point on the number-line.

By trying out different combinations of numerators and denominators, the user may construct various types of fractions and consequently explore their position on the number line as well as their properties. In cases where two or more equivalent fractions are constructed, the corresponding point on the number line appears automatically as orange marked without a "post it" label. When the cursor is passed over this point, a yellow frame appears underneath with the equivalent fractions visualised one beneath another (see, for instance, Figure 5b).

Additionally, the user has the ability to dynamically modify the size of the numerical unit (i.e. the distance between two successive integers) in both number lines and to perform basic arithmetic operations with fractions (i.e. addition, subtraction and multiplication). Specifically, the user can manipulate the size of the numeric unit on the number line by clicking on a specific button and dragging the arrow in a dynamic way along the line in the right direction. For the sum of (or the difference between) two fractions, the user has to select them by dragging the mouse from the point 0 to the desired point. A red arrow is visualised during the drag along the first length, and a yellow arrow for the second length. The tool determines the length of the sum (or difference) which is visualised as a blue arrow marked with a point on the number line and a "post it" containing the expression of the sum (or difference) (e.g. "3 + 5" or"1/2 - 1/5"). The user can also multiply a selected length on the number line by a number of times selected on the partition number line.

Some specific features of the FM may offer students opportunities to further appreciate the number-measure interpretation of fractions at the primary level. Two of these features were taken into account in the present study. The first one is that the FM offers a strong visual image of the number line as a container into which integers and fractions can be placed. Relating visual to symbolic representations of fractions provides students with opportunities to explore the relationship between numerator and denominator to construct meanings for the size of a fraction. The second feature is that the FM represents fractions constructed in specific ways on the number line but does not signal any kind of mistakes by means of visual feedback. This characteristic of the tool gives space for pupils' - qualitative and intuitive - interpretations of the provided display of fractions and opportunities to relate it to their existing knowledge. The FM thus may allow students to easily make connections between the structure of the number line, the fractions represented on it and the ways in which those fractions are related to each other. While this might take students beyond the usual computational approach indicated by the curriculum, it can also be thought of as providing another approach to supporting conceptual rather than instrumental understanding.

3.2 Research design and tasks

In research and task design, a perspective was adopted in which the novel character of the provided representations was considered as a challenge to design exploratory activities for primary school students based on the idea of 'measuring distances', which could provide opportunities for *using* comparison and ordering of fractions as well as operations with fractions in the number line context. When designing the tasks we decided to present the construction of fraction in the FM to students as a sequence of actions (e.g. pressing buttons and selecting numbers) avoiding explaining the underlying geometrical representation based on Thales Theorem, since it is not introduced at the primary level [4]. At the same time, we were interested to see if and how the use of the computer-based representations would be integrated with the use of other, more familiar means of representation (e.g. pencil and paper) as well as with students' existing knowledge. Thus, it was decided that during the implementation of the activities pupils would have the choice to use either pencil and paper or the FM. We expected that the possible differences in the answers on pencil and paper with those provided by the tool would challenge them to develop explanations and justifications which could create a fruitful context for meaning generation.

The design of tasks was based on integrating the visualisation of fractions as numbers indicated by specific points and labels on the number line with distances of places in the space (e.g. between one's house and a playground). The idea was that instead of seeing the numbers on the number line as static measures to view them as representing distances and engage students in measuring the distances between specific places. We took into account the fact that in such a context students would have the choice to convert fractions (of kilometres) to metres, which was familiar to them from their textbooks, although this might be less convenient in cases of more complicated calculations (e.g. sums of fractions). Actually, we made this choice for two reasons: firstly, to encourage students to work exclusively with fractions, triggering the consideration of them as numbers in their own right; secondly, to support the view of the number line as a container at any point of which there could be situated concrete 'things' (i.e. specific places) which may or may not correspond to integers.

We divided the activity sequence into two phases and developed for each of them a strand of tasks. The tasks designed for the two phases shared common features with the second, being more complicated in terms of the need to perform operations with fractions, thus requiring a more sophisticated identification and use of fractions for measuring and comparing specific distances between different points on the number line.

Phase 1: Placing and comparing/ordering fractions on the number line

The first strand of tasks (see Table 1) was chosen for introducing pupils to the way in which fractions could be represented on the number line and thus compared and ordered within the FM. Pupils were involved in two types of tasks: placing fractions on the number line, and finding a fraction between two others.

Task 1 concerned the identification and comparison of distances, expressed with the use of unitary fractions. Since at that time the students were not acquainted with the FM features and functionalities, we chose to engage them in working with simple unitary fractions whose numerical representation on the number line coincided with their part-whole representation; thus it was considered as closer to pupils' partitioning experiences (i.e. the position of the fraction 1/3 on the number line indicates also the respective part-whole relationship '1 piece of the 3', in which the unit 1 is divided).

In task 2, we intended to engage students in working with proper but not unitary fractions. We were particularly interested to stimulate students to relate their existing knowledge of these fractions to the visual representation of them on the number line.

Task 3 was designed in order to further provoke students' experimentation with finding proper fractions between two given ones. We expected that such an openended task would motivate students to explore the role of numerator and denominator in constructing and comparing/ordering fractions and to subsequently develop their own conclusions.

By designing task 4 we intended to see if and how students were able to identify and visualise improper fractions on the number line. We expected that through appropriate researchers' interventions, students would develop and test conjectures concerning the quantitative relationships between numerator and denominator for defining specific types of fractions on the number line (e.g. proper and improper).

(1) George's house is 1 kilometer far from his school. On his way to school he sees a square at 1/2 km, his friend's Chroni's house at 1/3 km and a sweetshop at 1/6 km. Can you say in which order he sees them when coming back home?

(2) Constantina's school is 1 km away from her house. On her way to school she sees a kiosk at 6/7 km, a super market at 2/5 km and a playground at ¾ km. Which is the order she sees them on her way to school?

(3) Lazarus is Constantina's best friend; his house is between the playground and the super market. Can you find some fractions indicating the position of his house?

(4) Efie and Constantina are friends. They meet each other at the playground. Efie says to Constantina: "You are very lucky. Your house is closer to the playground than mine." Discuss about the position of Efie's house.

Table 1 The first strand of tasks

Phase 2: Measuring and comparing distances by performing operations between fractions

The second strand of tasks (see Table 2) was chosen in order to engage students in measuring and comparing fractions, mainly produced by adding or subtracting fractional distances in different parts of the number line.

Task 5 was designed to engage students in subtracting fractional parts so as to calculate the distance between two different points on the number line, neither of them situated on zero. In task 6, we aimed at engaging students in dealing with comparing fractional distances resulting as sums of other distances between different points on the number line. Given that students were not expected to have some prior experience with similar tasks – especially within the number line context - we were interested to see if and how their experience in dealing with the previous tasks would be directed towards constucting meanings for the addition of fractions, taking into account their visualisation on the number line.

One critical point here concerned the fact that if students chose to use the FM for solving tasks 5 and 6, they had to recognise that for culculating the distance between two points by adding or subtracting fractions it was necessary to construct each of them separately on the number line starting from zero and then to continue with the addition or subtraction.

(5) Constantina says to Efie: "I think that you are lucky too. You walk only 2/3 kilometers to go to school." Why is Efie lucky? Can you find one fraction indicating the exact position of Efie's house? What is the distance of the two friends' houses?

(6) Maria, a friend of Constantina and Efie, says: "I believe that the luckier of you is the one who walks less in a day for going both to the school (in the morning) and to the playground (in the afternoon)." Who do you believe is luckier and why?

Table 2 The second strand of tasks

At any time during their work in both phases, students had the choice of using either FM or pencil and paper. In cases of combined or complemantary use of both means, students would be asked by the researchers to explain the ways by which they used both the computer and the pencil-and-paper facilities to reach (or not) a solution. We expected students to reflect on possible – probably unexpected - differences between pencil-and-paper and computer-based representations which might provide opportunities for them to link their ideas about fraction as number-measure with the use of different types of representations. We also expected students to exemplify the rules underlying comparison/ordering of fractions and operations with fractional distances.

3.3 Context and participants

The experiment was carried out by two researchers as a case study. It took place at the computer laboratory of a primary school in Athens with four 6^{th} grade (last grade of the primary level in Greece) students divided in two groups - Group 1 including the students S1 and S2 and Group 2 including the students S3 and S4, each consisted of one boy and one girl. Each pair of students was assigned to one computer. Four 90-minute meetings were conducted with each group of students at the same time.

The researchers occasionally intervened to pose questions, ask students to elaborate on their thinking and clearly express their ideas or strategies when appropriate with no intention of guiding them towards any particular activity or solution.

At the time of the study, the students had already had experience in working with fractions in the traditional classroom setting following the teaching sequence indicated by the primary mathematics curriculum. According to the curriculum, the introduction of fractions is primarily based on part-whole and equal partitioning relations, while the comparison of fractions as well as operations with fractions are limited to algorithms for symbolic manipulations. Placing numbers on the number line is not embedded in a systematic curricular activity but appears in a fragmented way in few exercises in the primary mathematics textbooks. So, we considered the students' experience with number lines to be rather limited.

3.4 Data collection and analysis

In data collection, one video-camera and two tape-recorders were used. All that was said in each group was captured by one tape-recorder. One of the researchers occasionally moved the camera to each group to capture the overall activity and any other significant details in students' work as they occurred. The camera focused each time on the area in front of the students, capturing the screen, gestures and pencil and paper actions. Verbatim transcriptions of all audio-recordings were made. Each transcription was structured to indicate simultaneous interaction with the FM (e.g. creating/deleting a fraction), screen display, gesture and pencil and paper products (including operations, pictorial representations, paints, etc.). Thus, the data analysed involved the transcriptions of all audio-recordings accompanied by the corresponding video-recordings. This was also supplemented by the student's FM files and written work produced when the students chose to use pencil and paper.

According to our research question, we aimed at analysing students' emergent ideas and thinking while incorporating in their activity the FM representations, rather than quantifying their responses to the tasks and classifying them as "correct" or "wrong". Extracts have been chosen for analysis on the basis that they appeared to be of interest in relation to addressing some aspects of the research question. We have to mention that since the research in the area of fraction as number-measure with the use of computational tools is rather limited we aimed at highlighting the students' possible learning trajectories and difficulties without any attempt to generalise the results, taking also into account that we just refer to two groups of students in a case study setting. More specifically, our objective in the analysis was to gain insight into: (a) the nature of the mathematical meanings constructed by pupils around the notion of fraction as number-measure concerning aspects of comparing/ordering fractions and conducting operations with fractions and (b) the ways in which meaning generation interacted with the use of the available resources (computational or not).

The elaboration of data was based on open coding (Strauss and Corbin, 1998) carried out on two levels. First, we looked for instances in which there was interaction concerning how fraction as number and/or measure was perceived by the students as well as how they put the available representations into use when they were working within the wider conceptual field involving comparison and ordering of fractions as well as operations with fractions. At an early stage in the analysis, transcripts were divided into 'episodes' defined as extracts of actions and interactions performed during a continuous period of time around a particular issue. Second, we focused on students' use of the provided computational tools and/or pencil and paper in these episodes. This kind of analysis involved both the application of *a priori* categories and the iterative definition and refinement of categories derived from the data, accompanied by ongoing interpretation of patterns in the coding. Initial categories of coding included:

1. Meaning of fraction as number-measure on the number line.
2. Comparing/ordering fractions.
3. Operations with fractions.
4. Interplay between pencil-and-paper and computer-based representations.

Further elaboration of the data led to other categories which were not predefined but which emerged from interaction with the data. For instance, in the first level of analysis, the category "Unit" arose, which concerned students' understandings of unit emerging through their interaction with the available tools. Having coded episodes throughout the data, patterns and relationships were identified within and between categories. We note that in the above analysis our focus was on the meanings generated by the two groups of students and not on the comments made by each pupil individually.

4 ILLUSTRATIVE EXAMPLES OF ACTIVITIES AND LEARNING INCIDENTS

In this section, we present some illustrative episodes of the two groups' activities to highlight the ways in which the students used the available representations to conceptualise fraction as number-measure in the number line context. In the first paragraph, we provide a short account of how students conceived fraction as number-measure throughout the activity sequence – involving also the initial phase of the experiment - by using the FM and/or pencil-and-paper representations. Next, we continue with students' construction of meanings for fraction as number-measure while working within the wider conceptual field involving students' engagement in comparing/ordering fractions as well as in approaching the nature of unit on the number line. The last paragraph of this section outlines how the conceptualisation of fraction as number–measure was interrelated to conceptual and instrumental aspects emerging from the students' use of pencil and paper and the FM.

4.1 Making sense of fraction as number-measure on the number line

In the initial phase of the experimentation, the students were engaged in placing fractions on the number line so as to compare and/or order them. At this phase of their work, the two groups of students did not use the FM to work with. Rather, they chose pencil and paper to construct their own pictorial representations on the number line as a means of visualising the contextual information provided by the given tasks. In task 1, for instance, both groups of pupils initially draw a picture of the number line, placing Chroni's house at the zero point and the school at point 1 (see Group 1's representation of the number line in Figure 3).

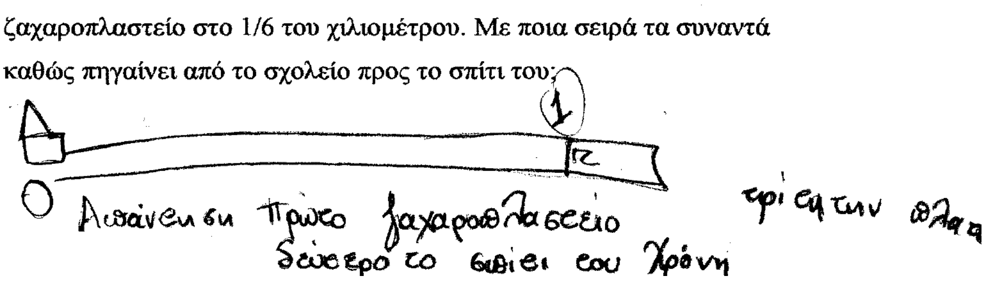


Figure 3 Individual representation of the number line (Group 1, Task 1)

For identifying the order in which George meets the places mentioned in task 1 on his way from school to his home both groups of students chose to convert fractions to integers, so that they could be more easily comparable. Thus, they multiplied the fractions by 1000 - a typical method of calculating distances in their mathematics textbooks - disregarding the fact that the kilometre itself as a unit could be divided into fractions (see Group 2's pencil and paper work in Figure 4). The conversion of fraction to metres is evidence of the use of 'mediating quantities' (Streefland, 1993) that students usually revert to in order to handle and ascribe meaning to fractions referring to natural continuous magnitudes such as time and distance.

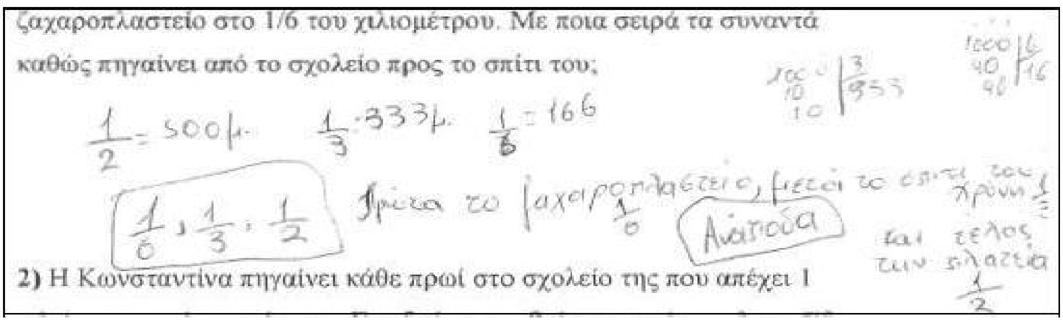


Figure 4 Students convert fractions to kilometers using paper and pencil

However, when they tried to use the tool to see how the respective fractions would be represented they started to form intuitively a correspondence between points, fractions and segments (as distances in kilometres). This became clearer as the tool functionalities became part of the students' activity and the fractions included in the problem situations were more complicated. Gradually, the students' interactions with the computer environment became strongly associated with the respective points on the screen while they could go back and forth between points, fractions and kilometres when they decided that they needed to provide an accurate measure of the distance covered by specific persons. This was facilitated by the students' interpretation of the feedback concerning the position of fractions on the number line as feedback concerning the representation of distances between specific places. This particular idea, in turn, facilitated the integration of the provided representation of fractions with pupils 'walking' on the number line to reach the places mentioned in the given problems. In the following lines, Group 1 students display an attempt at making sense of the total distance covered by Constantina as described in the task 5 (home-school-home-playground-home).

> S1: [*Showing with her index finger the distances on the number line of the screen*] So, it is twice this distance. She went from home to school, came back, went to playground and then came back home again.
> S2: [*Shows also with his index finger the same parts so as to confirm*] She went to school, came back and from home … to playground.

As their utterances reveal, their conceptualisation of fractions as measures of distances appears interrelated with the metaphor that the child "walks along" the number line, i.e. moving between places (points) and covering specific distances (segments). According to the language of the problem at hand, the students of both groups continued to refer to the numbers on the FM number line either as points or segments throughout the activity sequence. For instance, when they were calculating the position of certain places they were showing points, while when they were calculating certain distances, they were showing parts of the number line.

Indeed, pupils seemed to acknowledge this distinction, and provided indications that they were able to explain the mapping between specific fractions as points and the distances between them. For instance, in task 4, given that Constantina's house was considered on zero and the playground was at the point 3/4, the students had to find a fraction corresponding to the position of Efie's house which had to be more than 3/4 km from the playground. Group 1 students immediately suggested that this fraction would be "after 6/4" (taking two times 3/4 on the right direction of the line). After adding one more 1/4 - so as to take a fraction bigger than 6/4 - they constructed 7/4 in the FM as one answer to the problem.

> R: Are there any other fractions indicating the position of Efie's house?
> S2: Yes, many.
> S1: Lots of other places.
> R: How many? One hundred, one thousand?
> S2: More than these.
> S1: [*In a witty way*] Especially if she lives in another city.

These students seem to envisage the places in which Efie's house may be situated as numerous points on the number line which all correspond to fractions bigger than 6/4. By indicating the existence of many possible other places in which Efie's house could be located, S1 expresses more clearly a conceptualisation of the number line as a continuous segment being situated by ('many') fractions throughout its length (i.e. the other 'city' in which Efie may live can indicate any other position on the line 'after 6/4').

4.2 Working with fraction as number-measure within the wider conceptual field

4.2.1 Exploring the role of numerator and denominator for comparing and ordering fractions

Comparing and ordering fractions is considered as critical for the development of students' comprehensive understandings of fractions (Sowder, Bezuk and Sowder, 1993). In the present study one aspect of students' conceptualisation of fractions on the number line concerned students' explorations of the role of numerator and denominator in comparing fractions and ordering them by value based on the comparison of their position from zero (i.e. from the left to the right). Based on the coordination between their existing knowledge (i.e. the value of a fraction decreases when the denominator increases) and the computer feedback, both groups of students engaged in identifying fractions that exist between two other fractions when solving task 3. As an example, in the following part of this paragraph we focus on the work of Group 2 students.

In trying to specify the position of Lazarus' house, situated between the super-market and the playground (Task 3), Group 2 students were engaged in experimenting to find out some fractions between 2/5 and 3/4. In this case the students exclusively used the tool to construct and visualise the fractions on the number line. The following excerpt highlights the students' tool-mediated activity regarding the role of the denominator and numerator in the value of fractions.

> R: How did you manage to find out this fraction [*i.e. 3/5*]?
> S3: We played with the denominators. For instance, having the fraction 3/4 we chose a bigger denominator, which decreases the fraction, and we estimated that it will fall in between.
> R: Did you try to change the numerator?
> S3: [*To the Researcher*] We are going to do it now. [*To S4*] Let's try one.

There is a dynamic aspect in 'playing' with the values of the denominator as is expressed here by S3. The immediate access to the graphic representation of the constructed fractions seemed to play a critical role in triggering S3 to experiment with changing the numerical values of the numerator and the denominator so as to identify fractions between others.

The kind of exploration mentioned above evolved and Group 2 students were challenged to identify their own 'method' to generate as many fractions as possible between 2/5 and ¾, by experimenting with changes to the numerator and the denominator. Initially, they have concentrated on the fraction 2/5 and conjectured that one possible solution would be the fraction 3/5, which was accepted after being verified with the tool. Then, by increasing the numerator, they continued with the construction of fraction 4/5 but realised with the tool that it was bigger than 3/4. So, their subsequent experimentation was centred on the construction of new fractions by increasing the value of the denominator of 3/5, leading them to accept as answers to the task the fractions 3/6 and 3/7 after observing their visual representation on the FM number line. Again, they continued with the construction of new fractions in the FM by increasing the numerator of 3/6 with one -thus accepted 4/6- and then working in the same way by increasing only the denominator (i.e. producing 4/7, 4/8 and 4/9) and so on (i.e. producing 5/7, 5/8, 5/9, 5/10, 5/11, 5/12). Experimenting in this way with the tool, Group 2 students constructed 13 fractions located between 2/5 and 3/4, three of which were equivalent (i.e. 3/6, 4/8, 5/10).

At this phase pupils' experimentation was facilitated by the functionality of the tool to visualise, as marked in yellow, the points that were very close to one another according to the given scaling of the number line. In this case when the students passed the cursor over such a set of points, yellow frames appeared exactly underneath with the respective fractions ordered by value one beside the other (see Figure 5a). Additionally, in order to discern the constructed fractions in a number of similar cases, the students also used the scaling functionality of the FM according to which they could enlarge the numerical unit of the number line (see Figure 5b).

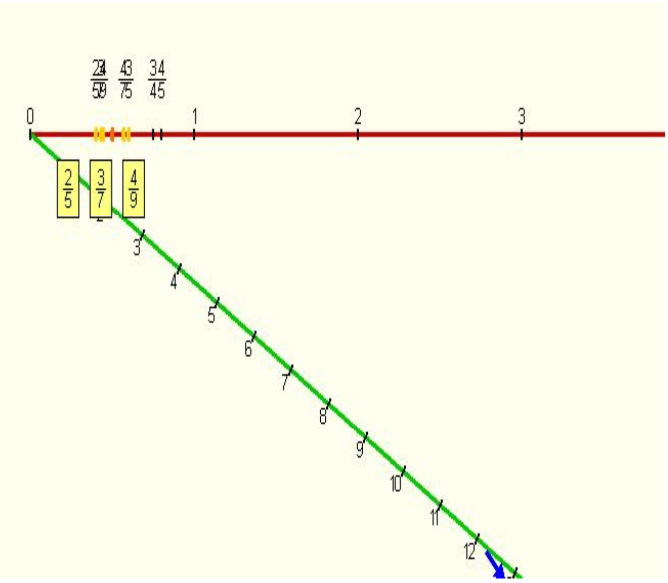


Figure 5a The visualisation of the constructed fractions very close to one another.

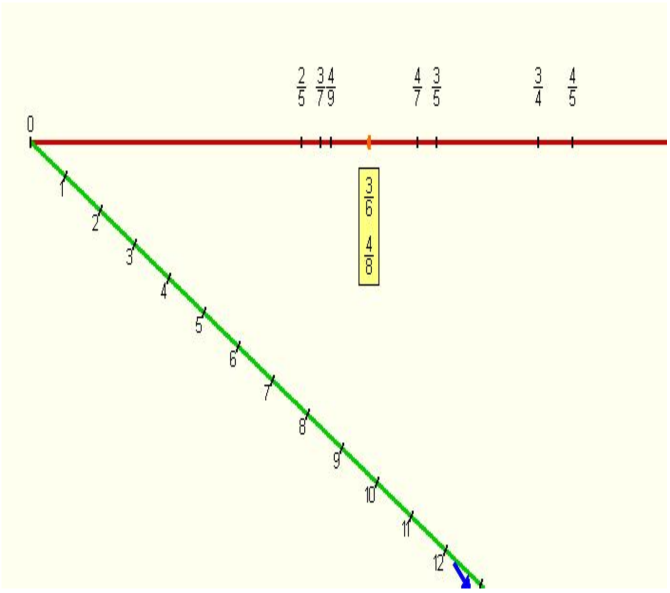


Figure 5b The representation of the fractions in Figure 5a after the enlargement of the numerical unit on the number line. It is also visualised the orange marked point corresponding to two equal fractions.

It is important to mention here that the ways in which any constructed fractions in the FM could be made visible (i.e. always visualising a 'gap' between any two fractions through the use of the functionalities described above) allowed students have a visual representation of the density of the number line. In the following excerpt Group 2 students appear ready to continue the process of locating more fractions between 2/5 and 3/4.

R: Do you think that you can create more?
S3, S4: [*Together*]: Yes.
S3: There could be many created. We can fill all this, from here up to there [*Shows on the computer screen with her index finger the segment between 2/5 and 3/4*], with fractions.

The researcher then probed whether the students were able to generalise the description of their 'method' to continue 'filling the gaps' for any two fractions in the space between the initial ones.

R: If you manage to construct more fractions between the two, do you think that you can continue creating fractions between any two of them?
S3: Yes, we should increase the denominator and increase the numerator a bit less, and in this way we will create new fractions between.
R: [*To S4*] What about you Spyros?
S4: That is correct, but some of those [*i.e. new fractions*] would coincide with others because, for example, they would correspond to the twofold or threefold values of other fractions.

The phrase "a bit less", referring to the increase in the numerator, obviously reflects S3's attempt to describe the above method of placing new fractions between two given fractions based on experimentation with changes to their numerator and denominator. According to this, the number of changes to the numerators was less than the number of changes in the denominators for controlling the generation of new fractions with the tool. It is particularly interesting in the above excerpt that S3's descriptions switch from referring to the 'balance' between the quantitative changes to both the numerator and the denominator while using a more generalised kind of language in order to capture the relation that underlies this specific process. At the same time S4 seems to exclude from the generated fractions the equivalent ones, knowing from his earlier interaction with the tool that they were represented at the same point on the number line. This type of pupil generalisation reflects Noss and Hoyle's (1996) notion of situated abstraction since mathematical rules that underpinned students' interpretations were rooted in action and articulated – quasi-mathematically – through the use of the available tools. We suggest that while engaged in ordering fractions by value Group 2 students progressively moved their focal point from the process of replacing specific numerical values on the initial fraction to trying to develop criteria to describe the useful manipulations to perform in the two terms of a fraction.

Yet given time, these students also related their own method to other aspects of their existing knowledge for equivalent fractions. When the researcher insisted on how they may proceed for discovering new fractions between any other two, S3 answered: '*We should convert them to have a bigger common denominator [i.e. than the existing]*'. Then they chose two of the identified fractions between 2/5 and 3/4 and after calculating their common denominator using pencil and paper, they verified the truth of their conclusion by creating them on the number line of the FM. Resorting to pencil and paper in that case can be seen as one way by which students attempted to link their existing knowledge of fractions with their ongoing experimentation with the FM, or, in other words, to legitimate the meanings emerging from the use of the instrument and relate them to the 'official' mathematical knowledge. In this case students seemed to easily move back and forth between standard mathematical algorithms and the visual mathematical representations provided by the tool in their attempt to construct meaning. It should also be noted that at this phase both groups of students used the scaling functionality of the FM (i.e. enlargement of the numerical unit on the number line) which allowed the visualisation of all the constructed fractions as distinct points on the number line (Figure 5b). Thus, the developed meanings here also concern the density of rational numbers approached through the iterative ordering of fractions by the students in an experimental way mediated by the combined use of pencil and paper and the tool.

In particular, the role of the tool can be articulated on two levels. At the first level, the students used the tool to generate and visualise fractions on the number line. At the second level, the visualisation of fractions was perceived by the students as a confirmation of their 'existence' and their actual position on the number line (i.e. between the given fractions).

Turning to the use of the scaling functionality in the FM by the students, it is important to say that this brings to the fore the notion of unit in understanding fraction as number-measure which is discussed in the following paragraph.

4.2.2 Approaching the size and the position of the numerical unit

In our experiment construction of meanings for unit appeared in different phases of students' explorations. In several cases there had been indications that students were in a position to abstract the independency of the length of the numerical unit from the fractional parts of it. This was mainly achieved through the extensive use of the scaling functionality of the FM by which students could dynamically change the measure of the numerical unit and thus zoom in on specific parts of the number line.

As mentioned in the previous paragraph, both groups of students approached the density of the number line through the generation of fractions, with the use of the tool, which were 'situated' between two other fractions. Particularly, the fact that both groups of students created many fractions in the range between 2/5 and 3/4, the labels of which appeared one over the other, made their symbolic representation invisible. Thus, the students enlarged the unit of the number line by clicking on the corresponding button of the Buttons Space and then dragged the new arrow-like representation of the segment from zero to one on the right (Figure 6), providing a clear view of the visualised fractions. When asked by the researchers to explain if the enlargement changed the relations between the constructed fractions, pupils of both groups in different phases of their activity, could explain that the ordering between the constructed fractions remained invariant. Although they did not refer explicitly to ratio scales, which would indicate a rather sophisticated consideration of unit, students in this case were able to precisely describe the correspondence between the points on the number line and their respective labels distinguishing that the ordering of fractions was not affected by the changes in the measure of the unit. At the same time they seemed to expect that the points corresponding to equal fractions seemed not to be 'removed' under the dragging of the unit (as happened with all the other labels corresponding to non-equivalent fractions) and linked it directly to the equality of the respective fractions. However, the fact that we just refer to four students in a case study setting makes us consider this finding as indicative only of students' implicit understanding concerning the role of the unit length in ordering of fractions on the number line.

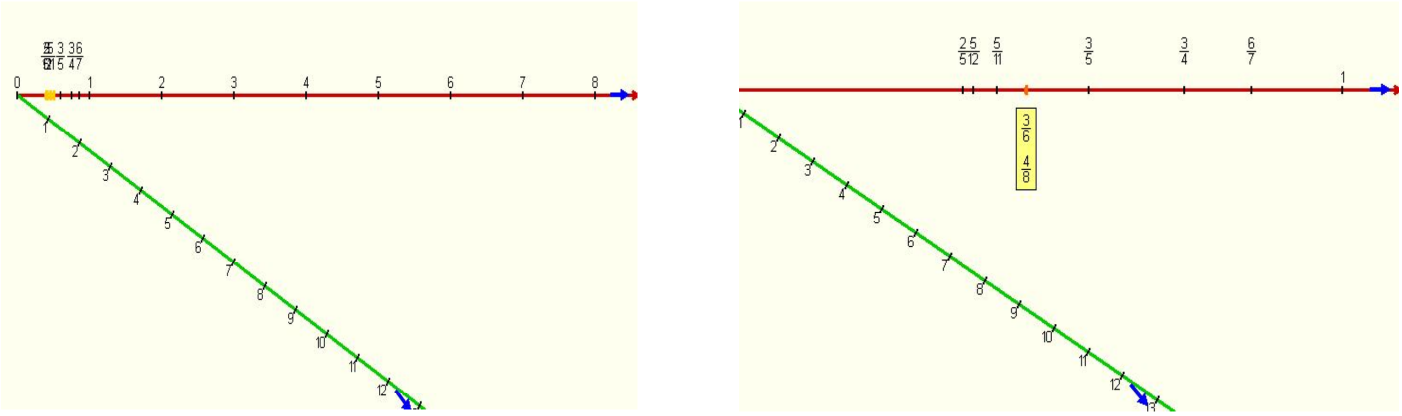


Figure 6 The enlargement of the unit on the number line

As far as the recognition of unit on the number line is concerned, only Group 2 students seemed to distinguish that the numerical unit of measure was regardless of its position on the number line. This was facilitated by the preceding interpretation of the proposed tasks by the students as ‘moving along’ the number line. In the next excerpt Group 2 students integrated the (unit) distance between Constantina’s house (corresponding to 0) and the school (corresponding to 1) in their approach and used it to bypass the constraint of representing numbers on the left of the zero point of the number line. After experimenting with different positions for Efie’s house in relation to Constantina’s house and to the school (Task 5), S3 tries to explain to S4 the possibility of considering Efie’s house on the left of Constantina’s house. In doing so, he indicates that the distances between different places (i.e. points) are independent of the position of the unit on the number line.

S4: Where is Constantina’s house? We know that it is 1 kilometre from school.
S3: At zero.
S4: How do we know that?
S3: We symbolised it here. [*Showing the distance from zero to 1*] We can do the same with 3 and 4 or 9 and 10. We’ve just preferred zero for the house and one for the school. Do you understand?

It is noticeable in the above excerpt that the indexical gestures of S3 on the number line appear as part of the situated abstraction concerning the position of the unit, to indicate in a precise way its imagined different positions on the number line. In the evolution of the above episode, S3 implemented his suggestion by considering, for a while, Constantina’s house on point 1 and the school on point 2. Again the unit, considered as the distance between two integers, remained constant but S3, at that time, was able to describe how the position of the other places was transformed on the number line as well: The playground was ‘moved’ to 7/4 (‘*I consider 4/4 as the whole. 3/4 is the playground. Thus, the new position of it is 4/4 plus 3/4 … 7/4*’) and thus S3 insisted that Efie’s house could be situated between zero and 1.

As we will see in the next paragraph, although both groups provided indications that they had made sense of the size as well as the position of the numerical unit on the number line, they faced difficulties in comparing specific segments produced as results of operations on the number line of the FM.

4.3 Conceptual and instrumental issues of working with fractions using pencil-and-paper and the FM

The introduction of new computational representations brings to the fore issues related to their use by the students, the means of manipulating them provided by a tool and those used in pencil-and-paper based work and most importantly issues related to how they may affect students’ learning activities. In the next paragraph we highlight these issues emerging in our study by focusing on the ways in which students performed operations with fractions using pencil and paper and the FM.

4.3.1 Performing operations with fractional distances on the number line

As far as calculations with fractions are concerned, it was observed that the students resorted many times to pencil and paper especially for doing calculations when performing operations with fractions. This was rather obvious when the students were engaged in solving task 6 and needed to add and subtract integral and fractional parts of the number line so as to calculate the total distance covered in a day by Constantina and Efie when following the route house-school-house-playground-house. Due to specific functionalities of the tool Group 1 students had difficulties in connecting the results of certain operations performed with the tool to the results they had found in their pencil and paper work. This was mainly related to the fact that the symbolic representation of a calculation in the FM includes only the numbers of the respective operation (e.g. “1/3 + 1/2”) and not the resulting equivalent fraction (e.g. “5/6”). Moreover, the user is not able to measure a specific part of the number line directly (e.g. between 1/2 and 3/4 using a kind of ruler) if it does not start from zero. For instance, in order to find the difference between 5/3 and 3/4 the user has to press the button "-"and then to select respectively the two fractions by dragging the mouse from zero to the point desired (i.e. first from zero to 5/3 and then from zero to 3/4). A red arrow is visualised during the drag of the first length and a yellow arrow for the second length indicating the consideration of fractions as distances measured form zero. So, the measure of a specific segment on the number line is realized by its equivalent transformation starting from zero which may be seen as complicating the calculation and demanding strong conceptual links between the final outcome and the initial problem situation.

In solving task 6, students of Group 2, however, preferred to use only the visual ordering of the points-fractions on the number line of the FM and bypassed doing any operations with the tool. Bear in mind that both groups considered Constantina's house on zero. Thus, they found that the playground corresponds to the point 3/4 and Efie's house to the point 5/3. Then these students chose to calculate the whole distance covered by the two girls by adding the respective fractions using pencil and paper and then by converting the result into kilometres.

S3: Efie walks 3 kilometres and 166 metres [*i.e. the total distance covered is 3 + 1/6*] and Constantina 3 kilometres and 500 metres [*i.e. the total distance covered is 3 + 1/2*].
S4: So, Efie is luckier.
S3: That's it.

On the contrary, Group 1 students who attempted to perform operations with the tool faced problems in coordinating the tool-mediated operations recognised as necessary for the solution with the representation of the respective fractions on the number line. In trying to provide an answer for task 6 these students were initially engaged in calculating and comparing the distances that the two girls cover when going from their homes to the playground. In the following excerpt, S1 needs to compare 3/4, corresponding to the distance covered by Constantina, with the result of the subtraction 5/3 - 3/4, corresponding to the distance covered by Efie. We have to note that the students had already created the fraction 5/3 - 3/4 on the number line by using the subtraction functionality of the FM (see Figure 7).

R: How did you find that point [*shows the point labeled as "5/3 - 3/4" on the number line*]?
S1: I subtracted 5/3 – 3/4. Because 5/3 is the whole, I subtracted the part from zero up to the playground. So, we've finished.
R: Can you now answer who walks further?
After calculating the result of the above subtraction (which is 11/12) with pencil and paper, S1 answers:
S1: Eleven twelfths.
R: So?
S1: …

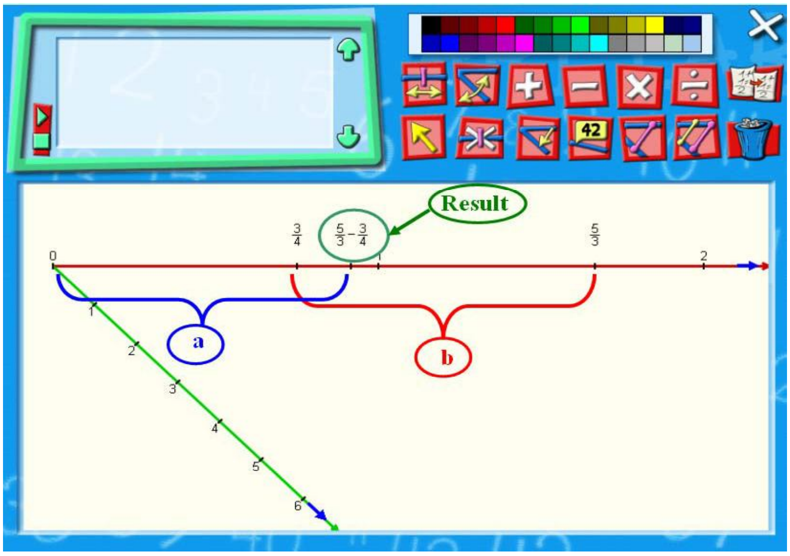


Figure 7 Two interpretations of the subtraction 5/3 – 3/4

Though she completed the requested calculations concerning the distance 'b' (Figure 7) and transformed it to the (equal) segment starting from zero (i.e. the distance 'a' in Figure 7), S1 is unable to connect it with the pencil and paper difference consisting of one fractional number (i.e. 11/12). This interesting difficulty, which was resolved in the experiment by the researcher's intervention, is discussed here as it highlights important conceptual and instrumental issues.

As far as the former is concerned, we have to underline the fact that these students managed to conceptualise the difference between the two points as corresponding to the desired distance covered by Efie but they were confused by the way that the transformation materialised in the FM. Without bringing into play the idea of converting fractions to kilometres – which might be useful - Group 1 students were not able to compare the two fractional results by considering them as numbers even in the pencil and paper context. The situation may have been further complicated for the students due to the fact that their previous interaction with the problem situation was in close interdependence with the specific points on the FM perceived as representing specific places. So, transforming a specific distance to another position – or relating it to a number calculated using pencil and paper - enhances the cognitive effort needed to effectively link these different types of mathematical representations for solving the task.

As far as the instrumental issues are concerned, we have to note that students from both groups often reverted to pencil and paper in order to perform an operation, especially in cases where they needed the exact arithmetic result of an operation (and not the whole arithmetic expression corresponding to it as it was provided by the tool) or in cases that they needed to perform operations between segments of the number line that did not start from zero. For example, remember that for calculating the distance between the points 5/3 and 3/4 it was not possible for pupils to measure it directly on the number line, but it was necessary to find the difference between the fractions 5/3 (marked as a length with a red arrow starting from zero) and 3/4 (marked as a length with a yellow arrow starting from zero). The difference was visualised as a blue arrow marked with a point on the number line and a "post it" involving the expression of the difference (e.g. "5/3 - 3/4"). The use of arithmetic operations with pencil and paper in such cases seemed to be more convenient and familiar practice in the context of the specific activities. However, we note that the way the tool represents the measure of fractions constituting sums or differences of distances between points on the number line is "correct" from a mathematical point of view. The visualisation of distances marked by arrows from zero, for instance, enhances the consideration of fractions as measures which is the underlying meaning of fractions on the number line when performing operations between them. Thus, the issue which can be highlighted here at the level of the tool design is that, what might be considered as a 'constraint' on one level can be a potentiality on another, in the sense that the mathematical epistemology built within the FM in this case brought to the fore a critical difficulty students had into coping with fractions as numbers representing specific quantities in this particular task.

5 SUMMARY AND CONCLUDING REMARKS

Our purpose in this paper was to illustrate a study investigating 12-year-olds' construction of meanings for fraction as number-measure represented by points on the number line. We have illustrated our analysis by looking at episodes in which students engaged in interaction with the available representations in situations involving comparison and ordering fractions as well as operations with fractions. Working within the FM, the dual representation of fractions as numbers and points on the number line, the order of fractions, the role of the unit and the operations with fractions came into play offering us an interesting terrain in which to investigate the nature of pupil's engagement with the fractional amounts involved in their activities.

The foregoing episodes illustrate the students' struggle to identify meaningful connections between the number-measure interpretation of fractions and other concepts, situations and representations of the relative conceptual field. The analysis provided indications that the proposed tasks created a context for the students to coordinate the interplay between the points as static numbers to the distances as dynamic measures from zero. The situated abstractions the students made regarding the density of the number line illustrate how they came to transcend the purely procedural view of placing fractions and associated the process of generating them by using the available tools with the mathematical structure of the number line. In this context both groups of students appeared to intuitively approach the density of the number line related to the number of fractions between two other fractions. Additionally, through the use of the scaling functionality of the software the concept of the numerical unit on the number line came to the fore and students were engaged in approaching its role in placing and ordering fractions on the number line.

While representations and mathematical content can be determined by inspecting the structure of a computational environment, their accessibility can be assessed only by observing its use in the context of specific activities. The ways in which the new representational resources were taken up and coordinated with more traditional and static means of representation constituted an interesting strand of our analysis. Initially, the students used pictorial representations with paper and pencil in order to consolidate the narrative of the word problems and the more abstract representation of the number line. Gradually, the role of the number-line in the FM was transformed from a tool to validate the place of specific fractions on the number line to a tool to explore the role of numerator and denominator in the size of a fraction and its corresponding position on the number line (e.g. in relation to other fractions). Many times, students moved back and forth between standard mathematical algorithms and the new representational media, using different possibilities to find an answer and different ways of reasoning. However, the students' difficulty in using the FM for making operations with fractions brings to the fore the cognitive demands of carrying out calculations based on the concept of fraction as number–measure on the number line as well as the issue of taking into account the relations between the tool functionalities and representations with the standard mathematical knowledge.

Concluding, of interest are those aspects of the representation of the number line in specially designed computational environments that can allow us access to students' interactions with new representational systems, and which could possibly have important potential for meaning-making in the domain of fraction as number-measure.

NOTES

1. Kaleidoscope (www.noe-kaleidoscope.org) was established by the European Community under the VI

Framework Programme (IST–507838 - 2003-2007) to promote the joint elaboration of concepts and methods for exploring the future of learning with digital technologies. It brought together 80 different European teams involving more than 1100 researchers and PhD students distributed over 23 countries.

2. The teams that constitute TELMA are: (DIDIREM) University Paris 7 Denis Diderot - Paris, France; (NKUA-ETL) National Kapodistrian University of Athens - Educational Technology Lab - Athens, Greece; (CNR-ITD) Consiglio Nazionale delle Ricerche - Instituto Tecnologie Didattiche - Genova, Italy; (LIG) Grenoble University and CNRS, Leibniz Laboratory, MeTAH, Grenoble, France (LIG); (UNILON-IOE) University of London - Institute of Education - London, UK; (UNISI) University of Siena - Department of Mathematics – Siena, Italy.

3. The FM is part of AriLab2 (http://www.itd.cnr.it/arilab/_english/index.html), a stand-alone version of an open system developed by the Consiglio Nazionale delle Ricerche - Instituto Tecnologie Didattiche (CNR-ITD) research team in Genoa (Italy) (http://www.itd.cnr.it). AriLab2 is composed of several interconnected microworlds based on the idea of integrated multiple representations and functionalities designed to support activities in arithmetic problem solving and in the introduction to algebra.

4. Within the cross-experimentation methodology of TELMA some teams coped differently with the fact that the geometrical representation of the FM based on Thales Theorem is not commonly familiar to students at the primary level. For the developing Italian team (CNR-ITD), the socio-constructivist perspective permits the use of such representations as black boxes giving responsibility to the teacher to manage the didactic situation and students' activities. Other teams such as the DIDIREM team which works within the French didactic culture, could not accept such an approach by raising issues related to its theoretical orientation as well as to curricular constraints which had to be respected. Thus, an emergent issue within TELMA work was that the use of technological tools designed by one research team and used by another research team highlighted differences in the ways in which different cultural and didactic contexts and theoretical frameworks handle epistemological aspects underlying the provided computational representations. In their forthcoming article, Morgan, Mariotti and Maffei (in press) offer an interesting theoretical elaboration of this issue and develop it in a way that can be of use in the design and employment of technological tools in mathematics education and in research into their use.

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BIOGRAPHICAL NOTES

Giorgos Psycharis holds a Ph.D in proportional reasoning using specially designed computational tools combining symbolic notation and dynamic manipulation of geometrical objects and relations. His fields of interest include the design of learning environments for mathematics with the use of computational microworlds in middle school with emphasis on the role of multiple representations in the construction of mathematical meanings. He has also been engaged in the design of exploratory software tools for mathematics, in classroom research into peer collaboration as well as in teachers' training courses for the use of computer technology in mathematics in the Greek educational system (Ministry of Education). He is currently working as a post doc researcher at the Educational Technology Lab (ETL, http://etl.ppp.uoa.gr) in the ReMath research project ("Representing Mathematics with Digital Media", http://remath.cti.gr, European Community, 6th Framework Programme, IST, IST-4-26751-STP, 2005-2009).

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Prof. Chronis Kynigos, is the Director of Educational Technology Lab (ETL, School of Philosophy, University of Athens) and a Professor of Educational Technology and Mathematics Education. In the past 15 years, he has led the pedagogical design of educational digital media and employed them in research involving a) aspects of designing and generating socio-constructivist learning environments in the classroom (emphasis on mathematics), b) design and implementation of innovative teacher education methods and c) the design and implementation of methods to infuse innovation in the educational system. These are: E-slate, a component kit to construct microworlds, MachineLab, a programmable 3D simulator and Cruislet, a 3D navigation system over a GIS based on the Cruiser platform. He has been responsible for ETL in the ESCALATE, C-Cube and Kaleidoscope projects and project director in the SEED and ReMath projects on behalf of another organisation (CTI – Patra). He is the author of an academic book titled 'The Investigations course' (in Greek) and has published more than 30 articles in refereed journals and research books. He is a member of the editorial board of the International Journal of Computers for Mathematical learning and a founder of the Greek Association for Research in Mathematics Education (http://www.ucy.ac.cy/enedim/).