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# NORMALISING GEOMETRICAL CONSTRUCTIONS: A CONTEXT FOR THE GENERATION OF MEANINGS FOR RATIO AND PROPORTION

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**Abstract:** *This paper describes aspects of 13 year-olds' activity in mathematics as emerged during the implementation of proportional geometric tasks in the classroom. Pupils were working in pairs using a piece of software specially designed for multiple representation (symbolic and graphical) of the variation in parametric procedures with dynamic manipulation of variable. In this paper we discuss children's use of normalising, an activity in which children 'correct' the geometrical figures while developing meanings for ratio and proportion. We discuss the potential of normalising for the construction of mathematical meanings in relation to particular aspects of the pedagogical setting including pupil's interaction within the computational environment as well as task design.*

## THEORETICAL FRAMEWORK

In this paper we report research aiming to explore 13 year-olds' mathematical meanings constructed during activity involving ratio and proportion tasks in their classroom. The students worked in collaborative groups of two using 'Turtleworlds', a piece of geometrical construction software which combines symbolic notation through a programming language with dynamic manipulation of variable procedure values (Kynigos, 2002). They were engaged in a project to build figural models of capital letters of varying sizes in proportion by using only one variable to express the relationships within each geometrical figure. We were interested to study the ways in which the students interacted with the provided computational tools and the ways in which the emergent meanings were structured by the tools (Noss & Hoyles, 1996).

We adopted a broadly constructionist framework (Harel and Papert, 1991) for our work taking also into account the situated cognitionist view about the complex ways by which knowledge is shaped within a particular setting (Lave, 1988). In this paper we discuss children's *normalizing* activity characterized by engagement with 'corrections' of distortions to figural representations (similar to the sense of Ainley et al. 2001). This kind of activity emerged as a coherent part of pupil's reflections on the graphical feedback resulting from the symbolic code and it was characterized by their gradual focus on relationships or dependencies between objects and representations and the emergence of mathematical meanings of ratio and proportion.

Proportional reasoning has been the object of many research studies (Hart, 1981, 1984, Tourniaire & Pulos, 1985, Hoyles & Noss, 1989, Harel et al., 1991). The results of most of these studies have revealed that children view ratio and proportion tasks as requiring addition and not multiplication and thus chose an 'additive strategy' for solving them. More specifically, it has been reported that geometrical enlargement settings provoke more addition strategies than any other one while students have great difficulty in identifying a ratio relationship regardless of context and numerical content (Kuchemann, 1989). Some explanations of these poor levels of performance have highlighted two areas of difficulties: (a) sometimes enlarging the sides of the original figure by addition still produces the same kind of figure (e.g. rectangle) (b) most pupils ignore that the resulting enlargement should be the same shape as the original because of being so "engrossed in the method to be used" and the arithmetic calculations (Hart, 1981). However, there have also been reported the benefits of the use of computational tools in children's proportion strategies derived from interacting within specially designed microworld settings that facilitate linkages between visual, numerical and symbolic representations of geometrical objects (Hoyles and Noss, 1989, Hoyles et al., 1989). Hoyles and Noss (1989). Discussing 'the qualitatively different kinds of work' facilitated within such a computer environment, they focus on the child-tools interaction which "directed the child's attention to key points in her problem solving and served to clarify the proportional relationship involved, p. 66". According to their analysis this interaction "is built on the synthesis between the child's need to formalize the relationship algebraically (i.e. to type a program) and to receive confirmation of intuitions (i.e. to perceive the intended geometrical effect on the screen, p. 65)".

In this report, we thus build on prior computer-based geometrical enlargement tasks with the aim to exploit kinesthetic control *as a process*. Our focus was on students' dynamic manipulations of the geometrical objects *during* the ongoing experimentation through actions with symbolic notations and representations (Kynigos & Psycharis, 2003). This kind of manipulation of the graphical outcome is also related to the following considerations: (c) In proportional tasks of that kind graphical representation of objects is tightly related to the use of algebraic relations and thus joint symbolic and visual control may have important potential for the construction of mathematical meanings of ratio and proportion. (d) According to the task pupils are asked to manipulate geometrical figures in a meaningful way i.e. to construct models of capital letters of different sizes that will not "distort" under size changes which keeps up with the functionality of any font both in and off the computer.

## RESEARCH SETTING AND TASKS

In our research perspective we attribute emphasis on two aspects of the pedagogical setting that are likely to foster mathematical learning: the choice of the computer environment and the task design. As far as the microworld artefact the pedagogical design of the software involved an integrated use of both formal mathematical notation and dynamic manipulation of variable values. In Turtleworlds, what is manipulated is not the figure itself but the value of the variable of a procedure. Dragging thus affects both the graphics and the symbolic expression through which it has been defined, combining in that sense these two kinds of representations which appear rather static in most of the enlarging geometrical settings. The second factor taken into consideration is the task design as offering a research framework to investigate *purposeful* ways that allow children to appreciate the *utility* of mathematical ideas (Ainley & Pratt, 2002).

The work reported here is part of an ongoing study on the generation of meanings of proportionality by groups of pupils working with 'Turtleworld' microworlds in the classroom setting. The research took place in a secondary school with two classes of 26 pupils aged 13 years old and two mathematics teachers. During the activity, which lasted for 32 hours in total over 9 weeks, each of the two classes had two 45-minute project work sessions per week with the participant teachers. Each class had the task to construct all the capital letters of the alphabet called '*The Dynamic Alphabet of your own class'*. The letters would be used in a following classroom activity by the pupils to construct 'dynamic posters' in which particular words or phrases can change size in the same way. During the classroom activity, the students were engaged in building models of capital letters of variable sizes, having initially been told that the aim was for each letter procedure to have one variable corresponding to the height of the respective letter. According to the task, each group of pupils was assigned to construct two letters while in a subsequent stage groups were asked to interchange their constructions so as to check and correct other pupil's work. Problems stemming form the use of different variables by different groups were left as a point of interaction among students and teachers. At the time of the study, the students had already had experience with traditional Logo constructions including variable procedures. During the study they were introduced to the dynamic manipulation feature of the software called 'variation tool'. After a variable procedure is defined and executed with a specific value, clicking the mouse on the turtle trace activates the tool, which provides a slider for each variable. Dragging a slider has the effect of the figure dynamically changing as the value of the variable changes sequentially. The graphics, the tool and the Logo editor are all available on the screen at all times. In spite of the task requiring a procedure with one variable for each letter, most of the groups initially experimented by choosing different variables for the segments of their constructions until they built their final one with one variable. For example, in the procedure of Figure 1 for letter "A" the first variable (:x) changes the length of the "slanty" sides, the second (:y) the length on the "slanty" sides from the base to the edges of the horizontal side and the third

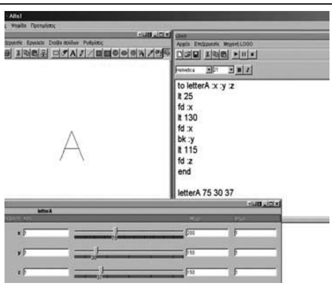


**Fig. 1: Constructing 'A' with three variables**

(:z) the horizontal side. The procedure for drawing the final model of a letter can be derived through the functional relation of the only variable to the ratios of the sides of a fixed model of the letter. Our general aim thus was to utilize the functionalities of the computer environment and the feedback it can provide so as to provoke children: (e) to construct relationships and figures according to proportional rules (not initially explicit to pupils); (f) to come up against visual conflict with common initial strategies e.g. the inclusion of an additive relationship in a procedure would result to a "distorted" figure for some numeric values on the variation tool; and (g) to engage in the dynamic manipulation of the enlarging process. Our objective was to gain insight into (h) the nature of the meanings of ratio and proportion constructed by pupils during their explorations and (i) the ways in which meaning generation interacted with the use of the available tools.

## METHOD

During the activity, we took the role of participant observers and focused on one group of students in each class (focus groups), recording their talk and actions and on the classroom as a whole recording the teacher's voice and the classroom activity. In our analysis we used a generative (Goetz and LeCompte, 1984) stance, i.e. allowing for the data to shape the structure of the results and the clarification of the research issues. Here we use data from the focus group in one of the classrooms. A team of two researchers participated in each data collection session as participant observers. We used two video-cameras and two microphones. One camera and one wired microphone were on the groups of students who were our focus (1 in each class). One researcher was occasionally moving the second camera to capture the overall classroom activity as well as other significant details in student's work as they occurred. A second wireless microphone was attached to the teachers, capturing all their interactions with all groups of students. Background data was also collected (i.e. observational notes, students written works). Verbatim transcriptions of all audio-recordings were made.

## VISUAL MANIPULATION OF THE INTERDEPENDENCE OF VARIABLES

In the following episodes we present different kinds of normalizing activity according to the criterion or motive of pupils' normative actions interrelated to the simultaneous emergence of mathematical meanings. In the first example, normalising is associated with the interdependence of the lengths of the construction. The focus group students made a model of the letter "A" using three variables as shown in Figure 1. Early in their work they had constructed the displayed figure – which we refer to as the *original pattern* of "A" – without using any variables. On the next stages of their exploration, pupils would try to change it proportionally. The three sliders were set in the values of the original pattern as displayed at the bottom of the screen: x=75, y=30 and z=37 when S1 started to move the slider of (:x) for the first time. S2 proposed to assign to it the value of 150.

S2: [To S1] Change it to 150.

S1: [The figure is distorted] But, I have to increase this one [i.e. the (:y)].

R: [To S1 who is dragging the slider of (:z)] What are you changing now with (:z)?

S1: With (:z) I was changing this [ i.e. the horizontal side].

R: Ok, but in your “a”, this little line is now too low.

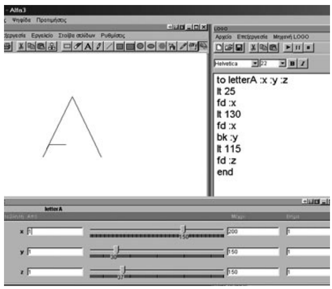


**Fig. 2: Distortions of ‘A’ constructed with three variables.**

[S1 drags the variation tool of (:y) to a higher value, thus pulling up the horizontal line]. In this case the normative process started after the ‘distortion’ of the figure when moving one of the sliders (Figure 2). S1 continued normalising by the appropriate change of the length of the horizontal lenght of “A”, so as to join up the side sections. However, those modifications had also changed the starting point for drawing the horizontal part, expressed by the variable (:y), as the researcher pointed out. Thus, S1’s normalizing action was to move the third slider of the variation tool to a higher value. Although S1’s suggestion of 150 being twice as much as the initial value of (:x) may be an indication of a proportional prediction for the values of the other variables, S2 did not change them proportionally. However, we may observe that pupils apparently connected at an intuitive level the articulation of the figure and the interdependence of the involved magnitudes. In this phase pupils seemed to give priority to complete the shape instructed by the visual outcome on the screen and not paying attention to some kind of relationship between the selected values. The next episode shows how this kind of experience was exploited in further exploring the construction.

## GRAPHICAL AND NUMERIC CONTROL OF THE SIMILARITY RATIO

For some time pupils seemed to move the sliders of the variation tool at random observing the visual feedback of the continuing changes in the variables. Gradually their dragging moved from this mode to become more systematic and focused in their attempt to discover some rule or invariant property so that they could create the original letter “A” in different sizes. S1, who used the keyboard, while trying to normalise an incomplete form of a figure had the idea to double and half the initial values of the original pattern. In the following excerpt, S1 set as initial value for each slider the half of the correspondent value in the original pattern and as the end value, its’ double.

R: [To S1] What are you changing now?

S1: I set it to the smallest.

S2: [After a while] Therefore, we have to study the relation to find what's right.

R: Which relation?

S2: That of the three numbers. To find a relation, if possible, in order not to think of it each time.

S1, by setting the specific limits on the sliders and dragging them at both ends, achieves to construct similar figures and take control of the similarity ratio equal to the values ½ and 2. S1's manipulation of the variation reflects the purposeful way in which the computational setting provided a web of structures which pupils could exploit in shaping the available resources to satisfy the emerging proportional rule (Noss and Hoyles, 1996). This time the exploration process became more 'focused' as if dragging had found its path to provide a form against with the normality of the shape was judged. However, when S2 speaks about relation, he refers to the correlation of each set of three numbers so as to solve the construction problem for all subsequent attempts. In his words, the emergent meaning is related to the functional interdependence between the construction magnitudes which is a necessary step for the completion of the final construction with one variable. This process is completed in the next episode through the use of appropriate function operators for the expression of the internal relationships built into the figure.

## GRAPHICAL DISCONTINUITY OF THE COVARIANT MAGNITUDES

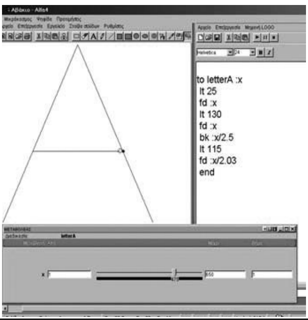


**Fig. 3: Changing 'A' with one variable**

Children have constructed the changing "A" based on the (:x) variable (Figure 3). For the expressions of the changing lengths corresponding to the variables (:y) and (:z), they have divided the variable (:x) by the numbers resulting from the division of the original pattern's lengths: 75:30=2,5 and 75:37=2,03 respectively. The students rounded off the second quotient, since its exact value is the periodic decimal 2,027027... As S1 moves the only slider of variable (:x) to gradually bigger values, the researcher points out a gap between the horizontal and slanted segments.

S1: Yes. Because 2,03 has been rounded off … It should have been 2,27.

S2: We have to make a more accurate … division.

They repeat the division and then they change the denominator from 2,03 to 2,027.

S2: However, it's better now. Before, the difference was bigger.

S1: Let's do it better.

S2: [By adding another 027 to the denominator] Another zero two seven.

As regards the mathematical content, the field where normalising takes place is the dynamic covariance of the sides of similar geometrical figures. The term 'better' used by S2 seems to refer to both the approximation of the function operator implying the use of more digits and to the best possible accuracy on the figure. The relationship between the figures' side and horizontal segments (interior vertex angle of 50°), is an irrational number and therefore, the figure will inevitably present a slight difference for high values of (:x). What is particularly noticeable in the excerpt is the emerging connection between symbolic and graphical representation and the way it is used to elaborate the internal relationships in the general procedure: pupils triggered by a little abnormality on the graphical outcome formed a utility in which symbolic notation helped them to extend the normative process. At the same time the episode is indicative of the dynamic nature of normalising in pupil's manipulation of relationships by exploring the dependencies between different objects and representations. As normalising develops the use of the variation tool in particular shifts to being an analytic tool connecting the various representations of the internal proportional relationships built into the figure and the experimental process to the results of the constructions themselves.

## DISCUSSION

These episodes illustrate the dialectic relationship between the evolution of normalising and pupil's progressive focusing on relations and dependencies underlying the current geometrical constructions and its representations. The key difference amongst the episodes is that in the evolution of the activity the appreciation of the feedback was much more closely bound into the articulation of the proportional relationships involved. We had hoped that children could see the construction problems in relation to symbolic changes each time – but this turned out not always to be the case. In the first episode an icon-driven interpretation of the task to build a bigger letter in proportion with the original pattern bypassed altogether the internal relationships of its structure and it was not related to any kind of proportionality. In the second episode children seemed to gain control of the normative process adopting intuitively the scalar proportional strategy of doubling and halving by the use of the symbolic interface of the variation tool. In the third episode pupil's previous experience with the computational tools had been moving in the direction of manipulating the graphical object and its symbolic relations as a source to bring new meanings to the questions arose by the current construction task. Indeed, the approximation of the horizontal segment in the last episode highlights the dynamic nature of normalising as a corrective activity since for high values of the only variable another corrective field could be introduced for further normalising actions likely to follow. What we took from these situations was not so much student responses to proportionality tasks, but rather, their progressive recognition and expression of relationships between the elements of the problem by playing with representations and relationships as well. The use of symbolic and graphical notation in conjunction with the dynamic manipulation of the way the figures evolved as variable values changed, played an important part in the generation of these ideas which was interwoven with the activity and the use of the tools.

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