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REIFYING ALGEBRAIC-LIKE EQUATIONS IN THE CONTEXT OF CONSTRUCTING AND CONTROLLING ANIMATED MODELS

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This paper reports on a design experiment conducted to explore 17-year-old students' constructions of meanings, emerging from their interpretations and uses of algebraic-like equations in the context of constructing and controlling animated models. We particularly focused on the students' engagement in reification processes, e.g. making sense of structural aspects of equations, involved in conceptualising them as objects that underlie their animated models' behaviour.

THEORETICAL BACKGROUND

In this paper we report on a classroom research [1] aiming to explore 17-year-old students' construction of meanings emerging from the use of algebraic-like equations employed as means to create and animate concrete entities in the form of Newtonian models. The students worked collaboratively in groups of two or three using a constructionist computational environment called "MoPiX" [2], developed at the London Knowledge Lab (http://www.lkl.ac.uk/mopix/) (Winters et al., 2006). MoPiX allows students to construct virtual models consisting of objects whose properties and behaviours are defined and controlled by the equations assigned to them. We primarily focused on how students interpreted and used the available equations while they engaged in *reification* processes (Sfard, 1991), e.g. making sense of structural aspects of equations, involved in conceptualising them as objects that underlie the behaviour of their models.

Recognising the meaning of symbols in equations, the ways in which they are related to generalisations integrated within specific equations and the ways in which a particular arrangement of symbols in an equation expresses a particular meaning, are all fundamental elements to the mathematical and scientific thinking. Research has been showing rather conclusively that the use of symbolic formalisms constitutes an obstacle for many students beginning to study more advanced mathematics (Dubinsky, 2000). Traditional approaches to teaching equations as part of the mathematics of motion or mechanics seem to fail to challenge the students' intuitions since they usually encompass static representations such as tables and graphs which are subsequently converted into equations. Lacking any chance of interacting with the respective representations, students fail to identify meaningful links between the components and relationships in such systems and the extensive use of mathematical expressions (diSessa, 1993).

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In the relevant research in the mathematics education field, a central question concerns the nature of equations and the ways in which they can be understood by students. Most of the respective studies are based on the distinction between the two major stances that students adopt towards equations: the process stance and the object stance (Kieran, 1992; Sfard, 1991). The process stance is mainly related with a surface "reading" of an equation concentrated into the performance of computational actions, following a specific sequence of operations (i.e. computing values). The object stance, however, is related to the actual entity -the equation itself- and the outcome of the computational actions performed (i.e. the computational product).

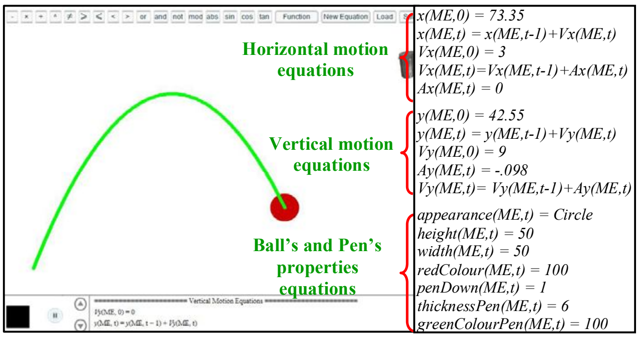
Elaborating further on the distinction between the above stances and the ways by which students understand algebraic expressions (and thus equations), mathematics educators brought into play the idea of students moving from process–oriented views to object–oriented ones via a process of abstraction which has been termed *reification* (Sfard, 1991) and has been considered to underlie the learning of algebra in general. Sfard's theory of reification (1991) describes three levels of mathematical conceptual development which eventually lead to the construction of a new concept. In the first level -the stage of *interiorisation*- the learner gets acquainted with processes involving operations performed on lower-level mathematical objects. At the second level -the stage of *condensation*- the learner is able to condense processes, viewing them as a whole. At the third level, corresponding to *reification*, the learner is able to view mathematical concepts as objects in their own right and use them as inputs in higher-order processes which might be precursors to new constructs.

Adopting a constructionist framework (Harel and Papert, 1991) in the present study we used a computational environment that is designed to enhance the link between formalism and concrete models, allowing us to study the ways in which the use of formalism, when put in the role of an expression of an action or a construct (a model), can operate as a mathematical representation for constructionist meaning-making. Our central research aim was to study students' construction of meanings, emerging from their uses of the available mathematical formalism, when engaged in reification processes. Particularly, we were interested to shed light upon the relationship between the evolution of students' understandings with their emerging engagement in different aspects of the abstraction processes (i.e. interiorisation, condensation and reification) concerning the conception of equations as objects.

THE COMPUTATIONAL ENVIRONMENT

MoPiX (constitutes a programmable environment that provides the user the opportunity to construct and animate models representing phenomena such as collisions and motions. In order to assign behaviours and properties to the objects taking part in the animations, the user attributes equations that may already exist in the computational environment's "Library" or equations that she constructs by herself. The MoPiX equations incorporate both formal notation symbols (i.e. Vx, x, t) as well as programming – natural language utterances (i.e. Circle, appearance).

However, their main characteristic is that they constitute functions of time. The environment constantly computes the attributes given to the objects in the form of equations and updates the display, generating on the screen the visual effect of an animation.



MoPiX offers a strong visual image of equations as containers into which numbers, variables and relations can be placed, allowing students to make easily connections between the structure of an equation and the quantities represented in it. It also allows the user to have deep structure access (diSessa, 2000) to the models animated as the equations attributed to the objects do not constitute "black boxes", unavailable for inspection or modifications. The manipulations performed to a model's symbolic facet (i.e. the equations) generally produce visual results on the Stage, from which students can get meaningful feedback.

TASKS

For the first phase of the activities we developed the "One Red Ball" microworld which consisted of a single red ball performing a combined motion (Figure 1). The students were asked to execute the model, observe the animation and discuss with their peers the behaviours generated. In order to stimulate students to start using the equations themselves, we asked them to try to reproduce the red ball's motion. In this process, we encouraged them to interpret and use equations from the "Library" and link the equations they used to the behaviours they had previously identified. As we deliberately made the original red ball move rather slowly we expected students to start expressing their personal ideas about their own object's motion (e.g. make it move faster) and thus start editing equations so as to ascribe it new behaviours.

For the second phase of the activities we designed a half–baked microworld (Kynigos, 2007), i.e. a microworld that incorporates an interesting idea but it is incomplete by design so as to invite students to deconstruct it, build on its parts and change it. The "Juggler" half-baked microworld (Kynigos, 2007) consisted of three interrelated objects: a red ball and two rackets. The ball's behaviour was partially the same as the "One Red Ball's", however, when it hit the rackets, it bounced, moving away in specific ways. We asked the students to execute the Juggler's model, observe the animation and identify each object's behaviour. The students were encouraged to discuss with their peers on how they would change the "Juggler" microworld and embed in it their own ideas. In this process, students were expected to deconstruct the microworld so as to link the behaviours animated to the equations and reconstruct it, employing strategies that would depict their ideas about their model's behaviours.

METHOD

The experiment took place in a Secondary Vocational Education school in Athens for 25 school hours with one class of eight 12^{th} grade students studying mechanical engineering and two researchers. The adopted methodological approach was based on participant observation of human activities, taking place in real time. A screen capture software was used so as to record the students' voices and their interactions with the MoPiX environment. The data corpus also included the students' MoPiX models and the researchers' field notes. We verbatim transcribed the audio recordings of two groups of students and also several significant learning incidents from other workgroups. The unit of analysis was the episode, defined as an extract of actions and interactions performed in a continuous period of time around a particular issue. The episodes presented were selected (a) to involve interactions with the available tool during which the MoPiX equations were used to construct mathematical meaning and (b) to represent clearly aspects of a reification process.

ANALYSIS AND INTERPRETATIONS

Conceiving the MoPiX equations operationally– The phase of Interiorisation

Attempting to make their objects move exactly like the "One Red Ball", the students interpreted and used during the previous phase of their experimentations several motion equations that they found categorised in the environment's "Library". However, as they gained familiarity with the MoPiX formalism, they didn't seem willing to confine themselves in merely reproducing a given motion. Expressing their own ideas about the way their objects should move, the students started modifying the pre-defined library equations.

The students of Group B, for instance, looking in the library for equations that would make their object move vertically, came across the "Vy(ME,0)=0". This equation prescribes the velocity of an object in y axis at the zero time instance (left part of the equation) to be 0 (right part of the equation).

S2	[*To S1 who attributed the "Vy(ME,0)=0"*] Press "Play". You didn't do anything. You just made the velocity 0 at the zero time instance. Its initial velocity is 0. You did nothing to it. It didn't change, to move downwards.
S1	Yes, yes.
S2	That's what I'm saying. Change it. Give it some initial…, we should give it an initial velocity. Isn't it better?
R2	Whatever you like.
S2	Give "3" as an initial velocity. The equation you used before, with the difference that after the equal sign, we will place a "3". There, move it up.

S1 No. That was stupid. Let's change the velocity. Increase it.

After attributing the "Vy(ME,0)=0" equation to their object and started the animation, the students realised that the equation they used didn't make their object move downwards as they had expected. This observation triggered the implementation of a series of changes on the right part of the equation, beginning with converting "0" into "3". The students went on replacing the arithmetic value on the right side of the equation with other ones, attributing each time the new equation to their object so as to verify its effect. However, this procedure seemed to be rather mechanical. All the new equations were in the form of "Vy(ME,0)=___" which means that they merely determined the object's velocity at the *zero time instance* and thus had no apparent effect to the object's motion. Nevertheless, the students continued replacing the arithmetic value on the right side of these equations with new ones, a process that implies an operational conception of the notion of equation. The continuous replacements indicate that students viewed the expression "Vx(ME,0)" on the left side as an unknown quantity (an "x") which had be equal to a specific arithmetic value ("x=____"). They seemed to disregard the structure of the "Vx(ME,0)" expression and the meaning its comprising symbols conveyed and considered the right part of the equation as a placeholder into which numbers should inserted in order to provide an arithmetic value for the unknown quantity.

Conceiving the MoPiX equations operationally – The phase of Condensation

Having determined the meaning of the symbols in the "Vy(ME,0)=0" equation, the same students, continuing their experimentations, sought for ways to make the velocity of their object to constantly be "3".

S2 How can we insert the 2nd, 3rd time instance… in there? [*the equation*]

S1 In the 0 time instance, it's 3…

S2 Do we need a symbol for this?

R2 Do we need a symbol? It's a good question. How you plan to express it?

S2 With symbols we usually express something that we can't describe accurately.

S1 Plus… t! [*He types Vy(ME,t)=3 and points at "t"*] So, when I look at this symbol

S2 I'll know it represents the infinity.

In the above extract, the students seem to have relocated their focus from just replacing arithmetic values to determine an unknown quantity into forming functional relations. Although they could have followed the previous strategy and start replacing the "0" on the left side of the equation with other numerical values (i.e. 2 , 3, …) so as to form several equations that would define the velocity at different "time instances", the students took a moment to think of ways to incorporate all the numerical values in one equation that would describe the velocity at all "time instances". This approach led them to introduce a symbol which they would "look at and know that it represents the infinity". The students seem to have dethatched their mathematical activity from simply attributing specific arithmetic values to an unknown quantity and to attempt to form a functional relationship among two varying quantities: the velocity and the time. Symbolising all the upcoming time instances with “t” –an in-built MoPiX symbol– the students seem to have taken a more decontextualised stance towards the notion of equation and at the same time to have condensed the up to that point number attributing processes into a whole.

Conceiving the MoPiX equations structurally – The phase of Reification

During the two previous phases of their experimentations, the students edited Library equations and manipulated in-built MoPiX symbols, such as the “t”. The changes performed restricted to the content of the existing equations (e.g. substituting a numerical value for another one or for a variable) while structure was left intact. The next episode describes how the Group A students, in the course of changing the “Juggler” microworld, didn’t just edit existing equations but constructed two new ones from scratch. The idea these students wanted to bring into effect was to “make a ball change its colour according to an ellipse’s position”.

S1	When it [*the ball*] is situated in a Y below the Y of this one [*the ellipse*] for example…
R1	I’m thinking… Will the ball know when it is below or above the ellipse?
S2	That’s what we will define. We will define the Ys.
S1	This. The: “I am below now”. How will we write this?
S2	Using the Ys. Using the Ys. The Ys. That is: when its Y is 401, it is red. When the Y is something less than 400, it’s green!

Having conceptualized the effect they would like their new equation to have, the students decided about two distinct elements regarding the equation under construction: its content (i.e. the symbols to be used) and its structure (i.e. the signs between the symbols). However, as there was no in-built MoPiX symbol to express the idea of an object becoming green under certain conditions, they had to invent a new symbol: the “gineprasino” (i.e. “become green” in Greek). To represent the ball’s position they used its Y coordinate in the form of a quantity varying over time (i.e. “y(ME,t)”), whereas to represent the ellipse’s position, its Y coordinate in terms of the constant arithmetic value (i.e. “274”) corresponding to the object’s position at the time. Adding a “less than” sign in between, the equation eventually developed was the “gineprasino(ME,t)=y(ME,t)≤274”.

Since this equation described the event to which the ball would respond (being below the ellipse) and not the ball’s exact behaviour after the event would have occurred (change its colour), the students decided to construct another equation. A Library equation which explains what happens to a ball’s velocity when it hits on one of the Stage’s sides, led students to duplicate this equation’s structure, eliminate any content and use it as a template to designate what will happen to the ball’s colour when it goes below the ellipse. The second equation encompassed in-built MoPiX symbols, the “gineprasino” variable in two different forms and numerical values (0 and 100) to express the percentage of the green colour the ball would contain in each case (i.e. being above or below the ellipse). The second equation developed was the: "greenColour(ME,t) = not(gineprasino(ME,t))×0 + gineprasino(ME,t)×100".

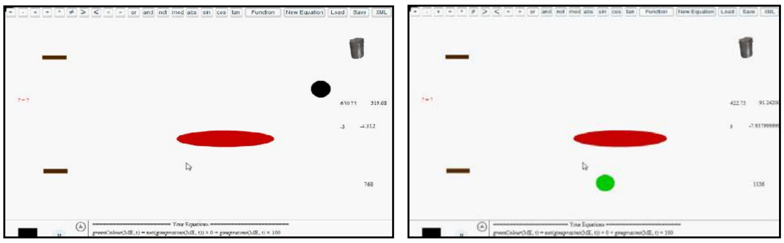


Figure 2: The ball's different percentage of green colour according to its Y position

The above episode implies a qualitative transformation of the students' mathematical activity form process-oriented into object-oriented. The operational conceptions regarding the MoPiX equations seem to have now given their place to more structural ones. In the process of constructing their first equation, the students invented a new –meaningful to them– symbol to which they attributed the properties of a varying over time quantity, used and manipulated in-built MoPiX symbols and inserted an inequality operator to specify the relation between their symbols. Translating their own ideas into algebraic equations, defining both their content (i.e. the symbols) and structure (i.e. the relation among the symbols) indicates that students have conceptualised the MoPiX equations as a "fully-fledged mathematical objects" (Sfard, 1991 pp.12).

A structural conception of the MoPiX equations is also advocated by the students' series of actions in the process of constructing the second equation. Striving to transform their idea into a MoPiX equation, the students indentified a mapping between an existing Library equation and the one they attempted to create. Subtracting the Library equation's structure and eliminating its content, the students formed a template whose fields they completed using terms relevant to the behaviour they wished to attribute to their object. This is a clear indication that the students were able to recognise the existence of structures external to the symbols themselves and to use them as landmarks to navigate their equations' construction process.

It is noticeable, however, that in the process of constructing the second equation, the students' conceptualisation of the first equation partially shifted to become operational again. Viewing the "y(ME,t)≤274" as a iterate comparing process between two numbers and the "gineprasino(ME,t)" as the outcome of this comparison, the students integrated the "gineprasino(ME,t)" varying quantity into their second equation treating it as an algebraic object. This aspect suggests the co-existence of both a structural and an operational conception of the MoPiX equations.

CONCLUSION

Our purpose in this paper was to illustrate the students' development of a structural conception of the notion of equations in the context of constructing and controlling animated models. Editing ready-made algebraic-like equations and constructing new ones so as to assign behaviours to their objects, the students reached different *degrees of structuralization* (Sfard, 1991) (i.e. interiorisation, condensation, reification) shifting gradually their view of equations from process-oriented into object-oriented, without, however, those two approaches being mutually exclusive.

Concluding, under the constructionist theoretical perspective, in the present study reifying an equation was not a one-way process of understanding hierarchically-structured mathematical concepts but a dynamic process of meaning-making, webbed by the available representational infrastructure and the ways by which students drew upon and reconstructed it to make mathematical sense.

NOTES

1. The research took place in the frame of the project "ReMath" (Representing Mathematics with Digital Media), European Community, 6th Framework Programme, Information Society Technologies, IST-4-26751-STP, 2005-2008 (http://remath.cti.gr).
2. "MoPiX" was developed at London Knowledge Lab (LKL) by K. Kahn, N. Winters, D. Nikolic, C. Morgan and J. Alshwaikh.

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