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MEANINGS FOR ANGLE THROUGH GEOMETRICAL CONSTRUCTIONS IN 3D SPACE

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Abstract. *The construction of meanings for the notion of angle in 3d geometrical space is studied during the implementation of specially designed geometrical tasks in the classroom. 13-year-old pupils were engaged in exploring the mathematical nature of angles while controlling and measuring the behaviours of geometrical objects in 3d simulations. Pupils worked in pairs using a specially designed computational environment that combines symbolic notation (by means of programming) and dynamic manipulation of graphically represented geometrical objects. The findings reveal a progressive coordination of different facets of angle in 3d space facilitated by the use of the available tools.*

THEORETICAL FRAMEWORK

This contribution reports classroom research [1] aiming to explore 13 year-olds' construction of meanings for the notion of angle in 3d geometrical space during activity involving the construction and manipulation of programmable geometrical figures. The students worked in collaborative groups of two using *MaLT (MachineLab Turtleworlds)*, a microworld environment for geometrical constructions which combines symbolic notation -through a specially designed version of Logo- with dynamic manipulation of graphically represented mathematical objects (Kynigos & Latsi, 2007).

Angles and turns are critical to 3d geometrical knowledge and since they are related to students' everyday experience they can be considered as a rich domain for mathematics meaning-making, not systematically studied up to now. Student's difficulties in the angle domains are well documented in the literature. Research results strongly suggest two major factors influencing students' use of angles: the physical presence or absence of the lines which make up its arms and the conceptualisation of turning as a relationship between two headings of segments (Clements et al., 1996). The situation is aggravated in 3d space by the apparent difficulty of students to recognise the basic elements of an angle. For example, an 'opening-closing door' situation incorporates a more obscure similarity of the standard angle domain since the arms of the embedded 3d angle are rectangles. Moreover, turning in 3d space can be considered as a sophisticated concept since the turning motion itself usually does not leave a trace and the 'heading' must be reconstructed from memory. It is also clear from previous research that students have great difficulty in coordinating the various facets of the angle embedded in various physical angle contexts involving slopes, turns, intersections, corners, bends,

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directions and openings (Mitchelmore & White, 1998). Taking this into account Mitchelmore and White (2000) highlight that a mature abstract angle concept depends essentially on learning to link the various physical angle contexts together through "the systematic attempt to investigate our spatial environment mathematically" (p. 233).

In the light of the above data, it makes no sense to perceive angle in 3d space as a mathematical notion on its own but rather it is more useful to consider it in terms of the concepts interrelated with it, the situations in which it may be used and the available representations, which constitute –in the words of Vergnaud (1996)- a *conceptual field*. For instance, a concept tightly related to angle in 3d space is that of a turn, a situation in which it may be used can be a situation evoked by a given task (e.g. an opening-closing door simulation) while the available representations can be based on the use of paper and pencil or on the use of computational tools.

Taking a constructionist perspective (Harel & Papert, 1991) in the present study we have selected the version of Logo developed in MaLT as one context to explore students' ideas around the concept of angle in 3d space based on turning and directionality. Our purpose was (a) to relate children's construction of meanings for the notion of angle in 3d space explicitly to their physical angle experiences and (b) to offer a framework in which to account specifically for their difficulties in coordinating different aspects of the notion of angle as well as to throw light on the paths by which students might come to integrate their various angle concepts in 3d space. More specifically, the main focus of the study concerned the ways by which students conceptualise angle as a spatial visualisation concept representing turn and measure when engaged in activities involving the construction and dynamic manipulation of 2d geometrical objects. Under this perspective our central research aim was to study how students used the available representations in MaLT to construct meanings for angle: (a) as a *geometric shape*, i.e. formed between two geometrical objects which can be either segments (in 2d geometrical figures) or 2d geometrical figures (in the 3d space, e.g. dihedral angles); (b) as a *dynamic amount*, indicating a change of directions both as a turn and as the result of a turn which can also be represented by a variable; and (c) as a *measure* represented by a number.

THE COMPUTER ENVIRONMENT

MaLT is a programmable environment for the creation and exploration of interactive 3d geometrical constructions consisted of the following interconnected components (see Figure 1).

• Turtle Scene (TS). This is a 3d grid-like projection of a simulated 3d space in which a 3d turtle is visualised when a command (or a procedure) is executed. Whenever the turtle moves it leaves behind it a trace which is a selectable 3d cylindrical line.

• Logo Editor (LE). This is the Logo-like programming interface which is linked to the TS. The user is able to write, run and edit Logo procedures to 'drive' the construction of geometrical objects by the turtle using a specially designed version of Logo. A distinctive feature of this specific version is that it provides an extension of Logo commands in 3d space including two new types of turtle turns: 'UPPITCH/DOWNPITCH n degrees' ('*up/dp n'*) which pitches the turtle's nose up and down and 'LEFTROLL/RIGHTROLL n degrees' ('*lr/rr n*') which moves the turtle around its trunk/vertical axis.

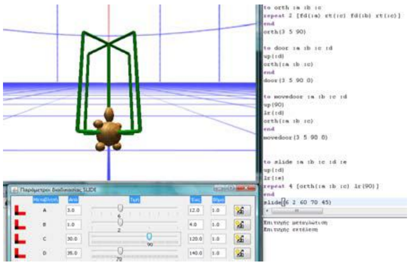


Figure 1: A revolving door simulation in MaLT.

• Uni-dimensional Variation Tool (1dVT). The main part of this component consists of 'number-line'-like sliders (see at the bottom of Figure 1), each corresponding to one of the variables used in a Logo procedure. After a variable procedure is defined and executed with a specific value, clicking the mouse on the turtle trace activates the tool, which provides a slider for each variable. Dragging a slider has the effect of the figure dynamically changing as the value of the variable changes sequentially.

RESEARCH SETTING AND TASKS

The work reported in this paper is part of a design-based experiment (Cobb et al., 2003) concerning the exploration of the mathematical nature of angles in 3d space by groups of 13-year-old pupils who work collaboratively in the classroom setting using MaLT to construct, control and animate 3d geometrical objects often encountered in everyday situations (e.g. doors, revolving doors). The experiment took place in a secondary school with one class of twenty 7 grade pupils (13 year-old) and one experimenting teacher who acted also as a researcher in the classroom. The class had totally 18 teaching sessions with the experimenting teacher over two months.

We divided the activity sequence in two phases and developed for each one of them a strand of two tasks. In designing the tasks for the first phase of the activity we took into account the constructionist theoretical implication that angle as turn (which inevitably involves directionality) in the simulated 3d space of MaLT is closely related to real world experiences as well as to body-syntonic activities such as walking or piloting a flying object. In task 1 students were asked to move the turtle in the right and left corner of the 3d space and then to bring it back at its initial position. It was not specified on purpose what was meant by 'the left and right corner', so as to leave students to explore 3d space and develop their own 'navigation' strategies. In task 2 students were asked to navigate the turtle in such a way so as to simulate the take-off and the landing of an aircraft. In the second phase of the activity angle was approached as a dynamic quantity through the simulation of real 3d objects in the TS.

In particular, in task 3 students were asked to construct rectangles using parametric procedures in at least two different planes of the TS simulating the windows of a virtual room. In task 4 students were asked to develop a parametric procedure so as to simulate the opening and closing of a door while in task 5 they were asked to use the 1dVT to control the four variables of a ready made procedure given to them so as to create the simulation of a revolving door.

METHOD

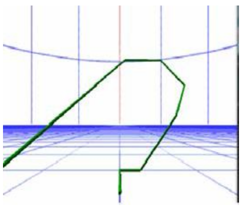
During the implementation of the activity a second researcher participated in the classroom acting as a co-researcher. The adopted methodological approach was based on participant observation of human activities taking place in real time. The researchers intervened in the children's work by posing questions and encouraging them to clearly explain their ideas and strategies. We used a specially designed screen capture software -called Hypercam- which allowed us to record students' voices and at the same time to capture all their actions on the screen. For the analysis we transcribed verbatim the audio recordings of three groups of students (focus groups) throughout the teaching sequence and we also selected significant learning incidents from the work of all groups in the classroom. Background data were also collected (i.e. observational notes, students' electronic and written works). In the analysis a phenomenological approach (Nemirovsky & Noble, 1997) was adopted according to which researchers' interpretations were focused on the student's interactions with the available representations and the meanings they attached to these from their own points of view while struggling to relate the turtle's move/turn and/or specific 2d representations in forming angular relationships in 3d space. The episodes were selected (a) to have a particular and characteristic bearing on the students' interaction with the available tools and (b) to represent clearly aspects of the students' construction of meanings for particular aspects of angle in 3d space (i.e. geometric shape, dynamic amount and measure) emerging from this use.

ANGLE AS A TURN IN 3D SPACE

During task 1 students have constructed simple crooked lines limiting turtle's motion either on the 'ground plane' or on the 'screen plane' (vertical to the ground plane) while their experimentation concerning the notion of angle was developed around how to find out the measure of turtle turns. Their preference in moving in a 2d plane was tightly linked to their difficulty in conceptualising the role of the rotation commands -especially rightroll/leftroll – while moving the turtle around in the TS. However task 2 seemed to have provided a more fruitful context for the experimentation of students with both new types of turtle turns. It seemed that the simulation of the take-off and landing of an aircraft brought in the foreground the concept of angle as a turn in 3d space with particular measure involving also angle as a slope, an aspect of angle difficultly recognised by students as the one supporting edge is missing (Mitchelmore & White, 2000). As they were engaged in navigating the turtle so as to simulate the take off of an aeroplane, in the initial stages of exploration pupils appeared not to focus on the change of planes or the angles (internal or external) of the crooked line that the turtle had drawn but rather at the angle that was drawn by the turtle in relation to the horizontal ground plane. This is evident in the following excerpt in Group A students' confusion over the way in which the commands up(45) and lt(50) affected/determined the generated graphical outcome visualized in Figure 2. These pupils were reflecting upon the commands given to the turtle so as to explain why the aircraft collided to the ground (Figure 2).

Researcher: Hey, nice take off!! I see you hit the ground!

Student: Look there is a slope up(45) and then a slope of lt(50).



fd(5)
rt(90)
fd(2)
up(45)
fd(5)
lt(50)
fd(6)
rr(30)
up(45)
fd(8)
fd(5)

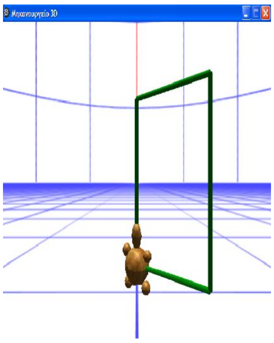
Figure 2: Simulating the taking-off and flying of an aircraft.

A closer look at the respective Logo code reveals that students, drawing upon their everyday experiences, considered the horizontal ground plane and the directions of up and down as fixed. This may be a possible explanation of their insistence to use the up command in order to get height regardless of the opposite graphical outcome. Apart from the lack of previous familiarisation with the new kinds of turtle turns, the preference of students to move the turtle only in one plane (during task 1) or their difficulty in coordinating turtle's turns and trace with the notion of angle as a slope could possibly be interpreted in the light of the fact that pupils, accustomed to work with 2d representations of geometrical figures, might have had difficulties in understanding either the conventions used to represent a 3d object in the computer screen (Lowrie, 2002) or the conventions that underlie the move of an entity – here the turtle – in a simulated 3d space. As we will see in the next paragraph simulating animated 3d geometrical objects with the use of variables was critical in helping students coordinate the various facets of angles present and conceptualise angle as a change of direction in 3d space while turning with particular measure.

ANGLE AS A DYNAMIC ENTITY FOR SIMULATING 3D OBJECTS

In task 3 the need to design figures in different planes of the 3d space challenged pupils to move the focus of their attention from directed turns between lines and planes to directed turns between two similar geometrical figures which is related to the conceptualisation of a dihedral angle in 3d space. For example, most groups of pupils recognised that in order to construct windows in two consecutive walls (planes in mathematical terms) the use of the commands 'rr/lr' or 'up/dp' was needed. During this construction process students easily identified the dihedral angle defined by the two consecutive windows (i.e. rectangles) and used the terminology familiar to them from 2d geometry in order to describe it. However, all groups of pupils had difficulties in identifying its measure. For instance, students of group B characterized the dihedral angle drawn by the turtle as an acute and not as a right one as it was the case, although they had commanded the turtle to leftroll 90 before drawing the second window. It seemed that students had focused more on the visual characteristics of the figural representation and were confused by the 'distortion' of the dihedral angle as a result of the use of a vanishing point in the line of horizon of the TS designed to strengthen the sense of depth in the representation.

However, the use of the two new types of turtle turns coupled with pupil's explorations of angle as a dynamic amount that could by dynamically handled and changed sequentially using the functionalities of 1dVT facilitated further the visualization of different planes in 3d space. For instance, in task 4 (Figure 3) students decided to use not a fixed turn measure but a variable after the rightroll command so as to simulate the opening and closing of a door.



```
to rect :a :b :c
up(90)
fd(:a)
rt(90)
fd(:b)
rt(90)
fd(:a)
rt(90)
fd(:b)
rt(:90)
rr(:c)
end
```

Figure 3. The opening-closing door.

The use of 1dVT allowed students to view their constructions from different perspectives which might have minimized the 'distorting' effects of static 3d representation and prompted them to focus on the measure of the turtle's turn in the Logo code. The more the students appeared accustomed to the conventions used in the 3d simulated space the more they were able to coordinate the visual characteristics of the drawn angles with their measure related to turtle's turns. For instance during the dynamic handling of the revolving door simulation (task 5) students were able to overcome the difficulties faced earlier during task 3 and to recognize the four consecutive right dihedral angles created between the four rectangles around the common vertical side of the four rectangles (see Figure 1). Experimenting with the variables of the procedure 'slide' (see below), which was given ready-made to them (task 5), so as to create a revolving door moving around group B students progressively got able to handle different aspects of angle simultaneously. Since for random values of the variable :c four parallelograms appeared around the common side of them, S1 compared the visual outcome with a door and recognized the need to construct initially four rectangles dragging on the slider :c on the 1dVT.

```
to slide :a :b :c :d :e
up(:d)
lr(:e)
repeat 4 [repeat 2 [fd(:a) rt(:c) fd(:b) rt(180-:c)] lr(90)]
end
```

S1: Wait, we should move it here first. It's the angle of the rectangle [*moves the slider :c to the value 90 so as to construct 4 rectangles*], so as to become like this (*i.e. the door*) and then probably turns with this [*moves the slider :e*]. Let's see…

S2: Yes, it definitely turns around with this [*i.e. slider :e*] as it has lr.

S1: Yes, but we don't only want it to turn, we also want it to move even further down.

S2: I should change here [*He puts the slider :d to the value 90 so as to have the simulation in a vertical position*].

S1: Yes, 90 is fine.

S2: Now, with this [*points to the slider :e*] it turns around normally.

The above excerpt accompanied by the respective Logo code indicates that students created meanings in relation to angle (a) as a constitutive element of a figure which is defined and stay fixed (variable :c), (b) as a means to move from the horizontal plane to the vertical one in relation to the viewing axis of the user which is again defined and stay fixed (variable :d) and (c) as a means of constantly changing planes in 3d space (variable :e) around the common vertical side of the four rectangles (Figure 1).

CONCLUSIONS

Under the constructionist theoretical perspective the above incidents illustrate the dialectic relationship between the evolution of students' interaction with the available tools and their progressive focusing on angular relationships underlying the current geometrical constructions and representations. As students were engaged in navigating the turtle in MaLT they gain a sense of the mathematical ideas related to the notion of angle through the construction and dynamic manipulation of 2d geometrical objects in 3d space by a process of hypothesising, experimenting and reflecting on the empirical observation of the graphical feedback on the screen. Struggling with the representational convention used and the identification of the angles corresponding to turtle's turning students progressively managed not only to construct dihedral angles defined by parallelograms in different planes but also to approach angle as a dynamic entity for simulating 3d objects through the use of variables. This kind of experience was critical for the coordination of the various facets of the angular relations (Mitchelmore & White, 2000) simultaneously present in the subsequent revolving door simulation where angle was approach either as a geometric shape with a fixed measure (e.g. between 2d geometrical figures) or as a dynamic amount with a varying measure. The above analysis describes angle concept development in terms of interconnected stages of abstraction which can be considered as representing a progressively more refined classification of students' experience based on their interaction with the available tools.

Notes

1. "Representing Mathematics with Digital Media", http://remath.cti.gr, European Community, 6th Framework Programme, Information Society Technologies (IST), IST-4-26751-STP, 2005-2008.

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