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Modeling in Education: New Perspectives Opened by the Theory of Mathematical Working Spaces

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Abstract. In this chapter the guiding line is the mathematical work that students develop in modeling tasks. We report first on MWS studies adopting the idea of a modeling cycle. In these studies the MWS framework gives insight into the processes involved in students' work and it also sheds light upon the complex relationship between reality and mathematics, and the multifaceted relationship between the process of problem solving and the underlying educational goals. We question the consistency of a sharp separation between mathematics and reality and, assuming that mathematical modeling is what mathematicians do when they work on models, we look at this work outside of education. Rather than steps of translation between reality and mathematics, we have to think of modeling as a coupling of reality and mathematics that should allow students to develop insight into, and understanding of, both mathematics and reality. We also look to epistemological studies that distinguish between modeling and mathematization, and characterize modeling by (1) plurality of models (2) operativity (3) subjective and social interpretation. The plurality of models for a given reality has been exploited in research studies to design tasks that put at stake transitions or coordination between specific domains corresponding to different a priori suitable working spaces. In these studies mathematical work contributes to clarify and operationalize models as well as to give meaning to abstract mathematical notions. A MWS perspective could then break with a conception of modeling as an activity pursued individually and for individual competencies. In addition, considering the three dimensions of a working space should help to avoid a reduction of modeling to a translation, and of mathematics to a language.

Keywords. Mathematical Working Spaces, Connected Working Spaces, Modeling in education, Plurality of models, Operativity of models, Social aspects of modeling.

11.1 Introduction: modeling and students' mathematical work

There are a variety of approaches to modeling in mathematics education that regard both educational aims (cognitive and social) and task conception. In this chapter the guiding line is the mathematical work that students develop in modeling tasks. How do researchers characterize this work and the way in which it contributes to students' development of their mathematical conceptualizations and personal and social capabilities? How does the theory of MWS with the idea of work and the three dimensions, semiotic, instrumental and discursive[1] help this characterization? How does it help to analyze a priori the design of modeling situations and to make sense a posteriori of students' work in these situations?

Our investigation aims to characterize the specific contribution of MWS within the variety of approaches to educational modeling. We do not try to take into account all the frameworks in this variety. Throughout this chapter, we reference particular frameworks when necessary. As a start, we consider the idea of a "modeling cycle" (Blum & Ferri, 2009) to describe students' potential and actual trajectories in problems involving "reality" and mathematics. In the rest of the chapter, we refer to other approaches, especially those to modeling in science and at work, and discuss learning objectives that various frameworks assign to modeling.

11. 2 MWS and modeling cycle

Despite researchers' growing adoption of a MWS perspective, research studies about modeling with this perspective are still scarce. We review in this section a majority of these studies and notice that the idea of a modeling cycle is part of their theoretical framework together with MWS. We examine the contribution brought about by MWS in these research studies. Further, we discuss assumptions underlying this idea of a modeling cycle in the light of mathematical work in order to try to explore more deeply the potential of MWS.

[1] A MWS is organized on two different planes, epistemological and cognitive. Here the three "dimensions" involve both epistemology and cognition and helps us to characterize mathematical work in modeling without considering two-way processes between the planes (geneses). We discuss this later.

The modeling cycle is organized in steps. Two initial steps consist of the elaboration of a real model, particularly by simplifying and structuring a real situation. It is followed by steps of translation, one of which associates the real model and a mathematical model. This is called mathematization. The next step, interpretation, associates real results and mathematical results. Among the existing MWS studies, Nechache (2018) analyzes a phase she named “describing reality” that corresponds to the first steps in the modeling cycle. Nechache’s study deals with a random experiment, a repeated Bernoulli trial with a stopping rule. She demonstrates that building a real model requires introducing probabilistic hypothesis (equiprobability of events, independence of successive trials) and then more or less formal knowledge in probability. Figure 1 summarizes this analysis. The MWS framework allows Nechache to analyze this work as participating in the discursive dimension of a pre-probabilistic working space. Blum and Leiss (2007) also evidence the construction of geometric relationships in the step of simplifying and structuring in a task related to distances in a real life situation. In their work, mathematical work starts before the step that they label mathematization. This demonstrates that regardless of how it is labeled in the modeling cycle, the real model is actually a mathematical model, provided that mathematics are not restricted to typical content knowledge.

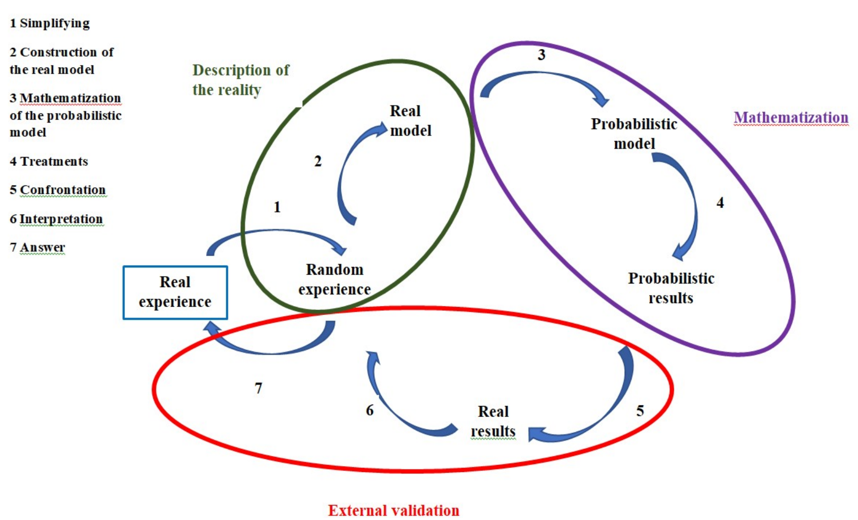

Fig.11.1 A modeling cycle in probability theory (adapted from Nechache 2018)

Nechache also analyzes the process of mathematization in probability. This process goes from a real model to a probabilistic one, allowing prediction making. It actually involves the interaction of the pre-probabilistic working space at stake in the previous phase, and a more standard probabilistic working space. She discusses three methods currently used in this process. The first method models the experiment by way of random variables and laws of probability. The work uses formalism and rules associated with random variables and is mostly in the discursive dimension. In the second method, the model is a weighted tree diagram and the work favors the semiotic and instrumental dimensions: Each node of the tree denotes a possible outcome at a given step. The computation of the probabilities follows "mechanical" rules rather than discursive reasoning. Finally, in the third method, the model is a simulation on a spreadsheet. Recognizing the spreadsheet's capabilities and obtaining adequate formulas and organization is not obvious, but requires reflection and combining instrumental and discursive dimensions. The outcome of Nechache's study was that the MWS framework allows an in depth a priori analysis of steps in the modeling cycle, and then of the stakes of modeling activity for students, especially with regard to mathematical conceptualizations.

Another study by Derouet (2019) analyzes a posteriori a process of mathematization in a classroom in light of MWS. The goal is to build a continuous probabilistic model for a phenomenon from statistical data. The analysis shows non trivial transitions between three domains (statistics, mathematical functions, and continuous probabilities), and the potential of the three dimensions of MWS for making sense of these transitions. Derouet explains particularly a step of mathematization using the diagram of figure 2. The students look for a function that best fits a histogram, namely, a candidate for a density function modeling the phenomenon. Derouet shows first a circulation between statistics and mathematical functions in the semiotic and instrumental dimensions. Students propose functions and the teacher operates a software displaying the graph of this functions and the histogram. Derouet notes that the work in the semiotic dimension (proposing functions that fit best) is at the students' initiative (in dark grey figure 2 left) while the teacher takes in charge most of the instrumental work (in light grey figure 2 left) . The discursive dimension of the continuous probabilities workspace is then solicited to check properties of the function with regard to probabilities (positivity, area under the curve) in connection with the semiotic work on mathematical functions (in dashed dark grey figure 2 right). Finally, there is work in the discursive dimension to adjust the function. This is performed primarily by the teacher (in light grey figure 2 right.)

In our interpretation, the outcome of the MWS framework is a fine grain analysis that brings deep insight into the stakes in this step of mathematization, i.e.: variety of knowledge activated by the students; choices about which actions should be left to the students as the teacher felt these were central to the situation; and the part the teacher takes in charge so that the situation progressed. Modeling is not seen as an individual endeavor to discover the right model, but rather as a social process where the parts respectively played by the teacher and the students are identified relatively to the dimensions.

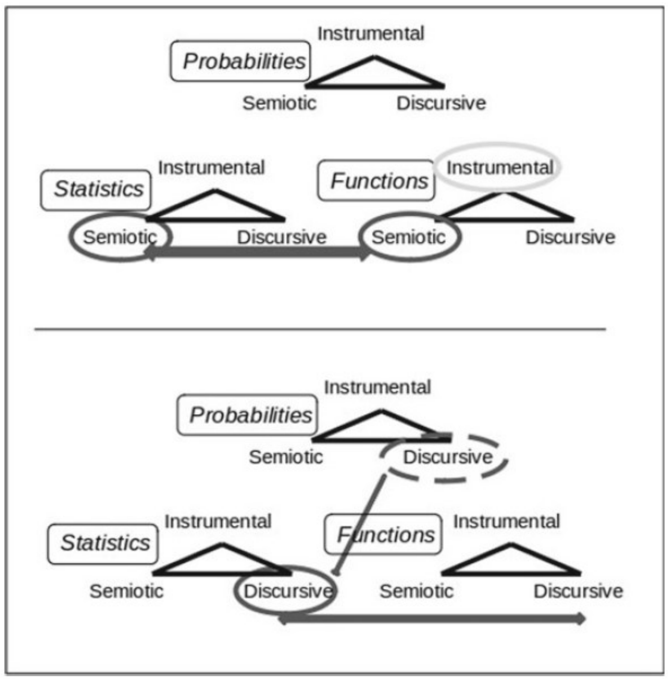

Fig. 11.2 A circulation between statistics, calculus and continuous probabilities working spaces.

Derouet et al. (2017) note that the use of the MWS framework can enrich and strengthen the analysis of the modeling process based on the study of a cycle in connection with the resolution of a problem. Beyond this, the authors note that in an educational context, a modeling task aims not just at solving a real problem, but more deeply at exploring and understanding the numerous uses of a mathematical notion. After solving a problem though modeling, students have to perform specific work in order to better understand the model and the mathematical objects involved in the solution. This thus enriches the MWS, in particular the theoretical referential in the discursive dimension.

Considering students' work in modeling situations, the MWS framework gives insight into the processes involved, and it also sheds light upon the complex relationship between reality and mathematics, and the multifaceted relationship between the process of problem solving and the underlying educational goals. This is good news but it also raises questions. If the steps of describing reality involve mathematical knowledge and the real model includes some mathematics, how is it consistent with a sharp separation between mathematics and reality? If working at the interface of reality and mathematics, as considered in probability theory by Derouet (2019) and by Nechache (2018), involves non trivial transitions and choices of dimensions, how can we think of the processes involved beyond a simple translation? What are the cognitive processes in work that aims at understanding the model and the objects involved? How can they be described, with geneses or with other cognitive notions?

11.3 The work of mathematicians in the social activity of modeling

Traditionally, statistics and probabilities are domains favored by secondary curricula for a mathematical approach to real-world phenomena. From the years 2000 the interest for involving real-world contexts in a wider variety of domains of learning, beyond statistics and probabilities, grew in mathematics education, as witnessed by the introduction of the OECD "Programme for International Student Assessment" (PISA) including a mathematics literacy test asking students to apply their mathematical knowledge to solve problems set in real-world contexts. Regarding math education research, the ICMI Study 14: Applications and modeling in mathematics education (Blum, 2002) was an important landmark. In it, Blum stressed that, "Today mathematical models and modeling have invaded a great variety of disciplines," and that "mathematising real situations as well as interpreting, reflecting and validating mathematical results in 'reality' are essential processes when solving literacy-oriented problems." The ICMI study started a very lively stream of educational research and put a growing focus on mathematics as an important activity in society. However nearly twenty years afterwards, much remains to be done in order to conceptualize the complex relationship between mathematics and modeling.

Derouet's and Nechache's studies deal with statistics and probabilities, and rely on epistemological and didactical knowledge in order to study modeling processes: It is more or less common knowledge that random phenomena cannot be approached without some mathematical apparatus. In contrast, in other domains researchers most often deal with modeling by starting from a simple conception of reality and its relationship to mathematics. For instance, Blum and Ferri (2009, p. 45) explain that by reality they mean "the rest of the world outside mathematics including nature, society, everyday life and other scientific disciplines." While this conception avoids being locked into philosophical debates about mathematics and reality, it does not shed much light upon the variety of intervention of mathematics in domains of modeling. This chapter aims to help close this gap, as the idea of mathematical work is central here. Mathematical modeling is what mathematicians do when they work on models and thus, it is useful to look at this work outside of education as a source of inspiration. One way is to build upon observations of mathematical modelers' practices. Another possibility is to look at epistemological studies of modeling and consider these as source of second hand information about the work of modeling.

11.3.1 Insights from the workplace

Frejd and Bergsten (2016) take a didactical transposition approach. They typologize workplace practices around the relationship between a client in need of a model for practical application, the modeler, and an expert that brings the relevant knowledge. While arguing that the typology could be a starting point for a transposition into educational settings, they recognize that there is still a long way to go. Their study allows them to criticize "policy makers referring to PISA league tables as an argument for curricular reform without providing analyses of how mathematical modeling is conceptualized and operationalized" (p. 31). In further interpretation, Frejd and Bergsten (2018) characterize mathematical modeling professional activity by four components: description, understanding, abstraction, and negotiation. From this characterization, they draw a "check-list" of didactic principles to be used in the design of modeling activities within mathematics teaching. An interesting principle is related to technology use "crucial for modeling work but also in itself (involving) a modeling activity.” This echoes the role of the computer as a key component in the instrumental dimension of the work in studies like Derouet’s.

In another study, Huincahue and Vilches (2019) define mathematical modeling as an activity in which two or more disciplines converge. For them, a mathematical modeler necessarily works in interdisciplinary teams, each discipline bringing its own contribution, but without hierarchical disciplinary preferences. Their empirical study shows modeling as a cyclical process of dialogue between mathematicians and specialists of a domain (like epidemiology or agriculture). The dialogue aims at elaborating a common understanding while discussing ways to approach a problem. The work is always directed by the need of a solution in the domain, but from the mathematician’s point of view, the actual outcome is the model(s) rather than the solution, and possibly a base for new developments in mathematics.

Finally, it is worth mentioning research focused on numeracy (i.e., reasoning and applying simple numerical concepts) from a modeling perspective. Wake (2015) is concerned that modeling activities in schooling are very different as compared to mathematical activity at workplaces. Rather than steps of translation between reality and mathematics, he thinks of modeling as a coupling of reality and mathematics that allows students to develop insight into, and understanding of, both mathematics and reality. Building from this, Wake goes on to identify (ibid, p. 13) three objectives for teaching/learning. Namely, students should become “able to:

• develop critically and mathematically informed models of complex realities;
• both construct and deconstruct (...) models of complex situations;
• understand how the structure of models of (workplace) realities and the structure of their mathematical counterparts (inter-)relate;”.

Wake then adds that education must

“• prioritize mathematical models that afford insight into typical structures important in workers’ practices and that are found in and across workplaces;
• involve learners in repetitive use of models in ways that ensures they engage substantially with issues of variability.”

11.3.2 Epistemological studies of modeling

Lagrange (2021) draws from studies by Francophone authors, Roux (2011), Bouleau (2014) and Varenne (2014). The idea of mathematization is questioned. It is necessary to distinguish between mathematizing processes (quantification, formalization, inductive and deductive reasoning, etc.) and the mathematization of a domain, which systematizes these processes until building a theory where deductive proof is the unique source of truth. This is the case for many domains taught in physical sciences at secondary level (think for instance of Newton's theory of universal gravitation), but modeling does not necessarily require this systematization and often it is not possible (think for instance of climate). While mathematizing processes can be important in modeling they do not by themselves differentiate modeling from mathematization. According to Roux (2011), modeling can be characterized:
(i) by locality: "whereas mathematization has a global aim, modeling is always local, and to say the least, fragmentary: the model assumes itself to be only one way among others to give an account not of reality, but of some of its aspects only";
(ii) by operativity: "whereas mathematization presents itself as a theory whose sole purpose is knowledge, modeling has an operative and interventionist component: a model is a device that allows one to act on what is being modeled, whether by supplementing, controlling or modifying."[2]

Drawing from Bouleau (2014), Lagrange (2021) extends this characterization:

> A corollary of (i) is the multiplicity of models for a given reality. The example of automobile traffic helps to identify a variety of models each taking into account aspects of traffic complexity and measurement (…) Concerning (ii) the operative character of the model should not hide an "interpretative" component which concerns the subject engaged in modeling and thus its social context. This component distinguishes modeling from mathematization, where deductive reasoning sets aside societal issues.

Distinguishing between modeling and mathematization, epistemological studies thus allow characterization of modeling by (1) a plurality of models which makes possible to account for local aspects of a reality (2)

[2] All translations from French sources are our own translations.

operativity and therefore proximity to simulation, (3) a subjective and social interpretative component.

Varenne (2014) insisted on varied types of simulation. As Varenne put it, a simulation can be just a “phenomenological model” giving account of visible functioning, or it can be a “second order model” when it is based on a previous modeling. This is consistent with different roles that digital simulation can play in modeling (Derouet et al., 2017). The first one occurs when building a real model with a simulation close to the initial situation (urn model or a calculator used to simulate rolls of dice or coin, in probability). The second role presupposes a stronger mathematical expertise in the MWS of the domain at stake as, for example, the implementation of an algorithm.

Finally for Bouleau (2014), the contribution of mathematics is in the work of the mathematician, that he describes as the search for simple ideas without which the models would be too complex to be interpreted. In other words, mathematicians quantify and formalize, but it is not for the sake of quantification and formalization, or to serve an idiosyncratic way to reach a solution, but as means to express simple ideas for other specialists. This is consistent with the Huincahue and Vilches (2019) observations reported above. Beyond simplification, the authors point out that mathematical modeling allows inexpensive experimental work as explained by a professional they observed: “Imagine the environmental engineers who model the movement of suspension particles from one continent to another … how can you measure that? Probe balloons in the stratosphere are very expensive ... then you model it!”

11.3.3 An example

Inspired by the above studies, Lagrange (2021) gives the example of a classroom situation for 12th grade students aiming at finding a function that models the main cable of a suspension bridge. In a first phase, the students looked at pictures of bridges and could easily conjecture that a parabola “fits” the cable and that they would be able to find a function for a given bridge by adjusting parameters. This would be a phenomenological model that possibly allows solving some concrete problems. In parallel, the students did not show a deep understanding of principles of bridge design. For instance, they overlooked the role of the suspenders (vertical cables equally spaced that connect the main cables and the deck). The goal of the situation was not principally to solve a problem, but rather to understand principles of bridge design and to develop the associated capacities in mathematics and physics. Therefore, further phases of the situation offered students tasks to investigate four models of a suspension bridge and learn about the cable properties. A first model was based on a mock-up of the bridge. The deck was represented by a succession of equal weights, suspended to a cable at equal horizontal distance. Such a mock-up embodies the idea that there is no compression in the deck, a fundamental property of suspension bridges. It is abstracted into a physical model of the evolution of tensions along the cable by way of recurrence formulas. A second model was a broken line in analytic geometry. Recurrence between tensions at suspension points at equal horizontal distance allows computing recursively the coordinates of these points. As a third model, a computer simulation made this computation dynamically. It displayed the broken line with dynamical variations, when parameters like the number of suspension points and the value of the horizontal component of the tensions were animated. Finally, a continuous model was obtained by considering an infinite number of suspension points. It was a quadratic function, as conjectured by students, but obtained by using fundamental concepts in physics and mathematics in interaction rather than by phenomenological observation. In this model, too, the horizontal component of the tensions could be a parameter. This showed the importance of this quantity: suspension bridge designers have to consider that increasing the height of pillars reduces the tensions and then the necessary resistance of the cable and of the anchor points, but also increases cost and constraints. The professionals, thus, have to make a balance between those two parameters in a given geological and geographical configuration.

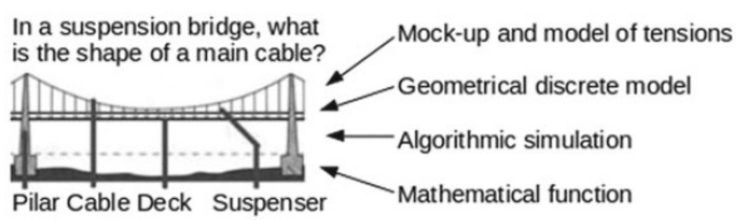

Fig.11.3 The suspension bridge. A question and four models (adapted from Lagrange 2021).

Although the four models can be seen as organized linearly from the mock-up to the mathematical function, this organization cannot be thought of in a modeling cycle, as all four models are both real and mathematical. In contrast, this characterization of four models echoes the above consideration of a plurality of models for a given phenomenon, as well as the role that mathematics plays in these models. Mathematizing occurs in each model. It simplifies, but does not hinder the physical nature of the bridge and the operativity of the model. It makes visible the parameters that are crucial for professional designers. The example is also consistent with the Huincahue and Vilches (2019) definition of mathematical modeling as an activity in which two or more disciplines converge, because a branch of physics (statics) is deeply at stake. Further, this example is also in line with Wakes' (2015) concern for modeling activities that couple reality and mathematics and reduce the gap between school and workplace.

11.4 Modeling: connecting a plurality of working spaces

In the preceding section the idea of work helped to overcome a too simple dichotomy between reality and mathematics and show the stakes brought about by insights from the workplace and from epistemology. In this section, we look more closely at the potential of the idea of working space. First, we consider the contribution of the three dimensions: semiotic, instrumental, and discursive. We begin with a famous citation from Galileo:

> "Nature is written in that great book which ever is before our eyes—I mean the universe—but we cannot understand it if we do not first learn the language and grasp the symbols in which it is written. The book is written in mathematical language."

Galileo's thinking is not far from the above-mentioned idea of mathematics as means to make the world intelligible, provided that "great book" and "language" are understood as metaphoric forms. Unfortunately, they are often taken literally, as mathematics is considered to be a language into which reality has to be translated.[3] A great strength of the MWS framework is that it supports avoiding this reduction. Mathematical work on a model is certainly not possible without means for representing. However, these means cannot be thought of as a standard preex-

[3] For instance, MEN/DGESCO-IGEN (2013), an official document for the French curriculum (lycée, grades 10 to 12) characterizes modeling as "Traduire en langage mathématique une situation réelle" (Translate a real situation into the mathematical language.)

isting mathematical language, as they have specificities and their elaboration is part of the work. Moreover, the work on a model is not reduced to symbolic manipulation. All studies mentioned above, both at the workplace and in the educational context, consider instruments which are specialized for a domain and have a mathematical basis. Finally, working on a model also involves reasoning in both the domain of modeling and in mathematics. In short, MWS is a good candidate for conceptualizing the work on a model. This is because the three dimensions take into account respectively the semiotic systems, the instruments, and the modes of reasoning used in this work. Furthermore, the notion of space encompasses the specificities of these dimensions with regard to the model beyond their mathematical nature.

In the MWS framework, the plurality of models for a given reality has led authors to consider a plurality of working spaces. In the “Alphonso” task (Kuzniak & Nechache, 2020, 2021), students have to compute the area of a piece of quadrilateral land in which the lengths of the sides are given. The observation shows each student building their own model after implicit assumptions. The student activity can be seen as the organization of personal geometrical working spaces adjusting their view of the task and their geometrical knowledge. This is also the case in the “Gutter” task, (Montoya et al., 2017) presented in Chapter 5.2.2 of this book. The study reports on pre-service teachers solving an optimization problem: finding the best way to fold a sheet of metal in order to create a gutter that maximizes water flow. The pre-service teacher course was in four years numbered from one (younger students) to four (older students) and the study involved year one, two and four students. Only year four students had been taught calculus. The researchers observed differences between procedures favored on the one hand by year one and two students, and on the other hand by year fours students. They analyzed these differences with respect to students’ personal working spaces. Year four students’ personal spaces were close to a reference working space in calculus. In other words, these students recognized the task as a standard problem for which official calculus procedures exist and have to be applied. For them it was clear that the water flow depends on the choice of the length on one side, and this dependency was modeled as a quadratic function with the symbolism, instruments, and mode of reasoning attached to mathematical functions. In contrast, the other students’ procedures varied. The researchers identified personal spaces the students arranged on purpose by using their knowledge in geometry and measure. Sometimes students based their answer on their (false) intuition about a model of an optimal gutter without considering variation. For students who did consider variation, the model was not a function but rather a collection of configurations in which to pick up an optimum. In these studies, researchers interpret students' procedures by identifying different underlying models and associated personal working spaces corresponding to different understandings of a reality and to different conceptualizations in various mathematical domains.

The possibility of varied models and associated working spaces for a given reality was exploited in other studies to explicitly design tasks that put at stake transitions or coordination between specific domains corresponding to different a priori suitable working spaces. Derouet (2019), for example, considered suitable working spaces, respectively in statistics, mathematical functions, and continuous probabilities. We mentioned above the situation of modeling a suspension bridge proposed by Lagrange (2021) that draws from a plurality of models of the main cable. Lagrange explains that the work on each model is related to a specific working space with a specific system of symbols, specific instruments, and specific ways of reasoning. In a first phase of group-work, a study of each model was devoted to a specific group of students. A second phase mixed the groups so the students—each one an expert of a particular model, could find together links between the models and then make connections between the working spaces. The a posteriori analysis of an experiment was done by considering the various connections the students had made and the dimensions implied.

In these examples, the plurality of working spaces with no hierarchy between them echoes the work in interdisciplinary teams of professional modelers. Mathematics is involved neither as "the queen of sciences," nor as a pure "servant." Mathematical work clarifies and operationalizes models, but it also contributes to giving meaning to abstract mathematical notions.

11.5 Learning by modeling: questions for the MWS framework

In this part, our goals are to investigate questions dealing with the objectives assigned to modeling tasks or situations based upon the MWS framework. Further, we explore here the methods and concepts used in this framework to analyze a priori and a posteriori what students learn and how they learn.
Objectives

We mentioned at the beginning of section 3 the stream of educational research following the 2002 ICMI study 14. With regard to objectives, this stream is characterized by a focus on modeling competences. It is reflected by the fact that six of the eight crucial issues proposed by Blum (2002) mention these competences. Issue five stresses particularly that modeling activities "will have to 'compete' with other components of the mathematics curriculum." This is a difference with the studies adopting the MWS perspective. First, the situations investigated by Derouet (2019) and Nechache (2018) are motivated by objectives clearly inscribed into the mathematics curriculum. Moutet (2019) is another example. His study puts the physics curriculum in view. It proposes and analyzes a teaching unit in relativity theory that puts at stake the physical notions of time and simultaneity. This is an example where mathematics is involved in order to make physical ideas accessible. As for Lagrange (2021) the objective is not that students reach a new curricular content, but rather that they understand more consistently notions already taught in calculus, algorithmics, analytical geometry, and physics. Also with regard to objectives, Psycharis, Kafetzopoulos, and Lagrange (2021) analyze a situation that does not aim at specific curricular notions. This study focuses on developing students' functional thinking by working on covariation in four different working spaces. The proposed situation is based on the same gutter optimization task described above in our discussion of Montoya, Viola, and Vivier's 2017 research. However, there is a difference: Whereas Montoya, Viola, and Vivier characterize a posteriori personal working spaces by observing students' self-generated procedures, Psycharis, Kafetzopoulos, and Lagrange start from the a priori identification of four possible models. One first model is a sheet of paper that can be folded in order to make a mock-up of the gutter. The second is a dynamic geometry (DG) construction representing a section. The third model is also in DG but involves the covariation of measures. The fourth one is a mathematical function. Four working spaces are associated with the four models in order to analyze a priori and a posteriori students' work on the four models and then examine student understanding of covariation.

There is then a common concern in studies adopting a MWS perspective about making students consider notions taught in varied fields in synergy. This concern is reflected by the consideration of working spaces specific to each field. This focus on the knowledge to be taught clearly differentiates these studies from research putting at stake modeling competences. It could denote a similarity with theories such as realistic mathematics education (RME). For RME, modeling is a means to bring students to mathematical concepts from a "reality" (Gravemeijer & Doorman, 1999). Modeling intervenes in the RME approach at the moments of first meeting (of "reinvention") in mathematics. A model of some reality is built, and after that it becomes an abstract tool for doing mathematics. This tool, labeled model for, has a representational and an instrumental dimension derived from the model of, but representations and instruments belong to mathematics. Gravemeijer and Doorman (1999) explain that “students’ final understanding of the formal mathematics should stay connected with (…) everyday-life phenomena.” However, this connection is conceived as a background reference for students’ practice of formal mathematics rather than as a relationship between a domain of reality and the mathematical knowledge at stake.

As a difference to RME, it seems that although modeling in the MWS framework is oriented towards mathematical knowledge, it has a potential for keeping a relationship between the mathematics learned and the various fields involved in the modelization and also, the social circumstances in which this relationship has been built. To go further, MWS studies on modeling could benefit from Wakes (2015, p. 684) reflection:
(…) workplaces provide rich sources of (...) realities, (...) providing purpose and a meaningful context in which to understand and explore an emerging mathematical model. The mathematical models developed in this way have the potential to provide insight, crucially into both the reality and the mathematics in contrast to the RME approach that focused ultimately on understanding the mathematics.

In line with this, studies on modeling could break with a narrow conception in which modeling is an activity pursued individually and for individual competencies. As noted above, modeling at the workplace is a social activity and implies the identification of specific working spaces and the collaboration of specialists of each space. In this latter conception, the stake for students is not to reach individual abilities for solving real world problems, but rather to participate in a collective task. Here, students take advantage of existing knowledge and develop this knowledge by interaction with others.

Consequence on design: an example

In order to exemplify the opportunities brought by workplace contexts, let us consider an example from professional sea navigation. Vroutsis, Psycharis, and Triantafillou (2018) proposed students a typical problem from this sector. Starting from a given position, a ship must pass through a dangerous area with underwater obstacles. The area was represented on a navigational chart and the captain had marked two landmarks on this chart as reference points. The students were asked to use the landmarks and specific professional measures (bearing, range, and horizontal angle) and tools (parallel ruler, divider) in order to find a way to steer the ship to ensure its safe passage. They then had to rediscover a navigational technique: describe an arc of a circle that avoids the obstacles by steering the ship and keeping a constant angle between the two landmarks. As a parallel aim, the students also learned about inscribed angles. The realistic context (original workplace problem), the practitioner's tools (measures) and the workplace constraints (rules) acting as boundary objects revealed the insufficiency of school taught knowledge. It thus highlights a discontinuity between formal mathematics and the genre of mathematics developed in the workplace.

When we analyze this task, we see that the problem involves two working spaces: one is professional navigation and the other is geometry. The professional context marks the three dimensions —representations, instruments used, and modes of reasoning— of the navigation working space. Thus, it differs from a geometrical working space in the school context. A third working space, programming, could take into account the prevalence of programming in professional modeling and of electronic devices in contemporary navigation. In order to bring the working spaces to life and help students develop connections, group-work could be organized in phases like in the suspension bridge situation reported above. In a first phase of group-work, three different tasks could be distributed to groups of students. One task, proposed to groups of students labeled sailors, would be in a navigation working space. A professional would present the problem and the technique used to go through dangerous areas. Students would have to implement the technique to a practical case of navigation and discuss its effectiveness. Another task, proposed to groups of students labeled geometricians, would be in a geometrical working space. In this task, for two given points A and B, students would have to conjecture with the help of DG the positions of the points M, such as angle (MA, MB) would be a given constant. A third task could be proposed to groups of students labeled programmers. They would work on programming a trajectory for a point M with the constraint that angle (MA, MB) is constant.

In a second phase, groups would be comprised of one sailor, one programmer, and one geometrician. The tasks would be to explain and confront the work done by each participant in the first phase and then, write a mathematical justification of the navigational technique and of the program (fig. 11.4). A collective classroom discussion could follow that emphasizes the contribution of mathematics to professional techniques. A further discussion point would be the care that should be exercised when applying to the real world: Good sailors will constantly double check!

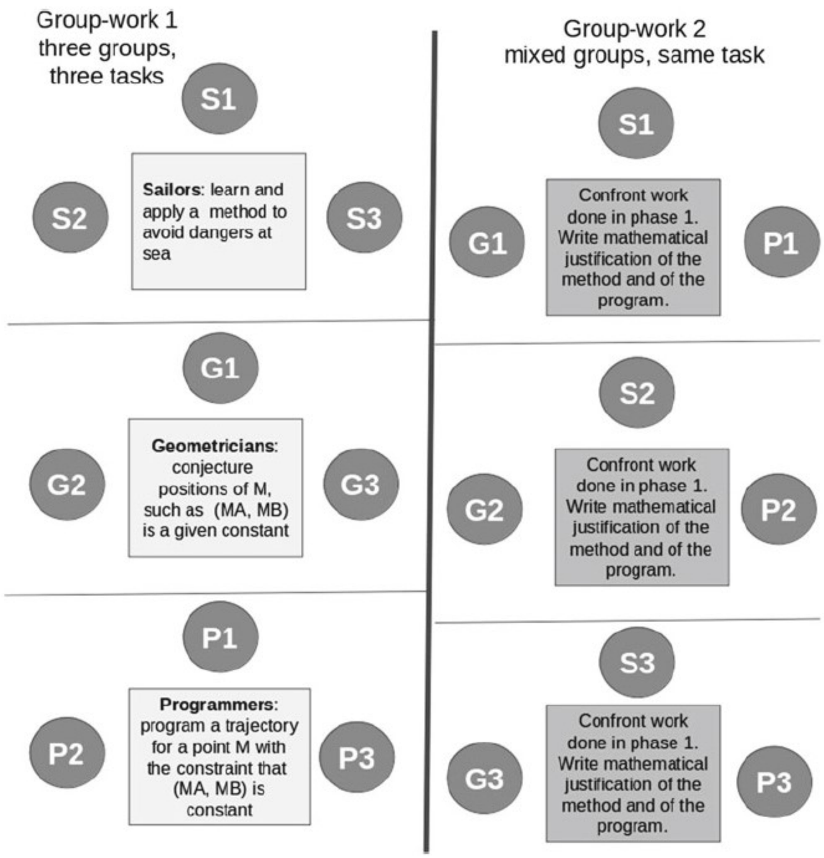

Fig. 11.4 A suitable group-work organization for the sea navigation situation.

With this type of design, modeling activities could articulate learning and socialization while contributing to the education of students as citizens and for their (future) professional/social life. This is a different way, as compared to modeling activities that aim primarily to foster individual modeling competencies.

Analyzing what students learn and how they learn

In the preceding sections of this chapter, we did not specifically separate epistemology and cognition. Furthermore, we characterized working spaces by the three dimensions (semiotic, instrumental, discursive). This is also the case of studies on modeling whose authors observe "connections" (Lagrange, 2021) or "circulations" (Derouet, 2019) between dimensions of the working spaces and analyze these connections or circulations as evidence of students' progress relative to the goals of the situation.

Beyond dimensions, the MWS theory (preceding chapters) considers three geneses that articulate poles of two planes labeled respectively epistemological and cognitive. The epistemological plane consists of human made entities (representamens, artefacts, rules of reasoning.) The cognitive plane is described in terms of processes (visualizing, constructing, proving). Each genesis can be viewed as the activation of a pole of the epistemological plane by an individual or a collective subject for the corresponding process in the cognitive plane and is analyzed with regard to the subject's cognition. Moutet (2019) considered an extended working space with a single cognitive plane, and two distinct epistemological planes, one in physics and the other in mathematics. Distinguishing two epistemological planes is consistent with the fact that in physics and mathematics, the three poles of the epistemological planes are specific to each field. A single cognitive plane presupposes that poles in the cognitive plane can be associated to poles of a mathematical epistemological plane, as well as to poles of a physical epistemological plane. This choice helped Moutet to identify how students develop knowledge through geneses involving alternatively poles of the mathematics epistemological plane and poles of the physics epistemological plane. With regard to modeling, Moutet's extended MWS is then centered on a single model (the Minkowski diagram) alternatively considered from a mathematical and a physical point of view.[4] Moutet also considered a model involving imaginary objects like a car traveling at nearly light speed.[5] However, the elaboration of this model and the connection with the Minkowski model is not conceptualized in the MWS theory.

This is different for Psycharis, Kafetzopoulos, and Lagrange (2021) who analyze how students progress in their understanding of covariation

[4] For another example of an extended MWS about a model in engineering, see Cosmes Aragón, & Montoya Delgadillo (to appear).

[5] Paradoxically labeled "real model" in reference to Blum's modeling cycle.

by working on a succession of models. At the theoretical level, their study draws from Minh and Lagrange (2016) that introduced the Connected Working Spaces (CWS) framework. Minh and Lagrange were concerned that although at both the epistemological and cognitive levels, functions make sense because of their occurrence in many dissimilar settings, teaching actually favors algebraic representations. They developed the CWS framework in order to take the following fact into consideration: For students and teachers to fully consider each setting where functions can make sense, they have to think of it as a particular working space allowing a specific type of scientific work and specific conceptualizations. Their hypothesis is that adequate teaching situations should organize students' work in various non-algebraic and algebraic working spaces and allow connections between these.

As explained above, in the situation analyzed by Psycharis, Kafetzopoulos, and Lagrange (2021), the students work in turn on four models of a gutter (sheet of paper, section in DG, measures in DG, mathematical function). Four working spaces are associated with the four models. The identification of connections between these working spaces in the three dimensions theorized in the CWS framework plays a role in the a priori and a posteriori analyses by focusing on key transitions occurring when students are introduced to a new working space and by specifying the dimensions involved.

Psycharis et al. (ibid) present episodes in various phases of work. We take the example of a phase where students were introduced to the DG working space. We extract here first a dialogue between the teacher (T) and students (S) while working to construct a dynamic rectangle (ABCD) to model a section of the gutter, and then the authors' analysis in two parts.

The dialogue:

> T: You will need one point for the lower part of the gutter and one point which describes the maximum folding. Then we need another point between these two points to describe every time the different folding, but first of all we need to find the restriction of the construction.
> S3: I propose to put point D in (0;0).
> S4: We have to create a point E as (0;10) in case we fold the metal plate in the middle so that we get a segment for positioning the free point C.

After creating C, the students observed the folding in order to find an expression for the x-coordinate of A. Most of the groups attempted to find it through solving the equation x+x+y=20 for y. Students from different groups commented: "I tried to create A, as (20-2*x ; 0) but it did not work!", "We created A as (20-2*yC;0) and it worked!", "As for us, we created A as (20-2*DC;0) and it worked also!

The first part of the analysis uses the CWS framework. In order for students to take advantage of the DG possibilities they need to identify key elements of the model as geometrical objects using geometrical notations. The teacher's intervention is crucial, insisting on the choice of points defining a rectangle but the students also have their part: propositions for creating points E and D involving the constraints of the sheet, expression of the dependency of point A to point C. The work is in the instrumental dimension: adequate use of the instrument (DG) is at stake. The work also combines a semiotic dimension: students progressively integrate the notations of DG in order to express the x–coordinate of A.

For the authors, this first part of the analysis alone does not give precise account of what students learnt and how. That is why they choose to combine with the activity theory framework Abstraction in Context (AiC), see Dreyfus et al. (2015). The abstraction in context theory was created to give account of conceptualization within contextualized tasks. This is how Psycharis et al. (2021) introduce AiC:

> AiC describes the process of abstraction by means of a model of three epistemic actions that researchers can be observe and analyze: recognizing (R), building-with (B) and constructing (C). Recognizing an already known mathematical concept, process or idea occurs when a student recognizes it as relevant in a given situation (...) Building-with comprises the combination of existing knowledge elements (i.e. recognised constructs) to achieve a goal (...) Constructing is carried out by integrating previous knowledge elements (constructions) by vertical mathematization to produce a new structure/construct (...)

Then the second part of their analysis uses AiC. Drawing from their experience with folding the paper sheet, the students see the need to define a rectangle through four points and also to distinguish the point that 'causes' the dynamic change of the construction (Recognizing). In order to organize the objects in the DG in a way consistent with the paper, the students make faulty and successful attempts to relate the coordinates of point A to measures dependent on point C (Building-with). In the end, different symbolizations for the coordinates of point A are suggested by different groups (Constructing) indicating students' progressive coordination of their preceding sensual manipulation of the paper sheet with the available notation structures of DG.

Analyses of the other steps confirm that at each step, students first connect a new working space to the former (Recognizing) and then develop conceptualizations (Building-with and Constructing) inside the new working space. Psycharis, Kafetzopoulos, and Lagrange conclude:

> AiC is powerful here to make sense of students' progress but could not be put into operation without the structure provided by CWS. In addition, CWS distinguishes between three dimensions (instrumental, semiotic, and discursive) in students' work that sheds light upon the AiC process of abstraction.

Reporting on four studies in this section, we observed varied ways of dealing with students' cognition in modeling work. Two studies (Derouet, 2019 and Lagrange, 2021) assume that the connections or circulations between working spaces are evidence of students' progress. Another one (Moutet, 2019 takes the classical MWS approach in considering a cognitive plane and geneses. Whereas the fourth study (Psycharis et al., 2021), combines an activity theory framework with the CWS framework and concluded on a productive combination. There is no definite best method to address the question of what students learn and how they learn in modeling situations. However, it is promising to note that MWS offers internal means with the idea of the cognitive plane, as well as the ability to combine with a cognitive oriented framework.

11.6 Conclusion

We will conclude this study of the new perspectives opened by the working space theory by focusing on two fundamental constructs presented in this chapter that have proven productive: the idea of mathematical work and the three dimensions of this work (semiotic, instrumental and discursive). The idea of mathematical work helps to think of modeling without artificial separation between mathematics and reality. It then looks at mathematics as contributing to various models of a reality rather than as being confined to mathematizing reality into pure mathematical models. It also motivates consideration of modeling practices outside education. These practices offer a wide landscape with common trends. These include the plurality of models for a given reality where each model contains some mathematics, and the conception of modeling as a social activity involving specialists from different fields and people with various points of view. Models are plural relatively to the fields of knowledge involved (profession, physics, mathematics…) but also with regard to their operativeness. Analytic models coexist with algorithmic simulations, and different models benefit from mutual connections. The idea of a plurality of working spaces reflecting this plurality of models, of fields, and of points of view is productive in the MWS studies. Further, this idea allows addressing traditional mathematical fields. These include 1) probability, 2) topics involving physics and mathematics like relativity, and 3) wider issues like the understanding of covariation as a basis for functional thinking.

At present, social aspects of modeling are not very developed in math education. The plurality of working spaces can help in an educational context to organize in the classroom collaboration and a discussion between fields and between points of view. Inspiration can be found from situations of modeling at the workplace. In this conception, there is no opposition between objectives in terms of socialization and in terms of knowledge. Socialization is based on the recognition of various forms of knowledge and the subsequent necessity of discussion and collaboration. Considering the three dimensions of a working space helps to avoid a reduction of the modeling activity to a pure translation— and of mathematics to a language. As Psycharis, Kafetzopoulos, and Lagrange (ibid) say, the identification of these dimensions in each of the working spaces involved in a modeling situation provides the "structure" needed for well-grounded a priori and a posteriori analyses. Building upon this structure, it is possible to study students' cognition along the modeling activity. We noted above two possibilities. One is to try to characterize "geneses" associating poles of epistemological planes and a cognitive plane. The other is to combine with an activity theory framework.

Studies on modeling based on MWS are still small in number. Analyzing some of these helped us in this chapter to open new perspectives for the design of modeling situations and for making sense of students' learning in these situations. Beyond providing elements of a theoretical framework for researchers, we hope that this chapter will also be useful for mathematics educators at large at the methodological level. This could include thinking of contents or notions to be taught in relation with social settings in which these contents and notions exist and contribute in understanding the world. This could also include identifying varied models developed in these settings and their connections, conceiving spaces for working on each of these models and finally, organizing students' work and social interaction inside and across the working spaces. A wide field for experimentation and research opens up.

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