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# DYNAMIC MANIPULATION SCHEMES OF GEOMETRICAL CONSTRUCTIONS: INSTRUMENTAL GENESIS AS AN ABSTRACTION PROCESS

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*Dynamic manipulation of geometrical constructions enabled by a specially designed computational tool, called variation tool, is studied during the implementation of proportional geometric tasks in the classroom. The analysis combined the use of two theoretical frameworks: instrumental genesis and situated abstraction. The Dynamic Manipulation Schemes (DMS) developed by 13-year-old students based on the use of the variation tool are reported in the paper. It is indicated that situated abstraction may complement instrumental genesis in analysing the links between student's behaviors and expressions of mathematical ideas within particular computational settings.*

## INTRODUCTION

This paper is reporting doctoral research aiming to explore 13 year-olds' dynamic manipulation of geometrical figures during activity involving ratio and proportion tasks in their classroom. The students worked in collaborative groups of two using 'Turtleworlds', a piece of geometrical construction software which combines symbolic notation, through a programming language (Logo), with dynamic manipulation of variable procedure values (Kynigos, 2004). In 'Turtleworlds', the elements of a geometrical construction can be expressed with variables (or functional relationships including variables) and dynamically manipulated by dragging on the 'number line'-like representation of these variables using a specially designed computational tool. Manipulation of geometrical objects in mathematics education has mainly been concerned with Dynamic Geometry Software (DGS) environments. In these environments manipulation can be characterised as dynamic since it is realised through dragging actions offering the ability to change constructed figures by interacting with particular features of them, while preserving specific mathematical rules (Hegedus, 2005). Some researchers have considered dragging as an instrument of mediation between the perceptual level of figures on the screen and the conceptual control on them (Hölzl, 1996, Arzarello et al., 1998), while others have confirmed its crucial role in supporting students to develop deductive explanations when encounter unexpected graphical results (Hadas, Hershkowitz & Schwarz, 2000). The present research aims to offer a different perspective on the process of *instrumental genesis* (Verillon & Rabardel, 1995) based on the kinesthetic control of figures in a computer environment combining two kinds of representation: dynamic manipulation and algebraic notation. The students were engaged in a project to build enlarging-shrinking figural models of capital letters of varying sizes in proportion by using only one variable to express the relationships within each geometrical figure. Thus,

proportional reasoning in this study is considered as a system of two variables with a linear functional relationship Y=mX (Karplus et al., 1983) which very often is perceived by students as additive rather than multiplicative (Hart, 1984) especially within geometrical enlargement settings (Kuchemann, 1989). The analysis elaborates the role of student's exploration with the 'dragging' modality of the computer environment in the process of instrumental genesis and describes how the notion of *situated abstraction* (Noss & Hoyles, 1996) could be used to make sense of pupil's evolving mathematical knowledge interrelated with the use of this specific computational tool.

## INSTRUMENTAL GENESIS AND SITUATED ABSTRACTION

The analytical frame of *instrumental genesis* is based on the distinction between *artefact* and *instrument* with the latter having a psychological component (*scheme*), indicating the dialectic relationship between activity and implicit mathematical knowledge, that a subject operationalises when using the artefact to carry out some task (Guin & Trouche, 1999). The activity that employs and is shaped by the use of instruments (*instrumented activity*) is directed towards the artefact, eventually transforming it for specific uses (*instrumentalisation*), as well as towards the subject leading to the development or appropriation of *schemes (of instrumented action)* in which the subject is shaped by actions with the artefact (*instrumentation*) (Artigue, 2002). The academic discussion on the above terms appears to admit as a key challenge for the integration of technology into classrooms and curricula to understand and to devise ways to foster the process of instrumental genesis (Trouche, 2003). However, it is has been recently highlighted (Hoyles, Noss & Kent, 2004, p. 314) that "although schemes of instrumented action recognise the crucial shaping of the learner by interaction with tools, their very generality makes it all the more important to take account of the *specific way* mathematical knowledge might be developed." (my emphasis). This is what the notion of *situated abstraction* (Noss & Hoyles, 1996) seeks to address, i.e. to describe how learners construct mathematical ideas drawn on the linguistic and conceptual resources available for expressing them in a particular computational setting as well as the ways in which learners exploit the available tools to move the focus of their attention onto new objects and relationships (which may be divergent from standard mathematics). In this paper instrumental genesis is considered as a process complementary to situated abstraction for effectively describing student's instrumented mathematical knowledge in terms of situated abstractions of mathematical ideas that are being developed and expressed during their interaction with a specially designed computational tool for dynamic manipulation of geometrical objects through dragging.

## RESEARCH SETTING AND TASK

In Turtleworlds, what is manipulated is not the figure itself but the value of the variable of a procedure by dragging on the dynamic manipulation feature of the computer environment called '*variation tool*'. After a variable procedure is defined

and executed with a specific value, clicking the mouse on the turtle trace activates the variation tool, which provides a slider for each variable (see at the bottom of Fig.°1).
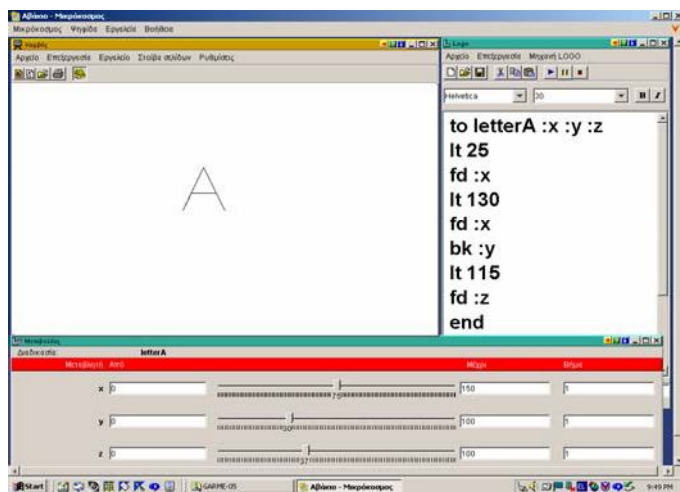


Figure 1: A model of "A" with three variables.

Dragging a slider has the effect of the figure dynamically changing as the value of the variable changes sequentially. In the procedure of Figure 1 for letter "A" the first variable (:x) changes the length of the "slanty" sides, the second (:y) the length on the "slanty" sides from the base to the edges of the horizontal side and the third (:z) the horizontal side. The graphics, the variation tool and the Logo editor are all available on the screen at all times. The user can change in each slider the initial value, the end value as well as the step of the variation (these numbers are shown in Figure 1 in the small boxes beside the sliders). The procedure for drawing a model of a letter with one variable can be derived through the functional relation of the only variable to the ratios of the sides of a fixed model of the letter. The research took place in a secondary school with two classes (A1 and A2) of 26 pupils aged 13 years old and two mathematics teachers. During the classroom activity, the students were engaged in building models of capital letters of variable sizes, having initially been told that the aim was for each letter procedure to have one variable corresponding to the height of the respective letter. According to the task, each group of pupils was assigned to construct two letters (for a more detailed description of the task see Psycharis & Kynigos, 2004). Having already had experience with traditional Logo constructions including variables, the students were introduced to the use of the variation tool at the beginning of the study by constructing basic geometrical figures (e.g. squares, rectangles) with variables.

**METHOD**

During the activity, which lasted four months, each of the two classes had two 45-minute project work sessions per week with the participant teachers. In the classroom a team of two researchers took the role of participant observers and focused on one group of students in each class (focus groups), recording their talk and actions and on the classroom as a whole recording the classroom activity. In each data collection session the researchers used two cameras: a first one was on the focus groups and a second one was occasionally moving to capture the overall classroom activity as well as other significant details in student's work as they occurred. Verbatim transcriptions of all recordings were made. We adopted an analytic stance integrating conditions (why) with interactions (how) (Strauss and Corbin, 1998) accompanying the use of the variation tool and the subsequent actions taken by pupils. The

researcher "read" each dragging on the variation tool as an incident directly linked to "before" (cause) and "after" (result). The unit of analysis was the episode, defined as an extract of actions and interactions developed in a continuous period of time around a particular issue. The extraction of the episodes was based on the following criteria: (a) the "initial motive" of the dragging, which mostly concerned distortions to the figural representations, (b) the children's "focal point" while dragging, recognized among what they said and did and (c) the "chain of proportional meanings", which accompanied the children's actions while or after dragging.

**RESULTS**

Early in their work most of the pupils constructed a model of their letter - which we refer to as *the original pattern* – without using any variables (Phase A). On the next phases of their exploration, pupils experimented to change it proportionally by choosing different variables for its segments (Phase B) until they built their final one with one variable (Phase C). Since none of the students had used the variation tool before, they were all at the genesis of instrumentation of this particular tool, beginning to form the partnership necessary to integrate its use into their experimentation so as to complete the requested tasks. Dragging on the variation tool was thus considered as an inevitable part of pupil's instrumented actions characterizing a number of qualitatively different *Dynamic Manipulation Schemes (DMS)* that our data analysis revealed. Along with Trouche (2003), I distinguish between 'dragging as a gesture' and scheme, considering the former as an observable part of the latter. Each scheme is considered below through representative examples.
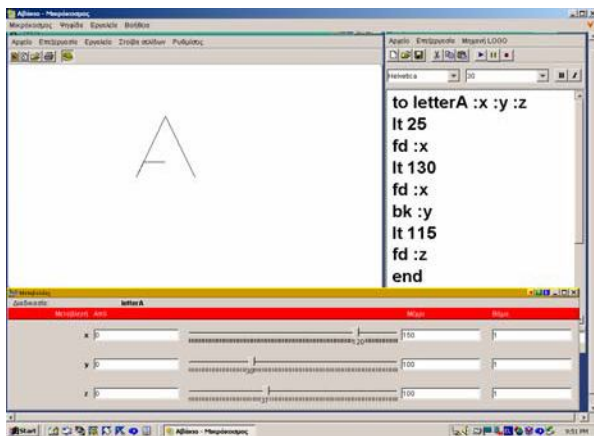


Figure 2: The 'distortion' of "A".

*Reconnaissance DMS.* In a number of pupils the initial draggings of the variation tool were associated with the changes on the figure when moving the existing sliders. In a construction of "A" (focus group-A2) with three variables (Figure 1) such a moving of a slider oriented students to recognize the interdependence of the lengths of the figure. The three sliders were set in the values of the original pattern as displayed at the bottom of the screen: x=75, y=30 and z=37. The 'distortion' of the figure (Figure 2) when moving the slider of (:x) for the first time lead students to move all the other sliders of the variation tool to higher values so as to 'close' the shape. In this phase pupils seemed to give priority to complete the figure instructed by the visual outcome on the screen and not paying attention to some kind of relationship between the selected values. However, we may observe that pupils apparently connected at an intuitive level the articulation of the figure and the interdependence of the involved magnitudes. The emergent *reconnaissance DMS* can be seen as a usage scheme (Trouche, 2003),

oriented towards the management of the variation tool (i.e. recognition of its functionalities) as well as an instrumented action scheme, implemented by the students to construct a bigger model of "A".

***Correlation DMS.*** Another scheme of the use of variation tool at first seemed to be another reconnaissance DMS emerging during student's transition from the construction of the original pattern to the dynamically changing constructions with the use of variables. However, further consideration showed that students were not simply using the variation tool to complete the shape of a letter instructed by the visual feedback, as seen above, but there was a partnership evolving with the variation tool assigned a defined role in their attempts to distinguish the relations underlying the interdependence of the involved values.
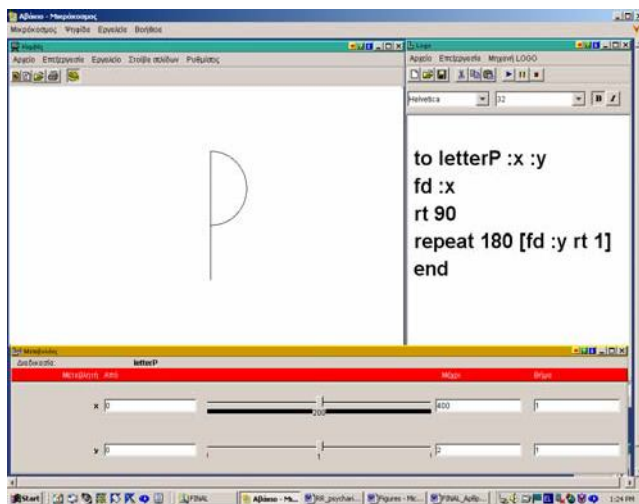


Figure 3: "P" with two variables.

In a "P" construction (Group 9–A2), the *correlation dragging* of the two sliders took its meaning via the equivalence of the ratios of the two variables involved in the construction. In the original pattern (x=400, y=2) students considered that the semicircle coincided with the middle point of the vertical segment. Experimenting to construct similar "P" models of different sizes, S1 had the idea to set as end value for each slider the correspondent values in the original pattern. He then constructed a (similar)

figure of "P" so as to preserve the property "intersection in the middle" by dragging the two sliders at half of the values in the original pattern (x=200, y=1) that corresponded to their middle points (see the current position of the two sliders in Figure 3).

> S1: When set at 200 [*i.e. slider x*] it means that it [*i.e. the semicircle*] is in the middle.
>
> R: And how do you know that the semicircle is in the middle?
>
> S1: We 'll also set this in the middle [*e.g. the slider y*]. It starts from 0 to 2. Therefore, we will set it exactly in 1.

The interrelation of the geometrical property with the arithmetic changes made by S1 is shown by the different meanings of the word "middle": at the beginning of the excerpt S1 uses it to refer to the figure, while in the end to the middle point of the slider y. Here, S1's specific draggings indicate the evolution of instrumental genesis: at the technical level he transformed the variation tool by moving both sliders on specific points (instrumentalisation) while -at the conceptual level- gained control of the similarity ratio (between the original and the new pattern of "P") by taking into account the preservation of a particular geometrical property.

**Testing DMS.** The *testing DMS* emerged as an indication of student's familiarization with the use of computational tools and it was characterized by qualitative differentiations in expressing both the geometric and algebraic properties of the requested geometrical constructions. Dragging within this scheme was mainly associated with testing student's conjectures based on indications or conclusions of preceded DMS. In an "N" construction with one variable, students (focus group–A1) integrated the variation tool into their approach and used it to test the situated abstraction of the relation between the two construction lengths (r and 1.5*r).
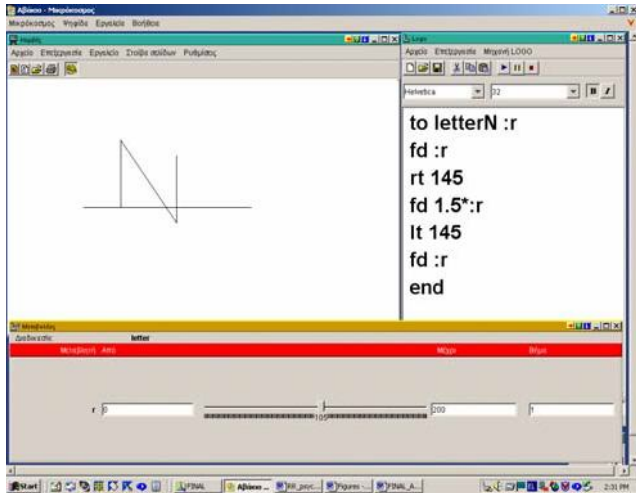


Figure 4: "N" with one variable.

Dragging the only slider r, students realized that the side length did not exactly coincided with the horizontal line that they had drawn at the letter base (Figure 4).

S2: It is exactly the same, or even worse [*i.e. the distortion*].

R: Therefore, this is probably not 1.5 times…

S2: Yes, it may be 1.45. [*S2 replaces in the procedure 1.5 by 1.45 and moves the only slider so as to test the new value*].

What is particularly noticeable in the above excerpt is that the suggested value for the functional operator by S2 precedes the new moving of the only slider indicating a shift in the use of the variation tool for validating the relationships described in the symbolic expression: students triggered by an abnormality on the graphical outcome formed a utility in which dragging in conjunction with the symbolic notation helped them to extend the elaboration of the proportional relation between the covariant magnitudes so as to prevent the distortion of the shape. At the same time the evolving DMS indicates the dynamic nature of the experimentation with the variation tool providing a basis for the development of subsequent correlations likely to follow.

**Verification DMS.** *Verification DMS* emerged as part of the evolution of the students' familiarization with the control of the mathematical concepts concerning the construction of enlarging-shrinking geometrical figures. The functional expression of one variable in relation to another was the most difficult type of correlation, especially in cases involving arithmetic values not resulting in integer quotients. In several cases forming such kind of relationships was facilitated by preceding correlations of values leading to integer quotients. For the construction of an enlarging-shrinking model of "B", the students (focus group-A2) chose to employ an already developed multiplicative strategy including integer quotients, that they had applied successfully in the construction of another letter. The original pattern was constructed for the values x=100 and y=0.44 (when replaced in the procedure shown in Table 1). In the final enlarging-shrinking model with one variable, variable y was

substituted by the expression x/227.3 since the result of the division 100:0.44 = 227.272727272 was rounded off by the students.

| To letterB :x :y |
| :---: |
| fd :x |
| rt 90 |
| repeat 180 [fd :y rt 1] |
| lt 180 |
| repeat 180 [fd :y rt 1] |
| End |

Table 1: The procedure of "B".

[S1 drags the only slider x for enlarging and shrinking the letter.]

S1: [To the researcher] You see?

S2: We divided 100 by 0.44 and got 227.3.

By dragging the only slider, S1 verifies the successful outcome of the multiplicative construction strategy, implying that it can also be followed in cases including non-integer correlations. In that sense, this specific dragging signals the use of the variation tool as an instrument mediating strategies based on properties and relations rather than on arithmetic values of a particular type. As far as the nature of the developed instrumented actions, we observe a complete shift of student's attention from the graphical to the symbolic representation of the computer environment.

## CONCLUSIONS

In this paper we have considered the different DMS generated as students begin to use the variation tool in constructing enlarging-shrinking geometrical figures by means of relations abstracted, i.e. constructed and expressed, within this particular computational setting. Under the situated abstraction perspective these DMS illustrate the dialectic relationship between the evolution of instrumental genesis and student's progressive focusing on relations and dependencies underlying the current geometrical constructions and its representations. According to the results, the key difference amongst the described DMS is that in the evolution of instrumental genesis the appreciation of the computer feedback was much more closely bound into correlations rooted in action (within the same or a new DMS) and inextricably linked with the use of the variation tool. As soon as the variation tool became part of student's activity, student's instrumented actions progressively evolved from the visual level (Reconnaissance DMS) to the conceptual level indicated by the development of mathematical practices involving the appreciation of the (scalar) relation between the lengths of similar figures (Correlation DMS), the testing of conjectures (Testing DMS) as well as the verification of employed multiplicative strategies (Verification DMS). In future papers, further elaboration of the interconnections between the above DMS and their evolution within specific groups of students is expected to enrich the analysis. However, the above results indicate that dynamic manipulation of figures in a kinesthetic way can be considered as a context in which the different instruments built by the students, based on the use of the variation tool, may reflect how the implicit emergence of proportionality as a concept-in-action (Trouche, 2003) might be explicitly operationalised and articulated in mathematical terms of situated abstractions as part of the instrumental genesis.

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