THE INTERNATIONAL JOURNAL FOR TECHNOLOGY IN MATHEMATICS EDUCATION

Volume 17, Number 1

Meanings Generated While Using Algebraic–Like Formalism to Construct and Control Animated Models

By Chronis Kynigos, Giorgos Psycharis, Foteini Moustaki

Educational Technology Lab, Department of Pedagogy, School of Philosophy, University of Athens
kynigos@ppp.uoa.gr; gpsych@ppp.uoa.gr; fotmous@ppp.uoa.gr

Received: 7 November 2008 Revised: 23 December 2009

This paper reports on a design experiment conducted to explore the construction of meanings by 17 year old students, emerging from their interpretations and uses of algebraic like formalism. The students worked collaboratively in groups of two or three, using MoPiX, a constructionist computational environment with which they could create concrete entities in the form of models by using equations and animate them to link the equations' formalism to the produced visual representation. Our aim was to further study the ways in which the use of formalism in constructionist environments can create contexts for the emerging of mathematical meanings. Some illustrative examples of two groups of students' work indicate the potential of the activities and tools for expressing and reflecting on the mathematical nature of the available formalism. We particularly focused on the students' engagement in reification processes, i.e. making sense of structural aspects of equations, involved in conceptualising them as objects that underlie the behaviour of the respective models.

1 INTRODUCTION

Researchers studying the construction of mathematical meanings in environments where students converse while engaged in the use of digital media have focused on a variety of aspects of this complex learning situation. One "big theory" concerning the use of computers for mathematical learning was that of *constructionism* (Papert, 1980). It was built upon the constructivist connotation of learning as "building knowledge structures" (Papert, 1991) in a context where students are consciously engaged in constructing and manipulating personally meaningful external artefacts (such as animations, geometrical figures etc.). In this theory, the notion of construction refers both to the public outcome of the students' activity, as well as to the process by which they come to develop more formal understandings of ideas and relationships. Papert's elaboration of the notion of constructionism, in situations where students program a computer to make graphical models by driving a digital entity (the turtle) on the screen, provided an insight into reconsidering the value of concrete thinking as a generator of abstractions rather than a pre-amble to abstraction as Piaget would argue. Constructions were initially considered to be the graphical models created by means of text-based programming formalism. Modelling, constructing, programming and the use of formalism seemed to be interwoven into a coherent set of parameters about doing mathematics with digital media. However, the emphasis on each of these parameters changed dramatically as new technologies became available and new ideas for doing mathematics with such media appeared. One strand of attention, for instance, retained constructionism but led onto meanings generated through the uses of dynamic manipulation of graphical representations, rather than the use of formalism, in Dynamic Geometry Systems (DGS). Conversely, another strand de-emphasised constructionism and modelling but enhanced the importance of formalism by using multiple, dynamically linked representations such as data, graphs and functions in Computer Algebra Systems (CAS).

Digital media made it progressively possible to consider by-passing traditional algebraic formalism and have students manipulating graphical and even more concrete-looking representations (Kaput, Noss and Hoyles 2002). Transcending some of these learning environments however, was the idea that mathematical formalism could now be put to use in different ways. The search was for ways in which using such formalism could become a means to express and develop mathematical meaning (Kynigos and Psycharis, 2003), sometimes based on innovative uses of algebraic formalism (Nemirovsky, 1994; Kaput, 1994). This would turn formalism from being an obstacle to students' understanding of mathematics to being an integral part of their representational repertoire for expressing mathematical meanings. As Dubinsky (2000) has put it, meaning can drive formalism and formalism can drive meaning. In both cases by-passing or exploiting formalism to do mathematics, however, the idea of learning through constructing models seemed to lose in importance. The prevalent digital media, DGS and CAS, afforded the use of mathematical representations in order to construct other mathematical representations.

For instance, algebraic formalism in CAS is used to interconnect symbolic computation and graphical representations, facilitating the execution of routine techniques. Creativity in the ways in which formalism can be put to use in order to construct models seems to have lost some of its importance in favour of strictly mathematical representations. Yet, the essence of constructionism was that, in constructing and conversing over concrete public entities, learning becomes a conscious progressive internalisation of actions which leads to the ability to dissociate from constructs and actions and think about more generalised, abstract entities (Papert 1991, p.1). Noss and Hoyles (1996) paid special attention to the role of concrete context on abstraction processes. They introduced the notion of *situated abstraction* to describe the process in which mathematical meanings are expressed as invariant relationships (possibly in ways diverging from standard mathematics), but yet remain tied up within the conceptual web of resources provided by the available computational tool and the activity system. In this perspective, when formalism can be put in the role of an expression of an action or a construct (a model) it can operate as a mathematical representation for constructionist learning through situated abstractions.

In the classroom research reported in this paper we were interested to study mathematical meanings emerging from the uses of algebraic-like formalism in equations used as means to create and control concrete entities in the form of animated models. The students worked collaboratively in groups of two or three using an interactive computational environment called "MoPiX", developed at the London Knowledge Lab (http://www.lkl.ac.uk/mopix/) (Winters, Kahn, Nikolic and Morgan, 2006). The MoPiX environment is designed to foster the construction of virtual models consisting of objects whose properties and behaviours are defined and controlled by equations assigned to them. Our general aim in this study was to investigate the possibilities of specially designed representation systems based on symbolic language, like MoPiX, for students' meaningful engagement with the use of algebraic-like formalism. Since the introduction of such systems affects both what can be expressed and how, it offers us new windows into the students' thinking and the abstraction processes that could be generated in context-specific situations. The study of emerging forms of learning through the use of digital media that may hold transformative potential and in which the representational system underpinning the tool is non-standard, such as in MoPiX, remains a major priority for research (Hoyles and Noss, 2008). At a more global level, we aimed to further study the ways in which the use of formalism in constructionist environments can create contexts for the emergence of mathematical meanings.

The activities we designed for this classroom research were intended to engage the students in the manipulation and use of equations that were provided to them or equations they constructed themselves in order to define and control the behaviour of animated models. The manipulation and construction of equations allowed the students to interpret and use the equations' algebraic formalism, while the respective models' generated animations which offered them opportunities to connect the available formalism to visual or graphical representations. We aimed to investigate if and how students were enabled to make sense of structural aspects of the equations, involved in conceptualising them as objects that underlie the behaviours of the animated models. In doing so, we did not intend to make some kind of comparison between the meanings emerging in this particular context and the knowledge related to the universal formalism of algebra or physics, targeted by the official school curriculum. Morgan and Alshwaikh (2008), on the other hand, reported in their study how the semiotic resources provided by MoPiX may support students' development of ways of operating with the notions of kinematics that are compatible with their formal definitions and the Newtonian laws of motion.

2 MEANING WITH EQUATIONS

Recognising the meaning of symbols in equations, the ways in which they are related to generalisations integrated within specific equations and the ways in which a particular arrangement of symbols in an equation expresses a particular meaning, is fundamental to mathematical and scientific thinking. Research has been showing rather conclusively that the use of symbolic formalisms constitutes an obstacle for many students even for those beginning to study more advanced mathematics (Dubinsky, 2000).

A central question of the respective studies in the mathematics education field concerns the nature of equations and the ways in which they can be understood by students. Most of these studies are based on the distinction between the two major stances that students adopt towards equations: the *process stance* and the *object stance* (Kieran, 1992; Sfard, 1991). The process stance is mainly related to a surface "reading" of an equation concentrated into the performance of computational actions following a sequence of operations (i.e. computing values). The equal sign in such a case can be interpreted by the students as a "do something signal". In contrast, according to the object stance, an equation can be perceived as describing a symmetric relation (specified by the equal sign) and can be treated as an object on its own right, which is critical for the development of the so-called *algebraic structure sens*e (Dreyfus and Hoch, 2004), i.e. the act that entails the ability to: "see an algebraic expression or sentence as an entity, recognise an algebraic expression or sentence as a previously met structure, divide an entity into sub-structures, recognise mutual connections between structures, recognise which manipulations it is possible to perform, and recognise which manipulations it is useful to perform." (Hoch and Dreyfus, 2004, p. 51).

Elaborating further on the process-object duality in students' understanding of algebraic expressions (and thus equations), mathematics educators brought into play the idea of students moving from process-oriented views (process stance) to structure-oriented views (object stance) via some process of abstraction which has been termed *reification* (Sfard, 1991) and has been considered to underlie the learning of algebra in general. In the present study, reifying an equation was considered as a dual process involving (a) the identification of the role of variables and numerical values as component parts of an equation used to represent and express specific or general quantities and (b) the development of understandings concerning the structure of an equation, based primarily on the conception of it as a system of connections and relationships between its component parts.

2.1 Formalism as a representation for constructionist learning

Recently, students' use and interpretations of symbolic formalism in understanding mathematical and scientific ideas have been studied in relation to the representational infrastructure of new computational environments designed to make the symbolic aspect of equations more accessible and meaningful to students, especially through multiple linked representations (Kaput and Rochelle, 1997). The interactive and dynamic character of the computer-based representations have brought to the fore the need to acknowledge both the transformative potential of the corresponding technologies and the opportunities provided by new symbolic systems for meaning generation. This need was also reinforced by the fact that such systems encourage different kinds of expression, that can be aligned or not with standard mathematical notation.

Some of these technologies are designed to enhance explorations of the conventional mathematical symbolism in experientially real contexts. A representative example is the SimCalc environment (Kaput and Schorr, 2002) which employs physical or cybernetic devices allowing the generation of real-time graphs in a variety of modes (e.g. algebraically, by clicking, dragging and or stretching segments or by importing data). This environment, however, although it emphasises the link between formalism and concrete models, does not give prevalence to the construction, modification and tinkering with these models which would encourage the constructionist aspect of learning discussed earlier.

Other design approaches took the challenge of building systems that could provide alternatives to the conventional formalism for the expression of mathematical and scientific ideas. During the WebLabs project, one such system was built around Ken Kahn's programming language, ToonTalk (Kahn, 1996; Kahn, 2004; http://www.toontalk.com). Programming in the ToonTalk environment is not realised by using a text-based language, but by training "robots" to perform a sequence of actions whenever presented with the right conditions. The behaviours of the objects are transparent and thus available for inspection and modifications and generalisable so as to be applied in a variety of contexts inside the virtual animated world. Simpson, Hoyles and Noss (2006), exploiting the idea that students could construct their own ToonTalk models to control an object's motion and export the data so as to plot position-time and velocity-time graphs, reported positive learning gains in the area of kinematics. Mor, Noss, Hoyles, Kahn and Simpson (2006) described how students, in the process of constructing ToonTalk programs to generate numeric sequences, gradually shifted their attention from a purely procedural view to a more structural one. These approaches bring into the foreground new representational systems that by-pass conventional formalism, making mathematics more learnable for students (Noss, 2004).

Other research approaches were based on the idea of using a programming language as a principle representational system for physics instruction, instead of the usual algebraic formalism. A notable example is the research reported in Sherin (2001) with the programming environment Boxer (diSessa, 2000) used as part of physics courses. The results discussed the nature of understandings associated with algebra-physics and programming-physics, indicating that programming privileges a different "intuitive vocabulary" than algebraic equations. Sherin reports that programming can define very different modes of knowing a domain, changing the nature of what is learned and thus allowing the creation of alternative practices. This approach seems to distinguish mathematical formalism and programming in contrast to Papert's view of programming as a formal mathematical representation in its own right (Papert, 1980).

Many studies that have reported positive learning potential in this area have been based on the use of microcomputer-based laboratory (MBL) tools. In most of these studies the symbolic activity is materialized through sensor equipment linked with computer software, allowing the real-time construction of graphs. In the analysis of their study Nemirovsky, Tierney and Wright (1998) described the processes by which the students' use of a computer-based motion detector affected the visualisation of the position versus time graphs and fed their conceptual understanding of the relationships involved. Arzarello, Pezzi, and Robutti (2003) emphasised how the students' use of motion detectors and symbolic-graphic calculators to create tables and graphs in the process of modelling motion, brought into the surface a dialectic between the mathematical concepts (such as the function) and their representations (tables and graphs).

All these design approaches are transcended by the attempt to embed a kind of mathematical symbolisation in the respective artefact and relate it to what is experientially real for students. However, as we discussed, there are differences in the emphasis that each system places on the use of conventional formalism and on constructionist learning. In our study we wanted to gain further understanding of a learning environment that puts an equal emphasis on both of these aspects. Formalism in MoPiX is materialised through conventional as well as programming notations, forming an algebraic-like symbolic system. Meaning, on the other hand, is not considered to be on the environment's symbolic system per se, but to emerge *in activity,* as students engage in interpreting, manipulating and using the available formalism to build animated models. Thus, we see our work situated within the approaches that regard the use of symbolic formalism as a process that has the potential to foster the construction of mathematical meanings (Cobb, Gravemeijer, Yackel and McClain, 1997; Noss, Healy and Hoyles, 1997; Nemirovsky and Monk, 2000; Gravemeijer, Cobb, Bowers and Whitenack, 2000, Kynigos and Psycharis, 2003).

In this view we aimed to address two main research aims. First, we were interested to raise students' awareness of structural aspects of equations involved in making sense of them as objects through reification. We wanted to see if and how novel and context-specific formalism might operate as a generator of situated abstractions developed by the students within the tools and activity structures of the setting. Here, the important point for us is to study how the ability to express relationships in innovative ways may provide new opportunities for emerging abstraction processes. Secondly, we were interested in exploring if and how, through carefully designed tasks, formalism can be approached by the students in meaningful ways that don't sacrifice the rigour inherent in them. Thus, the focus of our study was not on students' understandings of formalism in the context of conventional algebra or physics, but rather on how new ways of mathematical expression (which may diverge from the conventional ones) shape and define students' constructions of meanings.

3 THE COMPUTER ENVIRONMENT

MoPiX (Winters et al., 2006) constitutes a programmable environment that provides the user with the opportunity to construct and animate in a 2D space, models representing physical phenomena such as collisions and examples of motion. In order to attribute behaviours and properties to the objects taking part in the animations generated, the user ascribes to the objects equations that may already exist in the computational environment or equations that he constructs by her/himself. The equations ascribed to the objects are accessible and available for inspection and modification.

The MoPiX v.1 [3] environment consists of three main areas: the "Equations Editor", the "Stage" and the "Equations Library". The "Stage" is the area where the user places the objects to which she/he attributes equations and at the same time, the area in which the animation of the phenomena takes place. The user may select from the "Equations Library" the equation she/he desires and add it to her/his object by dragging and dropping it. The "Equations Library" contains several equations, classified into 15 different categories according to the kind of behaviours and properties they attribute to the objects (e.g. Horizontal Motion Equations, Vertical Motion Equations, 1D collision equations). If the equations available don't meet the user's needs, there is the possibility of editing equations that already exist in the environment or constructing new ones, using in both cases the "Equations Editor". The "Equations Editor" contains various "Operator Buttons" (e.g. logical, arithmetical, inequality) the combined use of which may result in the composition of numerous new equations. By trying out different combinations of equations, the user may construct various models which she/he may consequently animate in order to observe the visual representation generated.

The MoPiX motion equations look quite similar to the conventional physics equations one may find in a typical 10th grade schoolbook. An object's instantaneous velocity, for example, is defined in such schoolbooks as the "rate of change of position" and is calculated through the formula "$v(t) = (x(t) - x(t - \Delta\ t)) \div \Delta t$", where $x(t)$ is the object's position at the time t, $x(t - \Delta t)$ is the object's position at time $(t - \Delta t)$ and Δt is the time interval required for the object's displacement (Δx). In the MoPiX environment, however, this time interval is perpetually equal to 1 ($\Delta t = 1$), as time constantly increases by 1 time unit when the animation is running. Bearing this in mind for MoPiX, the regular equation $v(t) = (x(t) - x(t - \Delta t)) \div \Delta t$ becomes the equation $v(t) = (x(t) - x(t - 1)) \div 1 \Rightarrow v(t) = x(t) - x(t - 1)$. Solving the latter for $x(t)$, the equation "$x(t) = v(t) + x(t - 1)$" that comes up can be used to define the object's position at each time instance with regard to its position at the previous time instance and its instant velocity.

Attributing the "$x(t) = v_x(t) + x(t - 1)$" equation to an object whose horizontal velocity is predefined to be constant (e.g. $V_x(t) = 3$) and starting the animation (time runs forward), the object will appear to perform an horizontal uniform motion in the MoPiX environment. MoPiX computes every 1 time unit all the attributes given to the object in the form of equations - such as its x position and velocity - and updates the display correspondingly. As the object's x position at each time instance (i.e. $x(t)$) will be different to the one at the previous time instance ($x(t - 1)$), the update of the screen will make the object appear in a new position at each time unit, giving the user the impression that it has moved horizontally.

To make an object perform a non-uniform motion parallel to the Y axis (constant acceleration applied), one should use, apart from the "$y(t) = v_y(t) + y(t - 1)$" position equation, one more equation that would link the acceleration to the velocity. Going back to the schoolbooks, acceleration is defined as "the rate of change of velocity over time" and is calculated by the "$a(t) = (v(t) - v(t - \Delta t)) \div \Delta t$" formula. Taking into account again that Δt in MoPiX is always equal to one ($\Delta t = 1$) and solving for $v(t)$, the former equation becomes "$a(t) = v(t) - v(t - 1)$" $\Rightarrow v(t) = a(t) + v(t - 1)$. Attributing both the "$y(t) = v_y(t) + y(t - 1)$" and the equation "$v_y(t) = a_y(t) + v_y(t - 1)$" to an object whose vertical acceleration is predefined to be constant (e.g. $A_y(t) = -0.098$) and starting the animation, the object in the MoPiX environment will appear to perform a vertical non-uniform motion.

Figure 1 shows a red ball performing in the MoPiX environment a combined motion which appears to be uniform in the horizontal and non-uniform in the vertical directions, leaving at the same time a green trace behind to indicate the path. The equations that underpin the model's behaviour appear on the right part of the figure and are divided into three categories: Horizontal Motion Equations, Vertical Motion Equations and Ball's and Pen's Properties Equations.

As one may notice, all the MoPiX equations follow a two-argument function template (i.e. "f(α, β) =…."). The second argument on the template refers to "time", since MoPiX needs time values as an input to calculate any attributes given to the objects (e.g. its position, velocity or acceleration). The first argument, however, the personal pronoun "ME", refers to the object to which the equation will be attributed (it will be attributed to "me", the object) and serves not as a second variable for the function, but as a programmable parameter to the equation. As soon as the equation is ascribed to the object, the "ME" word changes into the object's name (e.g. "object_5"), allowing the user to attribute the prototype equation to another object. Thus, MoPiX equations are essentially functions of time and not functions of two variables.

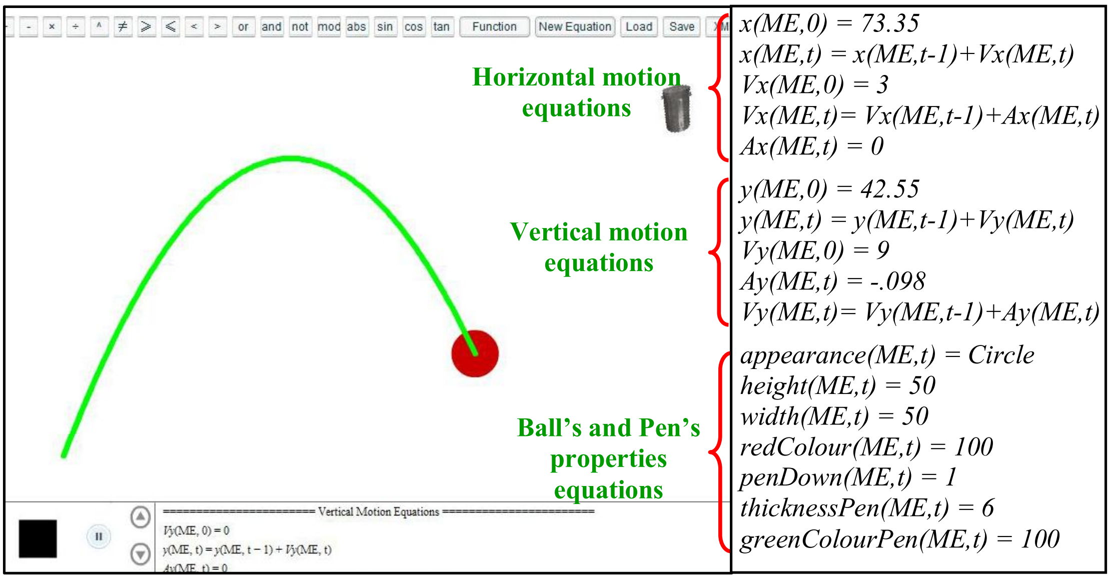

Figure 1 A ball leaving a trace while performing a combined motion in the X and Y directions and the equations attributed to it.

The first set of equations attributed to the red ball ("me") in Figure 1 are the Horizontal Motion Equations and defined in Table 1.

x(ME,0) = 73.35	the value of the horizontal position at the 0 time instance
x(ME,t) = x(ME,t–1)+Vx(ME,t)	the value of the horizontal position at any time instance
Vx(ME,0) = 3	the value of the horizontal velocity at the 0 time instance
Vx(ME,t) = Vx(ME,t–1)+Ax(ME,t)	the value of the horizontal velocity at any time instance
Ax(ME,t) = 0	the value of the horizontal acceleration at any time instance

Table 1 The Horizontal Motion Equations attributed to the red ball (left column) and the values that they define (right column).

The second set of equations are the Vertical Motion Equations and are defined in a similar way to the ones describing the Horizontal Motion: the value of the vertical position at the 0 time instance, the value of the vertical position at any time instance, the value of the vertical velocity at the 0 time instance, the value of the vertical velocity at any time instance, the value of the vertical acceleration at any time instance.

Although the Horizontal Motion Equations and the Ball's and Pen's Properties Equations attributed to the red ball were retrieved straight from the environment's "Equations Library", the same could not apply for the Vertical Motion Equations. To produce the specific parabolic trail appearing on Figure 1, it was essential to have some initial velocity in the Y axis. Thus, we picked from the "Library" the "V_y(ME ,0) = 0" equation, which sets the initial vertical velocity to be 0, sent it to the "Equations Editor" and changed it to "V_y(ME ,0) = 9" before assigning it to the red ball.

The third set of equations refers to the properties of the Ball and the Pen. Those equations seem to be closer to the natural language utterances, as their names (i.e. the first symbol on the left-hand side of the equation) provides the user with concrete evidence for the attributes it describes. For example, it is not difficult to discern that the equation "appearance(ME, t) = Circle" will set an object's appearance to be the one of a circle, while the "thicknessPen(ME, *t*) = 6" will set the pen leaving the trace to have a thickness of 6 pixels.

Nevertheless, as easy as the name of the programming-like equations may be, when looking at their right-hand sides, one may realise that they could be quite a bit more complicated than the ones presented in Figure 1. Take for example, the following equation: "amIHittingASide(ME, *t*) = (*x*(ME, *t*) ≤ 0 *or x*(ME, *t)* ≥ 799) *and* Vx(ME, *t*) ≠ 0". Judging by its name, the right-hand side of the equation should define whether the ball is hitting either one of the Stage's sides or not. To give an answer to the question (am I Hitting A Side?), MoPiX at each time instance calculates the ball's *x* position and V*x* velocity. It asserts that the ball has reached one of the sides only when "the object's x position is equal or less than 0" or "when its x position is equal or greater than 799" and at the same time its "x-axis velocity is nonzero". The output of this function will not be a number (e.g. "6") or a word (e.g. "circle"), as it was the fact for the Figure 1 equations, but rather a true or false statement.

Some specific features of MoPiX, underlying the novel character of the representations provided, may offer students opportunities to further appreciate the utilities of the algebraic activity around the use of equations. The first of these features is that MoPiX offers a strong visual image of the equations as containers into which numbers, variables and relations can be placed. The meaningful use of the MoPiX environment may allow students to easily make connections between the structure of an equation, the quantities represented and they ways in which those quantities are related to each other. The second feature of MoPiX is that it allows the user to have deep structure access (diSessa, 2000) to the models animated. The equations aggregated to the objects and underpin the models' behaviour do not constitute "black boxes", unavailable for inspection or modifications by the user (for a discussion on black and white box approaches see Kynigos, 2004). The third feature of MoPiX, inextricably interwoven to the second one, is that the manipulations performed to a model's symbolic facet (e.g. changing a numerical value or removing an equation from the model) produce a visual result on the Stage, from which students can get meaningful feedback. "Debugging" a flawed animation demands students' engagement in a back and forth process of constructing a model predicting its behaviour, observing the animation generated, identifying the equations that are responsible for the "buggy" behaviour and specifying which and how particular parts need to be fixed.

4 RESEARCH DESIGN AND METHODOLOGY

Our research approach is informed by the influential idea of "design" in learning (Cobb, Confrey, diSessa, Lehrer and Schauble., 2003) aiming to develop theories and an empirically grounded understanding of "how learning works" through experiments (i.e. design experiments), often based on the use of some technological innovation.

4.1 Tools and Tasks

Drawing on the idea of "layered learning design" (Kahn, Noss, Hoyles and Jones, 2006), we divided the activity sequence in two distinct phases and developed for each one of them a microworld (for a discussion of the term see Healy and Kynigos, 2010) in the MoPiX environment. The two microworlds shared common characteristics, with the second one being more complicated in terms of the kind of equations used and the numbers of objects participating in the animation. The activities designed for the second phase of the experimentation escalated in difficulty and required a more sophisticated manipulation of the MoPiX equations and the environment's features and functionalities.

Phase 1: The "One Red Ball" microworld

For the first phase of the activities we developed in the MoPiX environment the "One Red Ball" microworld. This microworld consisted of a single red ball performing a combined motion in the vertical and the horizontal axis, just like the one in Figure 1, only without leaving a trail. All the equations that descibed the red ball's motion were found ready-made in the environment's "Equations Library" and were attributed unaltered to the ball.

The students were initially asked to execute the model and observe the animation generated. They were invited to discuss with their teammates and with other workgroups the behaviours animated and write down their remarks and observations on a worksheet that we had prepared for them. At that time the students were not acquainted with the MoPiX features and functionalities as we had chosen not to implement a discrete "Familiarization with the computational environment" phase. Therefore, during one of the meetings, we carefully tried to draw their attention to the "Equations Library", aiming to provoke among them a discussion regarding these equations' specific role in the animation generated.

In order to stimulate the students to start using the equations themselves, we asked them to insert a new object on the "Stage" and try to make it move exactly like the red ball. In this process, we encouraged the workgroups to retrieve and use ready-made equations from the "Library" and by adding or removing them from their objects, to explore and interpret possible changes these equations and the symbols they comprised brought upon the animation produced on the screen. As we had deliberately made the original red ball move rather slowly, near the end of this phase, we expected the students to start expressing their personal ideas about their own object's motion (e.g. make it move faster than the red ball) and thus start editing the given equations so as to describe for themselves any new behaviours they might have in mind.

Phase 2: The "Juggler" microworld

For the second phase of the activities we designed a half-baked microworld (Kynigos, 2007), i.e. a microworld that incorporates an interesting idea but that is incomplete by design so as to invite the students to deconstruct it, build on its parts, customise and change it, eventually constructing a new artefact which could be distinctly different than the original one. The microworld, called the "Juggler" (Kynigos, 2007), consisted of three interrelated objects: a ball performing a combined motion in the X and Y axis when starting the animation (just like the "One Red Ball") and two mouse-driven rackets (as shown in Figure 2). In this game-like microworld, to prevent the ball from hitting the ground, one should manipulate the rackets and try to hit the ball to keep it continually "up in the air". However, this turns out to be quite demanding as the design choices we have made hinder the user from properly juggling the ball and stimulate him to look into the equations' formalism so as to re-establish the game rules. Contrary to the 1st Phase microworld, the equations underpinning the Juggler's ball behaviour did not derive excusively from the environment's

"Library", but were created by us, using the MoPiX formalism.

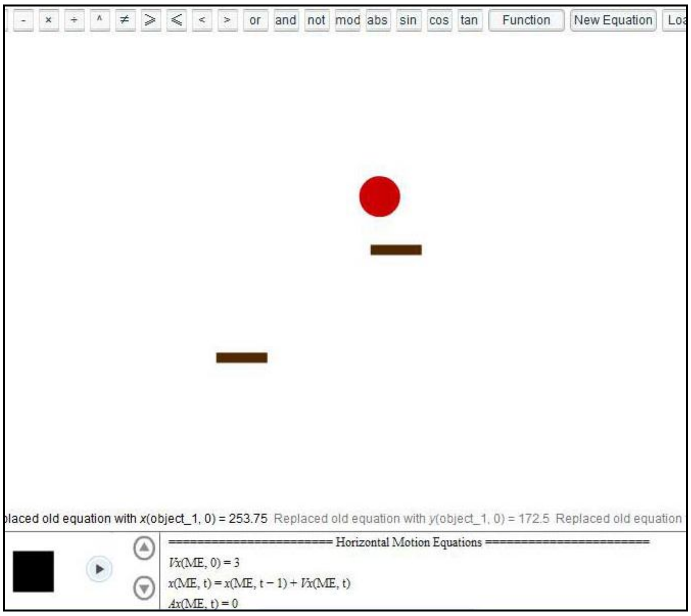

Figure 2 The "Juggler" microworld in MoPiX.

In Phase 2, we asked the students to execute the Juggler's model, observe the animation generated and try to identify the conditions under which each object interacted with the others, as well as the objects' possible changes of behaviour resulting from these interactions. The students were encouraged to discuss with their teammates how they would change the "Juggler" microworld and embed in it their own ideas regarding the behaviour of the objects involved. In the process of changing the half-baked microworld, students were expected to deconstruct the existing model so as to link the behaviours generated on the screen to its equations' formalism and reconstruct the microworld, employing strategies that would depict their ideas about the new model's animated behaviours.

4.2 Context and participants

The experiment took place in a secondary Technical and Vocational Education School in Athens with eight 12th grade students (17 years old), studying mechanical engineering. Through out their three years of tuition, the students had engaged several times in modelling activities, as their curriculum requested. However, in these modelling activities the mathematical aspect (the part of the mathematisation of a situation in hand) remained practically invisible to them. Using ready-made mathematical models the teacher provided, the students were mostly engaged in executing routine lab tasks (i.e. assembling parts) or at the very best performing standard numerical manipulations (i.e. solving an equation to find the unknown quantity). Mathematics, on the other hand, was taught as a completely separate "general education" subject, having no connections to vocational knowledge. Recognising in this context a gap between formal mathematics teaching and the established practices in modelling vocational situations, we introduced MoPiX environment as a computational medium holding the potential to bring together the use of algebraic-like formalism with the construction and manipulation of animated models.

The students worked in groups of two or three for 25 school hours. The members of each workgroup had at their disposal a PC connected to the Internet so as to save and retrieve models from a virtual library, translations in Greek of selected equations' symbols, a notebook for ideas and remarks and the MoPiX manual. The experimentation process was carried out by two researchers. One of them acted as a teacher-researcher since she had been a teacher in this school for several years and the other one as a co-researcher. The adopted methodological approach was based on participant observation of human activities taking place in real time. The researchers circulated among the teams, posing questions, encouraging students to explain clearly their ideas and strategies, asking for refinement and revision when appropriate and challenging students to express openly their thoughts and put into effect their ideas. Finally, at specific time points during the sessions, all groups of students were involved in classroom discussions orchestrated by the researchers.

4.3 Data collection and elements of observation

An ongoing analysis of the data, in parallel with the implementation of the tasks, provided the basis for the researchers to plan the upcoming sessions and facilitated the documentation of the types of interventions and activities that appeared to contribute to the students' constructions of meanings. The data collected and analysed were:

- Audio and video recordings (deriving from a screen capture software (HyperCam2) for the inter workgroup communications and from a camera/voice recorder for the intra workgroup communications).

- Students' notes and answers on the worksheets we provided at certain phases of the experimentation.
- Students' MoPiX models.
- Researchers' field notes.

4.4 Method of analysis

For the analysis we transcribed verbatim the audio recordings of two groups of students for which we had collected detailed data throughout the experimentation process, as well as from several significant learning incidents from the other workgroups. In analysing the data, we primarily looked for instances where meanings stemming from the students' interaction with the available formalism were expressed. The unit of analysis was the episode, defined as an extract of actions and interactions performed in a continuous period of time around a particular issue. In most cases, the transcripts of students' communications included in an episode were meaningless if they were not related to the sequences of actions that students had already carried out. For that reason we have developed detailed memos (Strauss and Corbin, 1998) of the students' activities, based on the screen capture files we gathered throughout the teaching sequence. The episodes which are the main means of presenting and discussing the data were selected (a) to have a particular and characteristic bearing on the students' interaction with the available tools during which the MoPiX formalism was used to construct mathematical meaning and (b) to represent clearly aspects of the reification processes emerging from this use (e.g. articulating variables and invariables within an equation or conceptualising the structure of an equation as a system of connections and relationships between its component parts).

5. ANALYSIS AND INTERPRETATIONS

To elaborate on the ways in which the students' understanding about the MoPiX equations evolved through their interaction with the environment, we present in this section some illustrative episodes of their work.

5.1 Interpreting existing equations' symbols

In the first phase of the experimentation, the students were introduced to the "One Red Ball" microworld which consisted of a single ball performing a combined motion both in the vertical and horizontal directions. As they were asked to reproduce this motion, the students focused their attention on the equations already existing in the environment's "Equations Library" and tried to detect those that would attribute to their object the desired behaviours. In this process, they often attempted to reveal the meaning conveyed by these equations' symbols.

5.1.1 Identifying the language of MoPiX equations

The first episodes document how students, trying to define the effect specific equations would have on their object, took into consideration only the "natural language" aspect incorporated in the MoPiX notation system. For example, the students in Group A used the "amIHittingGround(ME, t) = (y(ME, t) ≤ (height(ME, t) ÷ 2)) and Vy(ME, t) ≠ 0" equation, presuming that it would make their object "hit the ground". However, this equation describes just the conditions under which such an event would occur (i.e. when "y(ME, t) ≤ (height(ME, t) ÷ 2)" and "Vy(ME, t) ≠ 0") and not the action itself.

S2	*We want to ascribe the one that makes it "Hit A Side" [They search the Equations Library and they find the "amIHittingASide" equation which they ascribe to the object].*
S1	*I am hitting…*
S2	*"ME"? Me? What is that? Hitting A Side…. Ok. It's fine.*
S1	*Am I hitting ground? I'm hitting the ground, as well.*
S2	*The ground … [Searching the "Equations Library"] Where is this?*
S1	*That's it? That's it or… Oh, no. That's it. [They ascribe the amIHittingGround equation to their object and move to the next category of equations.]*

The students' decision to attribute to their object the equations mentioned in the above excerpt was exclusively based on the fact that the name of those equations (i.e. the first symbol on their left part) was very close to natural language utterances (e.g. the phrase "Am I Hitting the Ground?"). Although they also detected the presence of other symbols in the equations they used (e.g. "ME"), the students didn't seem willing to make any attempt to attribute meaning to those symbols. Apart from the equations' names, all the other symbols on the left or the right part of the equations were completely disregarded.

At this point of their work, these students also tended to disregard equations whose name didn't provide them a clear indication on the kind of behaviour they would attribute to an object if assigned to it.

S1	*[Pointing to the equation Vx(ME ,0) = 3] Vx [pronounces it vee ex], what is it? [To S2] The v x. Look it up to find out. The v x, what is it?*
S2	*[While searching for a translation in the Translations Sheet]. Should we care about those "Vx" s? It could be inside an equation. Let's look if an equation includes it. There it is, it's inside the equation. [He points to the "amIHittingASide (ME ,t) = (x(ME ,t) ≤ 0 or x(ME ,t)≥ 99) and Vx(ME ,t) ≠ 0" equation].*
S1	*Ahhh, it includes it, doesn't it?*
S2	*So?*
S1	*We don't need it.*
S2	*It says that the Vx means ME? Let it go. Let's not get involved with finding out what this means.*

The students' decision not to use the "V*x*(ME, 0) = 3" equation was based on the fact that they couldn't attribute meaning to the "V*x*" symbol, as they did before for the "amIHittingGround". Since "V*x*" was definitely not an utterance used in their everyday language, students sought for another criterion to conclusively decide whether they would assign this equation to their object or not. The fact that some other equation included the "V*x*" symbol seemed just to reinforce their decision not to use an equation whose name didn't satisfy the "natural language" criterion on which they had based their previous choices.

During those early phases of their experimentation, the students appeared to view and interpret symbols in isolation and to disregard the relationships among an equation's symbols as well as the different roles a symbol may play when used in different equations (e.g. the "Vx" symbol). Developing a "natural language" criterion in order to decide which equations to use, the students seemed to initially resort to "other experiential aspects, more accessible to them than the structural one". (Radford, 2000, p.240).

5.1.2 Articulating the role of variables

As they continued their experimentation with MoPiX, the students seemed to gradually abandon the "natural language" criterion and shifted their attention into identifying the meaning of the symbols and particularly into articulating their understanding about the variables and their specific role in the equations.

The students of Group B, for instance, detected in the "Equations Library" two equations that seemed to be describing the velocity in the *x* axis, the "V*x*(ME, 0) = 3" and the "V*x*(ME, *t*) = V*x*(ME ,*t* – 1) + A*x*(ME, *t*)". Their decision to attribute to their object the latter so as to define velocity at some other time instance besides "0", was the result of a comparison between the two equations' left parts. The symbols on the right part and the meaning they conveyed (e.g. the "A*x*") were still disregarded.

R2 *In the first equation [Vx(ME ,0) = 3] instead of "t", what do we have?*
S1 *The "0".*
R2 *That does this zero mean?*
S2 *That time is zero? No …*
S1 *That you don't define the time in this case.*
R1 *Ok. If I told you to talk about some other time here… Some other second.*
S1 *Yes?*
R1 *What would you do?*
S1 *[Attributing the Vx(ME, t) = Vx(ME, t – 1) + Ax(ME ,t) equation to the object they construct] We would say "with some velocity".*

The students' apparent focus on the equations' left hand sides in this excerpt indicates an implicit consideration of them as "templates" consisting of two distinct elements: the represented quantity (in this case the velocity in the *x*-axis) and time. The students seemed to recognise that in order to describe the represented quantity (i.e. the velocity) at the "0" time instance and at any other time instance, they just needed to maintain the represented quantity's symbol as it was (i.e. the "Vx") and change the one that corresponded to time (i.e. "0" or "*t*"). In the case of the initial velocity, time in the template was substituted by a specific arithmetic value (i.e. "0"), while in the case of the velocity at any other time instance, the independent variable "*t*" was required.

In a number of subsequent episodes, the same students seemed not just to articulate their understanding about the role of specific variables, such as the variable of time, or just about particular symbols in the equations, but eventually about the whole string of an equation's symbols and the relationships among them. In the following extract the students of Group B talk about the:

"*x*(ME, *t*) = *x*(ME, *t* – 1) + V*x*(ME, *t*)"

equation which refers to an object's position in the horizontal axis. Both S1 and S2 have already discerned that "*x*" stands for the object's position in the *x* axis, "ME" for the object to which the equation will be attributed, "V*x*" for the velocity in the *x* axis and "*t*" for time.

S1 *It [i.e. x(ME,t)] is the object [i.e. the "ME" part] in function with time [i.e. the "t" part].*
S2 *In function with the time you say.*
S1 *Always.*
R2 *What does this mean?*
S1 *[goes on disregarding the question and points at the x(ME,t–1)] It's your object [i.e. "ME"] in function with time minus 1 [i.e."t–1"].*
R2 *What does "in function with time" mean? Why do you say that? Can you explain it to me?*
S1 *How much... In every second, for example, how much it moves.*
R2 *Meaning?*
S2 *Wait a minute! [Showing both parts of the equation] The equation is this one. All of this. It's not just these two [i.e. the x(ME,t) and the x (ME,t–1)].*
S1 *And it says …. Minus 1, which means that in every second of your time it subtracts always 1, resulting to something less than the current time. Plus your velocity.*

Drawing on his previous experience with the MoPiX equations, S1 started to independently interpret the;

"*x*(ME, *t*) = *x*(ME, *t* – 1) + V*x*(ME, *t*)"

equation's symbols, moving from left to right. As he had already articulated his understanding about the variable of time and its role in the "template" presented above, he interpreted the "*x*(ME, *t*)" symbol using the phrase "in function with time". The functional relationship between the object's position and time was also stated as he interpreted the "*x*(ME, *t* – 1)" symbol, with the exception that in this case the phrase he used was: "in function with time minus 1". Having independently interpreted those two varying quantities, the position of the object at the current time instance (i.e. the "*x*(ME, *t*)") and the position of the object at the previous time instance (current time minus 1) (i.e. the "*x*(ME, *t* – 1)"), S1 attempted to also interpret the relationship between them. Subtracting the two quantities (i.e. "*x*(ME, *t*) – *x*(ME, *t* – 1)"), as they are located on different sides of the equal sign, S1 viewed a relation between them which he defined as the distance that the object has covered in a second of time. S2, who comprehended the kind of correlation S1 had made between those varying quantities, intervened to stress the fact that his team mate hadn't taken into account all the symbols present in the equation in hand. S1, who up to that point disregarded the "Vx(ME, t)" symbol on the right part of the equation, took an overall view of the equation and interpreted it, not by merely referring to the comprising symbols, but also by referring to the connection between them.

The developed meanings here concern the nature of the joint variation between the object's position and time and the specification of the corresponding functional relationship. It is noticeable that the existence of symbols standing for successive time units (i.e. "t" and "t – 1") seems to mediate the students' attempts to define the exact relation between the object's position, time and velocity and to raise their awareness to the structure of the equation as a system of connections and relationships between its component parts.

5.2 Exploring the distinction between variables and numerical values to control motion animations

As the students gained familiarity with the MoPiX formalism, they didn't just confine themselves to reproducing a given motion, as it was the initial task, but started expressing their own personal ideas about the ways their objects should move on the "Stage". In order to put into effect those ideas, the students turned to already existing equations which they edited and modified so as to describe new behaviours. In this process, they performed changes only to the equations' content (i.e. the symbols comprising the equations), while they left the structure of the equations completely intact.

One of the main elements they often altered in an equation was the arithmetic values present on either its left or right hand side. The following two episodes document the ways in which the students edited several equations' arithmetic values, so as to create new behaviours for their objects, and used the graphical representation generated on the screen to verify the effect these new equations had on their objects.

The students of Group B, in their attempt to reproduce the motion of the "One Red Ball", picked from the "Equations Library" and attributed to their object a set of equations that caused it to move horizontally on the "Stage". The next step in their experimentation was to look for equations that would also make it move vertically. The first equation they came across in the "Library" and attributed to their object was the "Vy(ME, 0) = 0" which prescribes the initial velocity of the object in y axis to be 0.

S2 *[To S1] Press "Play". You didn't do anything. You just made the velocity 0 at the 0 time instance. Its initial velocity is 0. You did nothing to it. It didn't change, to move downwards [The motion of the ball is exactly the same as the one before attributing the "Vy(ME, 0) = 0"equation to their object.]*
S1 *Yes, yes.*
S2 *That's what I'm saying. Change it. Give it some initial, we should give it an initial velocity. Isn't it better?*
R2 *Whatever you like.*
S2 *Give "3" as an initial velocity. The equation you used before, with the difference that after the equal sign, we will place a "3". There, move it up. [He takes the "Vy(ME, 0) = 0" equation and places it in the Equations Editor. He turns it into "Vy(ME, 0) = 3"].*

After attributing the "Vy(ME, 0) = 0" equation to their object and starting the animation, the students noticed that the equation they used didn't have the desired effect on their object. This observation triggered the implementation of a series of changes on the initial equation's arithmetic values. In the first place, S2 suggested that the "0" on the right part of the equation needed to be changed into "3" so as for the object to have "some initial velocity". The student appeared to regard the right part of the equation as a placeholder for specific numerical values, destined to define the object's initial velocity. In the following episode, the kind of conceptualisation mentioned above for the instantaneous velocity evolved to take into account "time" as a variable. This need emerged from the fact that the successive changes to the arithmetic value on the right hand side of the equation "Vy(ME, 0) = 0", didn't cause the object to move at all. The object remained still since both the original as well as new equations developed, referred to the initial velocity (i.e. the velocity at the 0 time instance) and not to the velocity over time. As students searched for ways to incorporate the "all the next time instances to come" element in their equation, they decided that they needed a symbol which they would "just look at and know that it represents the infinity".

S2 *That means that we have to express the "unlimited".*
S1 *Time… something. Always plus 1.*
S2 *Do we need a symbol for this?*
R2 *Do we need a symbol? It's a good question. How do you plan to express it?*
S2 *With symbols we usually express something that we can't describe accurately.*
S1 *Plus… t. [He writes down Vy(ME, t) = 3]. [Showing the "t"] So, when I look at this symbol.*
S2 *I'll know it represents the infinity.*

Replacing the "0" on the left hand side of the equation (i.e. an arithmetic value) with "t" (i.e. a variable), the students formed the "Vy(ME, t) = 3" equation to express the object's velocity in the y-axis for all time instances. In their new "template" the second argument in the parentheses constitutes a new placeholder which can be either completed with a specific arithmetic value or with the variable of time. This process was facilitated by the way in which the computational setting provided students a web of structures (Noss and Hoyles, 1996) which they could exploit in order to interpret the equations' formalism and identify its role in creating specific animations. In this case, the use of a variable appears as a result of the students' preceding observations on the ways in which specific changes in the equations formalism affected the animation generated on the screen. We suggest that the students moved their focal point from the process of successively replacing specific numerical values in the initial equation, which indicates a process stance to the equations, to considering the construction of a functional relationship. This required a recognition of which manipulations it was possible and useful to perform in the initial equation, indicating a developing focus on the structure of the equations and the gradual acquisition of structure sense (Hoch and Dreyfus, 2004). This kind of activity evolved in the subsequent phases of the students' experimentation as they used the available formalism to construct new equations.

5.3 Relating different objects' behaviours by constructing new equations

The next episodes describe how the Group A students, in the course of changing the "Juggler" half-baked microworld, didn't just use or edit already existing equations to define their objects' behaviour, but constructed from scratch two new equations using the MoPiX formalism. The two interconnected equations were developed in response to their need to determine the colour of an existing object (i.e. the ball) according to the position of a newly inserted one (i.e. an ellipse). In this process, the students invented new symbols with which they encoded meaning and determined not only the new equations' content but also their structure.

The main idea that the students wanted to bring into effect was to "make the ball change its colour according to the ellipse's position on the Stage". Due to their familiarity with the computer environment, the students were in position to already know that there was no such equation in the "Equations Library" and that they had to build a completely new one so as to express their idea. The first equation developed for that purpose was the one that described the event to which the ball would respond and change its colour. Talking about how they would achieve this goal, the students decided to include in their equation the two objects' Y coordinates and relate them to each other so as for the ball to know "I am below now" (i.e. below the ellipse).

S1 *What I want to happen is that: when the ball is above the ellipse to become red and when it is below the ellipse to become green. I don't care about when it hits [i.e. the ground]. Can we do this?*

S2 *You have to define something. How did you define the plane which is the ground? How did you define that on the right side there is a wall and that you can't go beyond this wall? [The "ground" and the" wall" are elements of already existing equations that the students had used].*

S1 *[To R1] Excuse me … The x, y coordinates. Can't the environment recognize them? Their values. Where the objects are situated. Can't it recognize them?*

R1 *Yes.*

S1 *It can recognize them. So I can say that I want this [i.e. the ball] to change colour.*

R1 *Yes?*

S1 *When it is situated in a Y below the Y of this one for example [i.e. the ellipse].*

R1 *You know … I'm thinking … Will the ball know when it is below or above the ellipse?*

S2 *That's what we will define. We will define the Ys.*

S1 *This. The: "I am below now". How will we write this?*

S2 *Using the Y. Using the Y. The Y. That is: when its Y is 401, it is red. When the Y is something less than 400, it's green!*

S1 *Let's start on that. Let's do it.*

Having conceptualised the effect they would like the new equation to have on their object, the students in the above extract came to a decision about two distinct elements regarding the equation under construction: its content (i.e. the symbols it would include: the Y coordinates) and its structure (i.e. the ways in which those symbols would be related to each other: using a "less than" sign). Moreover, what is noticeable here is that students were able to concretize their decisions and relate the content and the structure of the equation directly to the formal mathematical properties inherited by the XY coordinate system.

Starting developing the equation on the environment's "Editor", the students came across the fact that there was no in-built MoPiX symbol (such as the "x", "Vx", that respectively represent the position and the velocity parallel to the x-axis) to express the idea of an object becoming green under certain conditions. The first thing they did so as to overcome this problem was to invent a new symbol that would express a varying quantity. The "gineprasino" (i.e. "become green" in Greek) symbol was decided to represent in the new equation's "template" a varying quantity and the "t" variable to be used so as to describe the "at any time instance" aspect.

Having completed the left part of the equation (i.e. the "gineprasino(ME, t)=_______") and in order to complete the right part as well, the students used (as noted before) the Y coordinates of the two objects and the less than sign to relate them. Surprisingly, the way in which they used the Y coordinate concept for each object was completely different. The ball's Y coordinate was expressed in terms of a quantity varying over time (i.e. the "y(ME, t)"), while the ellipse's Y coordinate was expressed in terms of the constant arithmetic value corresponding to the object's at that time position on the Stage (i.e. the "274"). Adding the "less than" sign in between, the first equation eventually developed was the:

"gineprasino(ME, t) = y(ME, t) ≤ 274".

Unexpectedly, this equation didn't cause the ball to become green since it solely described the event to which the ball would respond (being below the ellipse) and not the ball's exact behaviour after the event would have occurred (change its colour). To overcome this obstacle, the students decided to construct another equation in which they tried to find out ways to integrate the "gineprasino" variable. The structure of an equation they had previously used:

"Vx(ME, t) = (not(amIHittingASide(ME, t – 1)) × (Vx(ME, t – 1) + Ax(ME, t – 1)) + (amIHittingASide(ME, t – 1)) × (Vx(ME, t – 1)× –1)" which explains what happens to a ball's velocity when it hits on one of the "Stage's" sides and the way in which the "amIHittingASide(ME, *t*)" variable was incorporated in it, were the two elements that the students identified as helpful in the construction of their second equation. Recognising the "V*x*(ME, *t*)" equation's similarity to the one they were trying to develop (instead of what happens to the velocity under certain circumstances they would determine what happens to the colour) and a similarity between the "amIHittingASide" and "gineprasino" variables, led students to duplicate this equation's structure, eliminate the content and use it as a template to designate what happens to the ball's colour when it is below the ellipse. The equation to be completed was the "greenColour(ME, *t*)= ______", which they had used in the past in the form of "greenColour(ME, *t)* = 100" in order to give 100% green colour to their objects.

To link the first equation which encompassed a new symbol to the second one which included symbols that were in-built in the MoPiX environment (i.e. the "greenColour"), the students utilised the "gineprasino" variable in a similar way to the "amIHittingASide", exploiting the fact that this variable may receive two distinct values (1 or 0) according to the ball's position. To complete the equation, students used two arithmetic values, the "0" and the "100", to express the percentage of the green colour the ball would contain in each case (i.e. the ball being above or below the ellipse). Thus, the second equation developed was the:

"greenColour(ME ,*t*) = not(gineprasino(ME, *t*)) × 0 + gineprasino(ME, *t*) × 100".

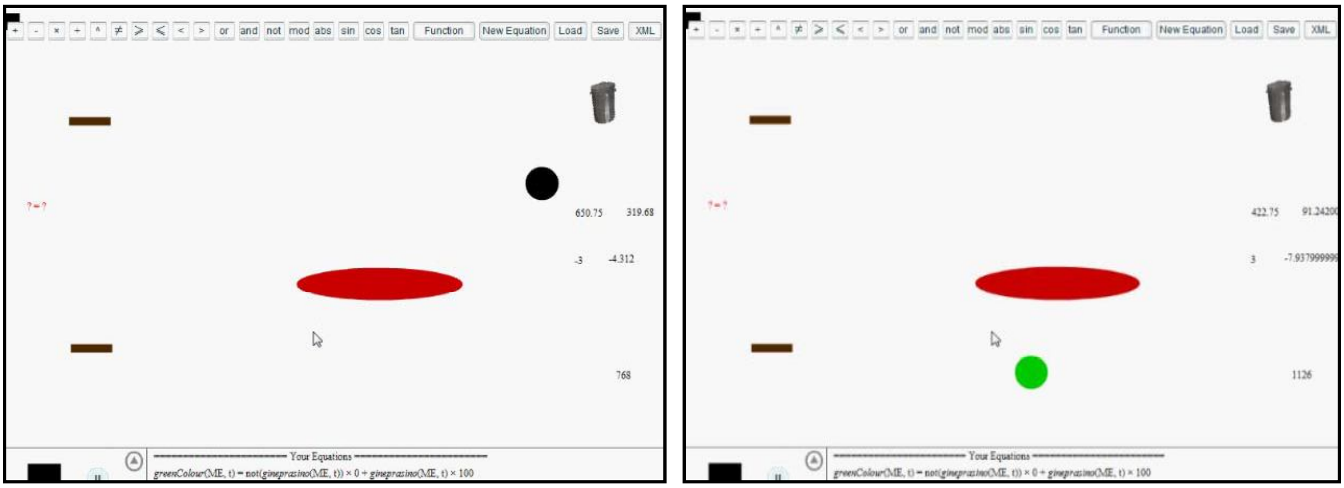

Figure 3 The ball's different percentage of green colour according to its Y position

The above episode contains many interesting events which indicate the existence of a qualitative transformation of the students' mathematical experience in reifying equations, emerging from their interaction with the available tools. These events suggest that the students were able not just to develop insights into the use of algebraic-like equations as means to create and control animated models but also to focus on the structural aspects of these equations, eventually conceptualising them as objects on their own right (Sfard, 1991). Moreover, the mathematical ideas generated and expressed in this episode, although they appear to be situated in the context of their genesis and use, entail the potential to be extended or transferred beyond the boundaries of the available symbolic system.

While building the first equation the students got engaged in processes such as inventing and naming variables (i.e. the "gineprasino" variable), relating symbols with mathematical systems (i.e. the XY coordinate system) and manipulating variables and numerical values as well as inequality symbols to produce the equation's structure. However, in building the second equation, the meaning generation evolved to include the students' view of equations as objects. The students, in the first place, extracted mathematical meaning from an equation that seemed to describe a behaviour similar to the one they intended to attribute to their object. Conceptualising a mapping between the idea "the ball should change its velocity when it hits on one of the Stage's sides" and the idea "the ball should change its colour when it is situated below the ellipse", the students duplicated the similar equation's structure and inserted new terms so as to define a completely novel behaviour for their object. This constitutes a clear indication that the students were able to relate the MoPiX equation under construction with a previously met equation's structure and recognise mutual connections between those two structures.

The manipulation of the second equation's terms reveals further their developing structural approach to equations. Inserting the "gineprasino" variable and providing it with new forms (i.e. the "not(gineprasino)"), the students seemed to have conceptualised the first equation as an object and used it as a means encode meaning and structure in the second one. We think that this reflects a kind of mathematical thinking that relates to the development of a good algebraic structural sense, accompanied with the acquisition of a functional outlook to equations as objects, which is considered to be crucial to the relational understanding. Furthermore, viewing the output of the first function as an input for the second, the students seemed to have developed understandings concerning higher-order mathematical processes, such as the composition of functions. Although the formalism of the equations used remains essentially context-bound, we see in this episode that the situated abstraction (Noss and Hoyles, 1996) of mathematical ideas shaped by the available infrastructure and expressed within it, can be "mapped onto" parts of formal mathematics (Hoyles, Noss and Kent, 2004). This form of abstraction cannot be considered as necessarily linked to decontextualisation, but it can be characterised as a process possibly containing within it "the seeds of something more general" (Noss and Hoyles, 1996, p. 49) (i.e. the composition of functions) that may be transferred in other contexts.

6 SUMMARY AND CONCLUDING REMARKS

Our purpose in this paper was to illustrate a particular approach to studying the construction of meanings emerging from the students' manipulations of algebraic-like formalism used to create and control animated models. Our aim was also to study the ways in which the use of formalism, when put in the role of an expression of an action or a construct (a model), can operate as a mathematical representation for constructionist meaning-making in context-specific situations. A central idea in our approach was that algebraic-like formalism, as a coherent part of constructionist computational environments, affords an opportunity to make visible important aspects of equations and thus to facilitate the development of structural conceptions of them as objects, rather than to relegate equations into black-box tools whose functionalities remain opaque.

We have illustrated our analysis by looking at episodes in which students engaged in processes of manipulating the available formalism to generate and control animated models. MoPiX offered an environment in which the students could interpret and use ready-made algebraic-like equations to define their models' behaviour, inspect and modify those equations by editing them and construct new ones, employing in any case the available formalism. The interpretation, use, editing and construction of MoPiX equations required focusing on the connections and relationships among the component parts of the equations and thus fostered the emergence of a structural conception of them as objects through a reification process.

In the first part of the analysis an initial "natural language-driven" conceptualisation of the MoPiX equations seemed to have been leading students towards the development of criteria for an isolated interpretation of the MoPiX formalism. However, the second part of the analysis highlighted the development of a dialectic relationship between action and meaning that was mediated by the visual and symbolic representations provided by the tool. As soon as the students became familiar with testing their models and observing the animations generated on the "Stage", their interactions with the computer environment became strongly associated with editing the content of the respective equations. This revealed a subtle shift in meaning generation from a process-oriented view to equations into an object-oriented view and the progressive development of algebraic structure sense. In the last part of the results, students' previous experience with the MoPiX formalism seemed to become part of their repertoire, allowing them to experiment and to develop their own equations so as to relate the positions of two different objects and determine the one's colour according to this relation. This kind of problem demanded a more analytic use of MoPiX equations than they had developed up to that point, such as constructing new equations following specific structural rules, specifying the values of the variables that form these equations and finally using the equations as objects to represent variables in other equations (i.e. a function composition process).

The foregoing examples demonstrate students' progressive building of connections between the syntax and semantics of the MoPiX equations and the objects' animated behaviours. The generation of a dialectic relationship between the reification of the equations and the meaning-generation through the mediation of the computational environment, indicated that students were provided with an appropriate symbolic framework which facilitated the recognition of equations as tools for controlling and creating animated models. The students' attempts to develop a structural approach to the equations were primarily facilitated by the computational environment's feedback. Students had the opportunity to run their models, observe their visual outcome - linking it with the available formalism - and consequently refining them by modifying the existing equations or constructing new ones, gradually focusing on the equations' structural aspects.

As it is becomes clear from the analysed episodes, students were not engaged in mechanical symbolic manipulations. Rather, they seemed motivated to actively struggle for meaning in every stage of their interaction with the available formalism, which facilitated their engagement at different layers of complexity (Simpson et al., 2006) in the exploration and symbolisation process and potentially afforded the emergence of a variety of learning trajectories. Our results indicate that most of the students had a multiplicity of ways by which they accessed different layers of complexity while interacting with MoPiX formalism at different times and for different purposes. Under this perspective, reifying an equation in this context was not a one-way process of understanding hierarchically-structured mathematical concepts but a dynamic process of meaning-making, webbed by the available representational infrastructure (Noss and Hoyles, 1996) and the ways by which students drew upon and reconstructed it to make mathematical sense. Of course, the issue that is raised concerns the 'distance' between the mathematical meanings emerging from students' interactions with MoPiX formalism and the actual curriculum. Our aim was not to provide a comparison between the meanings generated in the MoPiX environment and official curriculum knowledge but rather to highlight the mathematical possibilities for constructionist learning in context-specific situations through the use of innovative formalism. This may be provide a rationale for subsequent research studies involving, for instance, the design and use of new computational tools in which the available formalism would be closer to standard mathematical notation.

We finally turn to the issue of design, attempting to highlight the link between formalism and constructionist learning through the construction and modification of models. A main finding of our analysis is that the trajectory of the students' thinking about the equations as they interacted with the MoPiX environment was interrelated with aspects of the available symbolic system (e.g. the fact that parts of the equations in several cases were treated as templates). Thus, it is interesting to reassess which aspects of the formalism, (conventional or not) as a coherent part of specially designed constructionist environments, can assist students to deal with the symbolic and structural aspects of the equations and thus facilitate them in developing their understandings of the role the symbolic formalism plays in generating animated models.

Concluding in this line, this may suggest that formalism in the role of an expression of a model - the basic tenet of constructionism - can open the access to situations of students' interactions with genuine algebraic-like representational systems which facilitate meaning-generation in activity.

ACKNOWLEDGEMENTS

The research took place in the frame of the project "ReMath" ('Representing Mathematics with Digital Media'), European Community, 6th Framework Programme, Information Society Technologies, IST-4-26751-STP, 2005-2009 (http://remath.cti.gr). The participating research teams are: (DIDIREM) University Paris 7 Denis Diderot - Paris, France; (NKUA-ETL) National Kapodistrian University of Athens - Educational Technology Lab - Athens, Greece; (CNR-ITD) Consiglio Nazionale delle Ricerche - Instituto Tecnologie Didattiche - Genova, Italy; (MeTAH) MeTAH and Leibniz - IMAG - Grenoble, France; (UNILON) University of London - Institute of Education - London Knowledge Lab - London, UK; (UNISI) University of Siena - Department of Mathematics - Siena, Italy. The present study was designed as one part of a *cross-experimentation methodology* developed in the project (see Artigue, Bottino, Cerulli, Kynigos, Lagrange, Maracci, Maffei, Mariotti and Morgan 2006) according to which each team experimented with an interactive learning environment for mathematics designed by another team. In the present study NKUA-ETL team experimented with MoPiX designed by the IOE - LKL team (K. Kahn, N. Winters, D. Nikolic, C. Morgan and J. Alshwaikh). This is a new approach to collaboration, seeking to facilitate deeper understanding across research teams with diverse practices and cultures, and to progress towards integrated views of technology use in mathematics education.

NOTES ON MoPiX

MoPiX v.1 is available at:
http://remath.cti.gr/content_by_cat.asp?ContentId=273&CatID=39,
while version 2 is available at:
http://modelling4all.nsms.ox.ac.uk/Resources/MoPiX/en/about.html.
To describe the computer environment as accurately as possible we used the MoPiX v.2 online help (http://modelling4all.nsms.ox.ac.uk/ModelOldVersion/?MoPiX=1&session=new), as well as the LKL Technical Report (Winters et. al, 2006) and the MoPiX v.1 User Manual.

REFERENCES

Artigue, M., Bottino, R. M., Cerulli, M., Kynigos, C., Lagrange, J–B., Maracci, M., Maffei, L., Mariotti, M.–A. and Morgan, C. (2006) *Integrative Theoretical Framework*, Project Number: IST4–26751, ReMath.

Arzarello, P., Pezzi, G. and Robutti, O. (2003) Modelling body motion: An approach to functions using measure instruments, in Henn, H. W. and Blum, W. (eds) *ICMI Study 14: Applications and Modelling in Mathematics Education*, Dortmund: University of Dortmund, 23-28.

Cobb, P., Confrey, J., diSessa, A., Lehrer, P. and Schauble, L. (2003) Design experiments in educational research, *Educational Researcher*, **32**(1), 9–13.

Cobb, P., Gravemeijer, K., Yackel, E. and McClain, K. (1997) Mathematizing and symbolizing, in Kirshner, D. and Whitson, J. (eds) *Situated Cognition*. Mahwah, NJ: Lawrence Erlbaum Associates, 151-233.

diSessa, A. (2000) *Changing Minds, Computers, Learning and Literacy,* Cambridge, MA: MIT Press.

Dreyfus, T. and Hoch, M. (2004) Equations - A structural approach, in Hoines, M. J. and Fuglestad, A. B. (eds) *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education,* Bergen, Norway: Bergen University College, **1**, 152-155.

Dubinsky, E. (2000) Meaning and formalism in mathematics, *International Journal of Computers for Mathematical Learning*, **5**(3), 211–240.

Gravemeijer, K., Cobb, P., Bowers, J. and Whitenack, J. (2000) Symbolizing, modeling and instructional design, in Cobb, P., Yackel, E. and McClain, K. (eds) *Symbolizing and Communicating in Mathematics Classrooms classroom. Perspectives on discourse, tools, and instructional design*, Mahwah, NJ: Lawrence Erlbaum Associates Publishers, 225-274.

Healy, L. and Kynigos, C. (2010) Charting the microworld territory over time: design and construction in learning, teaching and developing mathematics, *Zentralblatt für*

Didaktik der Mathematik (ZDM) - The International Journal on Mathematics Education, **42**, 63–76.

Hoch, M. and Dreyfus, T. (2004) Structure sense in high school algebra: The effect of brackets, in Hoines, M. J. and Fuglestad, A. B. (eds) *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education,* Bergen, Norway: Bergen University College, **3**, 49-56.

Hoyles, C., Noss, R. and Kent, P. (2004) On the integration of digital technologies into mathematics classrooms, *International Journal of Computers for Mathematical Learning*, **9**(3), 309-326.

Hoyles, C. and Noss, R. (2008) Next steps in implementing Kaput's research programme, *Educational Studies in Mathematics,* **68** (2), 85-94.

Kahn, K. (1996) ToonTalk - An animated programming environment for children, *Journal of Visual Languages and Computing*, **7**, 197–217.

Kahn, K. (2004) ToonTalk - Steps towards ideal computer - based learning environments, in Tokoro, M. and Steels, L. (eds) *A Learning Zone of One's Own: Sharing Representations and Flow in Collaborative Learning Environments*. Amsterdam: IOS Press, 253-270.

Kahn, K., Noss, R., Hoyles, C. and Jones, D. (2006) Designing digital technologies for layered learning, in Mittermeir, R. T. (ed) *Informatics Education – The Bridge between Using and Understanding Computers, Proceedings of International Conference in Informatics in Secondary Schools – Evolution and Perspectives,* Berlin: Springer, 267-278.

Kaput, J. J. (1994) Democratizing access to Calculus: New routes using old routes, in Schoenfeld, A. (ed) *Mathematical Thinking and Problem Solving*, Hillsdale, NJ: Erlbaum, 77-156.

Kaput, J. J., Noss, R. and Hoyles, C. (2002) Developing new notations for a learnable mathematics in the computational era, in English, L. (ed) *Handbook of International Research in Mathematics Education,* London: Lawrence Erlbaum, 51-75.

Kaput, J. J. and Roschelle, J. (1997) Deepening the impact of technology beyond assistance with traditional formalisms in order to democratize access ideas underlying calculus, in Pehkonen, E. (ed) *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education,* Helsinki, Finland: University of Helsinki, **1**, 105-112.

Kaput, J. J. and Schorr, R. (2002) Changing representational infrastructures changes most everything: The case of SimCalc, algebra & calculus, in Heid, K. and Blume G. (eds) *Research on the Impact of Technology on the Teaching and Learning of Mathematics*, Mahwah, NJ, US: Erlbaum, 47-75.

Kieran, C. (1992) The learning and teaching of school algebra, in Grouws, D. A. (ed) *Handbook of Research on Mathematics Teaching and Learning*, New York: Macmillan, 390-419.

Kynigos, C. (2004) Black and white box approach to user empowerment with component computing, *Interactive Learning Environments ,* **12** (1-2), 27-71.

Kynigos, C. (2007) Half-baked Logo microworlds as boundary objects in integrated design, *Informatics in Education,* **6** (2), 335-359.

Kynigos, C. and Psycharis, G. (2003) 13 year-olds meanings around intrinsic curves with a medium for symbolic expression and dynamic manipulation, in Paterman, N., Dougherty, B. and Zilliox, J. (eds) *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education,* University of Hawaii, Honolulu, U.S.A. Hawaii, **3**, 165-172.

Mor, Y., Noss, R., Hoyles, C., Kahn, K. and Simpson, G. (2006) Designing to see and share structure in number sequences, *International Journal for Technology in Mathematics Education*, **13**, 65-78.

Morgan, C. and Alshwaikh, J. (2008) Learning about motion in a multi-semiotic environment, in Joubert, M. (ed) *Proceedings of the British Society for Research into Learning Mathematics,* University of Southampton, Southampton, UK, **28**(2), 73-78.

Nemirovsky, R. (1994) On ways of symbolizing: The case of Laura and the velocity sign, *Journal of Mathematical Behavior*, **13**, 389-422.

Nemirovsky, R. and Monk, S. (2000) "If you look at it the other way...": An exploration into the Nature of Symbolizing, in Cobb, P., Yackel, E. and McClain, K. (eds) *Symbolizing and Communicating in Mathematics Classrooms,* Mahwah, NJ: Lawrence Erlbaum Associates, 177-221.

Nemirovsky, R., Tierney, C. and Wright, T. (1998) Body motion and graphing, *Cognition and Instruction,* **16** (2), 119-172.

Noss, R. (2004) Designing a learnable mathematics: A fundamental role for computer?, *10th International Congress on Mathematics Education, Abstracts, Plenary and regular lectures.*

Noss, R., Healy, L. and Hoyles, C. (1997) The construction of mathematical meanings: connecting the visual with the symbolic, *Educational Studies in Mathematics*, **33**(2), 203-233.

Noss, R. and Hoyles, C. (1996) *Windows on Mathematical Meanings*: *Learning Cultures and Computers*, Dordrecht: Kluwer.

Papert, S. (1980) *Mindstorms: Children, Computers, and Powerful Ideas*, New York: Basic Books.

Papert, S. (1991) Situating Constructionism, in Harel, I. and Papert, S. (eds) *Constructionism*, Norwood, NJ: Ablex Publishing Corporation, Chapter 1.

Radford, L. (2000) Signs and meanings in students' emergent algebraic thinking: A semiotic analysis, *Educational Studies in Mathematics*, **42**(3), 237-268.

Sfard, A. (1991) On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics*, **22**, 1-36.

Sherin, B. L. (2001) How students understand physics equations, *Cognition and Instruction*, **19**, 479-541.

Simpson, G., Hoyles, C. and Noss, R. (2006) Exploring the mathematics of motion through construction and collaboration, *Journal of Computer Assisted Learning*, **22**, 114-136.

Strauss, A. and Corbin, J. (1998) *Basics of Qualitative Research: Techniques and procedures for developing grounded theory* (2nd ed.), Thousand Oaks, CA: Sage Publications.

Winters, N., Kahn, K., Nikolic, D. and Morgan, C. (2006) Design sketches for MoPiX: A mobile game environment for learning mathematics, London Knowledge Lab, Technical Report.

BIOGRAPHICAL NOTES

Chronis Kynigos, is the Director of Educational Technology Lab (ETL, School of Philosophy, University of Athens, http://etl.ppp.uoa.gr) and a Professor of Educational Technology and Mathematics Education. In the past 15 years, he has led the pedagogical design of educational digital media and employed them in research involving a) aspects of designing and generating socio-constructivist learning environments in the classroom (emphasis on mathematics), b) design and implementation of innovative teacher education methods and c) the design and implementation of methods to infuse innovation in the educational system. These are: E-slate, a component kit to construct microworlds, MachineLab, a programmable 3D simulator and Cruislet, a 3D navigation system over a GIS based on the Cruiser platform. He has been responsible for ETL in the ESCALATE, C-Cube and Kaleidoscope projects and project director in the SEED and ReMath projects for RACTI. He is the author of an academic book titled 'The Investigations course' (in Greek) and has published more than 30 articles in refereed journals and research books. He is a member of the editorial board of the International Journal of Computers for Mathematical Learning and a founder of the Greek Association for Research in Mathematics Education. (http://www.ucy.ac.cy/enedim/). He is responsible for the creation of an ETL spinoff, a hi-tech collaborative games park in Athens called 'Polymechanon' http://www.youtube.com/watch?v=d8AJwADKd90.

Giorgos Psycharis holds a Ph.D in proportional reasoning using specially designed computational tools combining symbolic notation and dynamic manipulation of geometrical objects and relations. He has also been engaged in the design of exploratory software tools for mathematics, in classroom research into peer collaboration as well as in teachers' training courses for the use of computer technology in mathematics in the Greek educational system (Ministry of Education). He has worked as a post doc researcher at the Educational Technology Lab in the ReMath research project ("Representing Mathematics with Digital Media", http://remath.cti.gr, European Community, 6th Framework Programme, IST, IST-4-26751-STP, 2005-2009). He has recently taken a full-time position as a lecturer in mathematics education at the Mathematics Department of the University of Athens.

Foteini Moustaki is a vocational education school teacher and has a MA degree in "Didactics of specialised subjects with ICT" from the School of Philosophy, Department of Pedagogy, University of Athens. She is currently a PhD student in the University of Athens and a member of the Educational Technology Lab. She has participated in the Escalate (Enhancing SCience Appeal in Learning through Argumentative inTEraction, http://escalate.org.il/, European Community, 6th Framework Programme, Science education and careers. Project number: 020790, 2006-2007) and the ReMath project (Representing Mathematics with Digital Media, http://remath.cti.gr, European Community, 6th Framework Programme, Information Society Technologies, IST-4-26751-STP, 2005-2009).