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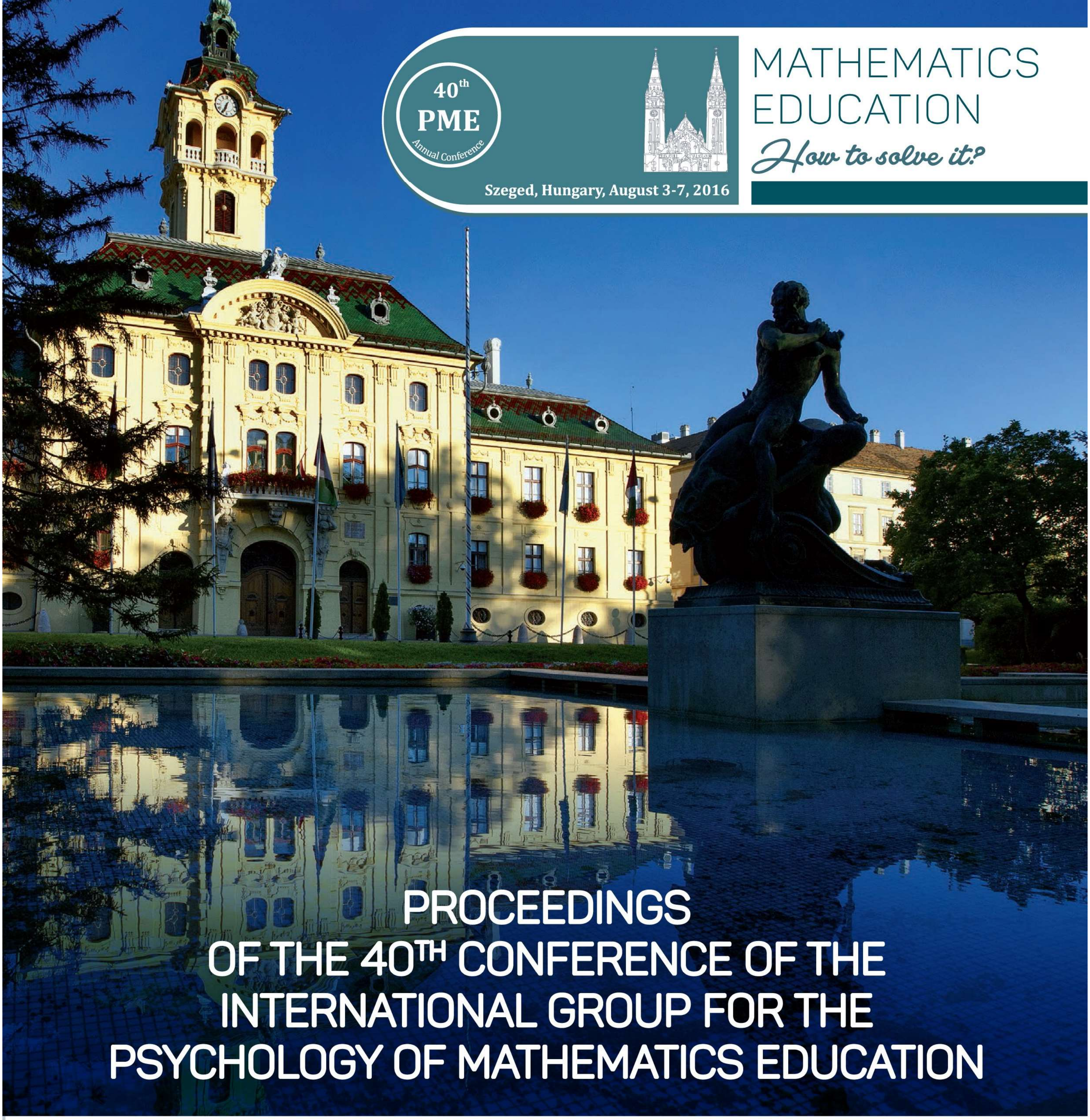

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CONCEPTUALISING FUNCTION AS COVARIATION THROUGH THE USE OF A DIGITAL SYSTEM INTEGRATING CAS AND DYNAMIC GEOMETRY

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This paper reports ongoing classroom research focusing on 16 year-olds' construction of meanings for function as covariation. The students worked on modelling tasks related to optimization problems using a digital environment that connects CAS and dynamic geometry. We analysed students' activities on functional dependencies in different settings including physical device, dynamic geometry, magnitudes and mathematical functions. The results indicate students' transition from covarying quantities to mathematical functions as an abstraction process mediated by the use of the available tools.

THEORETICAL FRAMEWORK

This paper reports classroom research aiming to explore 16 year-olds' construction of meanings for function as covariation while exploring and solving contextual problems through the use of the digital environment Casyopée (Lagrange, 2010), which links Computer Algebra Systems (CAS) and dynamic geometry. The students worked in groups of two to solve problems involving modelling geometrical dependencies. Casyopée offers opportunities for students to experience variations and covariations of quantities and to decide, through appropriate feedback, if couples of corresponding covarying magnitudes can define functions. In this case, students can create a function and make sense of it using multiple integrated representations.

The notion of function occupies a central position both in school mathematics curricula and research in mathematics education. Existing research confirms the complexity of issues involved in students' conceptualisation of function (Thompson, 2011). A number of researchers described students' transition from a focus on actions and processes to a gradual focus on structure and vice-versa in terms of the distinction between the process view and the object view of functions (e.g., Sfard, 1991). A gradual consideration of this distinction as a dynamic interplay led, from the middle nineties, a number of approaches to emphasize the covariation aspect of function (Carlson et al., 2002; Thompson, 2011; Lagrange, 2010; Psycharis, in press). The essence of a covariation view is related to the understanding of the manner in which dependent and independent variables change as well as the coordination between these changes. According to Carlson et al. (2002), *covariational reasoning* consists of "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (ibid., p. 357).

Existing research on students' covariational reasoning indicated that coordination of two covarying quantities is embedded within an evolving/developing process in which

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students initially construct an image of the coordination of two quantities that gradually moves in the direction of the continuous coordination of both quantities (Saldanha & Thompson, 1998). Carlson et al. (2002) studied college-level students' ability to reason about two covarying quantities when interpreting and representing dynamic situations (e.g., a bottle filling with water) by constructing their graphs. The outcome of their research was a covariation framework with five levels characterizing students' engagement in making sense and representing functional relationships through corresponding mental actions: (a) coordination (coordinating the change of one variable with changes in the other variable); (b) direction (coordinating the direction of change of one variable with changes in the other variable); (c) quantitative coordination (coordinating the amount of change of one variable with changes in the other variable); (d) average rate (coordinating the average rate of change of the function with uniform changes in the independent variable); (e) instantaneous rate (coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function).

A critical point in students' conceptualisation of function as covariation is related to their ability to pass from quantitative reasoning - "based on quantities themselves and images of them that entail their values varying" (Thompson, 2011, p. 48) - to algebraic reasoning that involves describing the relation between two covarying quantities formally. This is a step far from trivial for the students. Lagrange (2014) indicated modelling dynamic situations with the use of specially designed digital tools (e.g., integrating geometrical and algebraic representations) as a context favoring students' transition from relations among quantities to mathematical functions. Based on the work done for the development of Casyopée, he described students' work with dependencies through a *modelling cycle* (ibid.) involving four settings: (a) a physical device allowing dependencies of items to be experienced by humans; (b) a dynamic figure modelling these (physical) dependencies into a digital tool (e.g., dynamic geometry); (c) magnitudes standing for measures of quantities independently of the unit in which they are measured; and (d) algebraic functions. In this approach, students' transition from experiencing dependencies in a physical system to the world of functions is expected to be mediated by their work with covarying magnitudes and the use of multiple representations such as formulas, graphs and tables. In the present study, we adopted this approach and we considered students' passage from physical dependencies and covarying quantities to mathematical functions as an abstraction process of meaning generation (described by the idea of *situated abstraction*, Noss & Hoyles, 1996) evident in students' identification and expression of relationships through the use of the available tools.

The general aim of this study is to shed light on students' conceptualisation of function as covariation, as they are engaged in solving modelling tasks involving geometrical dependencies with the use of concrete materials (e.g., manipulatives) and Casyopée. Our focus is on how the students used the available representations in order to attribute meaning to two covarying quantities from the level of physical dependencies to the level of magnitudes and mathematical functions. We were also interested in exploring the role of the available tools (digital and non-digital) in shaping students' activity towards more abstract conceptions of the relation between two covarying quantities.

THE DIGITAL ENVIRONMENT

Casyopée (Lagrange, 2010) is a digital environment that combines CAS (i.e. a symbolic window with registers: numeric, graphic and symbolic, Fig. 4) and dynamic geometry allowing students to treat functions using interconnected representations.

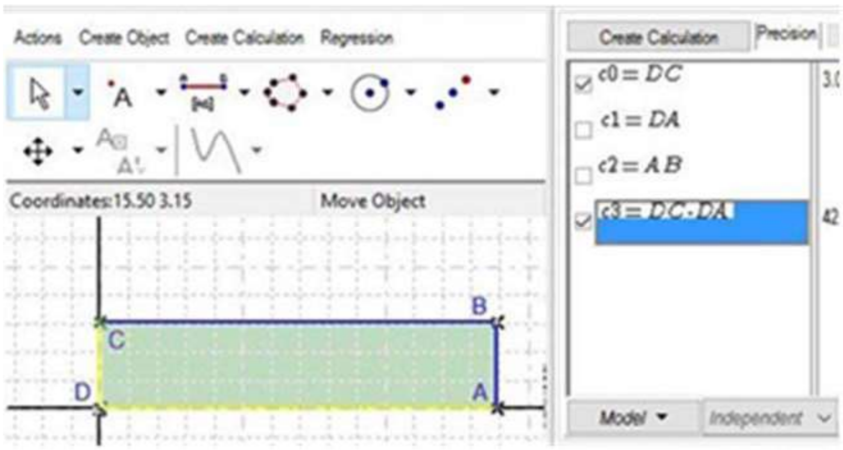

Fig. 1 Windows of dynamic geometry and geometric calculations in Casyopée

Students can observe covariation of quantities in the dynamic geometry window and define independent magnitudes (related to free points) and dependent ones (using the "geometric calculations" window, Fig. 1) involving distances (e.g., lengths), *x*-coordinates, *y*-coordinates, areas, etc. In Casyopée, magnitudes are labelled as *c0*, *c1*, *c2*,... in the geometric calculations window. The movement of a free point changes the measures of all magnitudes depending on it. Casyopée provides the opportunity to check if two covarying magnitudes (i.e. chosen one as independent and the other one as dependent) are in functional dependency or not through the "automatic modelling" functionality. In the former case, the system automatically exports the formula of the corresponding function to the symbolic window, while in the latter case an error message indicates that functional dependence is not possible. The new function can be treated by students using its formula, a table and a graph (Fig. 4).

METHODOLOGY

The research reported in this paper is the first part of an ongoing classroom-based design research (Cobb et al., 2003) aiming to study meaning generation for function as covariation by 16 year-old students, who work in groups with concrete materials (e.g. manipulatives) and Casyopée to model a series of dynamic real life situations. The experiment took place in a secondary school with one class of twenty 11th grade students (10 groups of two), one researcher who acted as teacher (called teacher in the paper) and another one who had the role of participant observer in the classroom. The class had totally 14 teaching sessions (45 min each one) over 3 months (one teaching session per week). At the time of the study, the students had been taught about function as correspondence (according to the curriculum), monotonicity and extreme points.

The activity sequence was divided in two phases and for each one of them we designed a series of tasks related to optimization problems. In the sequence of tasks, covariation appeared from simple to more complex situations. In the first phase (2 teaching sessions), after an introduction to main features of Casyopée (e.g., dependencies emerging by moving free points, definition of geometric calculations, automatic modelling) the students were asked to find the minimum distance of a point M on a parabola from a given point A. They had to create a function by defining the x-coordinate of point M and the length of AM as covarying magnitudes and use that function to solve the problem. In the second phase (12 teaching sessions), the students were engaged in modelling three realistic situations through the tasks: *Gutter*, *Front of a Store*, *Oil Tank*. In this paper, we analyze the first task and its implementation (4 teaching sessions) as well as the work of three groups of students (groups 1, 2, 3).

Fig. 2 Gutter design

In the *Gutter* task, the students were engaged in exploring the construction of a gutter model (Fig. 2) that allows the biggest amount of water passing over it. The task was divided in subtasks corresponding to students' activities on functional dependencies in the different settings of the modelling cycle outlined previously: (1) experiment with folding a piece of paper (10cm X 20cm) to explore the construction, notice if some quantities change together and express their relation formally with the use of one variable; (2) design a dynamic figure with one free point that models the situation in Casyopée and explore; (3) use the software tools to propose a function modelling the problem (e.g., create appropriate geometric calculations of magnitudes and test if functional dependency can be defined); (4) solve the problem using the available tools.

The data, which consisted of video and audio recordings, outcomes of students' work and screen captures software files, were analyzed under a data grounded approach (Strauss & Corbin, 1998). Through the analysis, episodes were selected: (a) to have a particular and characteristic bearing on the students' interaction with the available materials/tools; (b) to represent construction of meanings for function as covariation highlighting students' transition towards more abstract conceptions of the relation between two covarying quantities. Then, the selected episodes were categorized in themes indicating different aspects of the students' work with dependencies as well as their progressive passage from covarying quantities to mathematical functions.

MEANING GENERATION FOR FUNCTION AS COVARIATION

In this section, we provide an account of the emerging themes of episodes characterising students' activity throughout the implementation of the *Gutter* task.

Identifying dependencies in the paper model

Fig. 3 Experimentation with the paper model

In the first teaching session, the students were given a piece of paper (10cm X 20cm) and were asked to explore the creation of a gutter model (Fig. 2) that maximizes the amount of water passing over it. They experimented by folding the paper in different ways. For instance, they folded it in different points along its length (Fig. 3) and width and through this they recognized that the amount of water depends on the dimensions of the rectangular cross section of the gutter. Until the end of this teaching session the students conceived the interdependence of the dimensions of this rectangle and its area perceptually.

In the second teaching session, all groups were engaged in using algebraic notation to represent the length and width of the rectangle (cross section). Most of them used two variables *x* and *y* without linking them. Only students of group 1 and 2 in the class used one variable to express the dimensions of the cross section. Group 2 students recognized that folding the paper along its length (in three segments symbolized as *x*, 20-2*x*, *x*) gives a better solution than folding it along its width (symbolized as *x*, 10-2*x*, *x*) by comparing 20-2x and 10-2x, without providing a solution to the problem. Group 1 students noticed the interdependence of the covarying quantities (i.e. width of the cross section, cross-sectional area) and recognized that the optimization of the area leads to the solution of the problem.

To sum up, in the first two teaching sessions students' interaction with the given piece of paper allowed them to experience the problem sensually and through this to conceive the dependencies of the covarying quantities perceptually. Only two groups of students took a step further in identifying the interdependence of the covarying quantities, whose covariation leads to the function that models the problem.

Modelling dependencies in dynamic geometry

In the third teaching session, the students were asked to construct a dynamic figure in Casyopée modelling the problem (i.e. a rectangle with one dimension depending on the other) with one free point. Then, they were asked to define appropriate magnitudes as geometric calculations (e.g., the area of *ABCD*) so as explore further their covariation for solving the problem. Four groups of students in the class (including groups 1, 2, 3) completed the construction successfully in the dynamic geometry window. First of all, the students of these groups renamed the origin as point *D*, then, they constructed a free point *C* on the y-axis and represented the height of the gutter through the segment *CD* (Fig. 1). In some cases (e.g., group 1), the students did not take into account that point *A* was dependent on point *C* at the beginning of their work and constructed it as a fixed point. The teacher had to intervene indicating that the coordinates of the defined points (e.g., *xC*, *yC* in Casyopée) could be used for defining new ones. Finally, the students defined successfully all points and expressed the dependence of points *A*, *B* and *D* to the free point *C* through expressions such as *A*(20-2**CD*,0) or *A*(20-2*yC,0), *B*(*xA*, *yC*). Then, according to the task, the students were engaged in exploring further the given problem by creating magnitudes as geometric calculations in the corresponding window so as to observe how changes in the dynamic figure changed their numeric values. For instance, group 1 students created four magnitudes (i.e. c0=*DC*, c1=*DA*, c2=*AB* and c3=*DC***DA* in Fig. 1). Next, we provide an episode showing how students' conceptualisation of covariation was influenced when they engaged in manipulating the dynamic figure. The group 1 students started to observe how dragging of *C* changes the numeric values of *c0* (i.e. *DC*), *c1* (i.e. *DA*) and *c3* (i.e. area of *ABCD*) in the geometric calculation window. In the next excerpt, students were moving point *C* continuously and the teacher asked them to describe their observations.

415 R: [*To S1*] What do you observe by dragging the point *C*?

416 S1: I look at the cross sectional area [*i.e. c3*] to see when it [*i.e. the gutter*] has maximum capacity. I see that this seems to happen when the height [*i.e. DC*] is nearly half of the base [*i.e. DA*].

417 R: How did you find it?

418 S1: Here, we see that if we increase *DC* more than 5, the capacity of gutter decreases more and more. If we decrease it less than 5, the capacity decreases again. When it is equal to 5 or 4.9, the capacity seems to take its maximum value.

By continually moving the point *C*, S1 focused on the direction of change of *DC* (more or less than 5) and the corresponding changing of the area *ABCD*. Through coordinating changes in the figure with numeric changes of the corresponding magnitudes, she approaches dynamically the value of *DC* that seems to give the maximum value of the cross-sectional area. S1 appears not to be completely sure that the measure of the magnitude that maximizes the area is 5 or 4.9 since her dragging on point *C* is not stable. At the level of covariation, this episode indicates how the available tools supported students to link covariation of measures (in the dynamic geometry) and covariation of magnitudes (in the geometrical calculation window).

Conceptualising magnitudes as dependent and independent variables

After designing dynamic models of gutters in the dynamic geometry window and defining the covarying magnitudes as geometrical calculations in Casyopée, the students decided to explore the problem through the definition of a new function in the "automatic modelling" window (fourth teaching session). One challenge that emerged at this point concerned the selection of a dependent and an independent variable. Students' selection of variables was facilitated by their interaction with the dynamic figure (e.g., as in the episode presented above). For instance, group 3 students came to select *c0* (=*yC*) as an independent variable and *c2* (=area of *ABCD*) as a dependent one by referring to their preceding dragging of the free point *C*. Explaining this choice, one student (S6) said: "By moving point *C*, we concluded that what is [i.e. independent] variable is *C* and what it changes is the area [i.e. of *ABCD*]". Based on her dragging on point *C*, S6 seems to conceptualise the dynamically changing magnitudes *c0* and *c3* as a pair of independent-dependent variables. This is expressed by the students through a situated abstraction indicating their transition to conceptualizing covariation at the level of variables.

Conceptualising function as covariation by connecting different representations

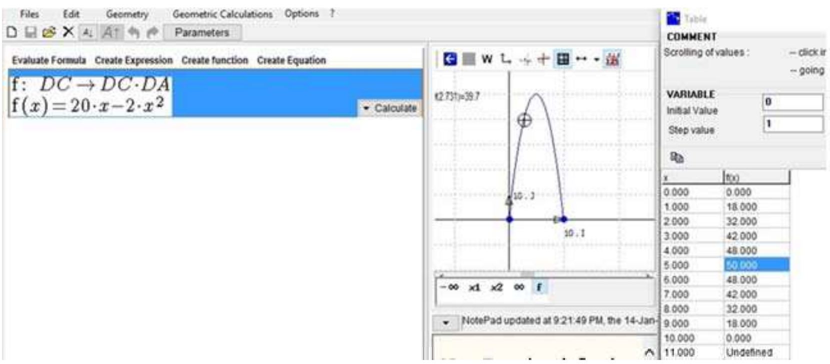

Fig. 4 Windows of Algebra, Graphs and Table

In the fourth teaching session, two groups of students (groups 1, 2) obtained a formula for a function modelling the situation ($f(x)=20x-x^2$, see Fig. 4) through "automatic modelling". This indicated their transition from working with covariation of measures and magnitudes to working with mathematical functions. The students' exploration of the problem at that time was characterized by the progressive coordination of different representations of function. In the next episode, group 2 students were able to interpret their answer to the problem by linking three representations of function in Casyopée: formula (symbolic window), table and graph (Fig. 4).

620 S3: We see in the table that the area is maximized when the coordinate of the free point *C* [*i.e. yC*] is 5. That is, we have the maximum area when one side [*i.e. of ABCD*] is half of the other.

621 S4: In the graph of the function, *x* is between 0 and 10. When *x* is equal to 5, the area is maximized.

622 S3: In the upper point of the graph, we have the greatest area … when we are in the maximum of the function.

Here, meaning generation is progressive. The students conceptualise function as covariation by observing the graph and the values of the two variables in the table and, at the same time, they refer to: (a) the *y*-coordinate of *C* (as independent variable); (b) the relation between the two sides of the dynamic rectangle modelling the problem; (c) the domain of the function; and (d) how the extreme point in the graph is related to the solution of the problem.

CONCLUSION

By analyzing students' activity, we could trace their conceptualisation of function as covariation throughout their engagement in working with a contextual task. The analysis was structured around four themes of episodes indicating students' progressive passage towards more abstract conceptions of the relation between two covarying quantities. Meaning generation in these themes of episodes involves: making sense of the interdependence of two covarying quantities sensually by modelling the problem with a piece of paper; making sense of the dependency between the free point (i.e. *C*) and the area (i.e. *ABCD*) influenced by its move; conceptualising the creation of relevant magnitudes as geometric calculations; linking covariation of measures (in the dynamic geometry) to covariation of magnitudes (in the geometrical calculation window); conceptualising two dynamically changing magnitudes as a pair of independent-dependent variables; using this pair to define a function through "automatic modelling"; and conceptualising function as covariation by connecting different representations. Besides, the analysis indicates the critical role of "geometric calculations" and "automatic modelling" in facilitating students' passage from the level of quantities and magnitudes to mathematical functions.

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