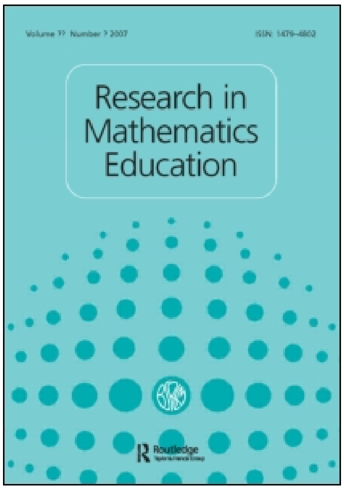

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Normalising geometrical figures: dynamic manipulation and construction of meanings for ratio and proportion

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Normalising geometrical figures: dynamic manipulation and construction of meanings for ratio and proportion

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Enlarging-shrinking geometrical figures by 13 year-olds is studied during the implementation of proportional geometric tasks in the classroom. Students worked in groups of two using 'Turtleworlds', a piece of geometrical construction software which combines symbolic notation, through a programming language, with dynamic manipulation of geometrical objects by dragging on sliders representing variable values. In this paper we study the students' *normalising* activity, as they use this kind of dynamic manipulation to modify 'buggy' geometrical figures while developing meanings for ratio and proportion. We describe students' normative actions in terms of four distinct *Dynamic Manipulation Schemes* (*Reconnaissance*, *Correlation*, *Testing*, *Verification*). We discuss the potential of dragging for mathematical insight in this particular computational environment, as well as the purposeful nature of the task which sets up possibilities for students to appreciate the utility of proportional relationships.

Keywords: enlarging-shrinking; dynamic manipulation; ratio and proportion; Turtleworlds; normalising

Introduction

The domain of ratio and proportion occupies a central role in the wide range of mathematical topics studied at both primary and secondary levels, and teaching ratio and proportion is known as an exceptionally difficult task. Many research studies provide considerable evidence which indicates that students perform poorly in proportion related tasks since they tend to solve them by following additive rather than multiplicative strategies (Hart 1981, 1984; Behr et al. 1992; Lamon 1993; Kaput and West 1994; Singh 2000).

In the present study we took into account Papert's view (1980) that learning environments based on the use of dynamic digital tools are much *richer* in opportunities for generating meanings. Taking a constructionist approach (Harel and Papert 1991), we intended to move in the direction of identifying tools and tasks to facilitate students' meaningful engagement in enlarging-shrinking geometrical figures. We focused on 13 year-olds' construction of meanings for ratio and proportion emerging *in activity*, in the sense of Papert (1980), that involved the identification of linear functional relationships (Y = mX), the cornerstone of proportional reasoning (Karplus, Pulos and Stage 1983), for enlarging-shrinking geometrical figures with the use of specially designed computational tools. The

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students worked in groups of two using 'Turtleworlds'[1], a piece of geometrical construction software which combines symbolic notation, through a programming language (Logo), with dynamic manipulation of variable values (Kynigos 2004). The students had to connect formal and graphical descriptions of geometrical figures continually, and through manipulating variable segments or angles, to appreciate the inappropriateness of additive strategies. In the spirit of constructionism (Harel and Papert 1991), these figures were their own productions and would be used by others later.

In this paper we focus on how students manipulated geometrical figures dynamically by manipulating the values of variables defined through Logo procedures. We discuss students' use of *normalising*, similarly to the sense of Ainley, Pratt and Nardi (2001), in an activity in which students 'correct' abnormalities in distorted geometrical figures through the use of the available tools. Normalising emerged as an integral part of students' exploration of graphical feedback, revolving around the dynamic manipulation of the geometrical figures. The analysis highlights how normalising facilitated students' gradual focus on relationships and/or dependencies between objects and representations, and the emergence of mathematical meanings for ratio and proportion.

Theoretical framework

Ratio and proportion as a conceptual field for meaning-making

The component of proportional reasoning that is critical for the study of ratio and proportion is the development of multiplicative thinking, characterised by the ability to understand the multiplicative relationship underlying the comparison of two quantities (Behr et al. 1992). As suggested by Vergnaud (1983), ratio and proportion should be considered as being at the core of the *multiplicative conceptual field* which integrates several proportion-related concepts such as multiplication, division, fractions and linear functions, various situations in which they may be used, and the available representations of these concepts and situations.

Vergnaud (1983) cites three subtypes of multiplicative structures within the multiplicative conceptual field. One subtype, the isomorphism of measures, involves direct proportions between two measure spaces (M1 and M2) and characterises many situations, including enlarging-shrinking geometrical figures. In the enlargement-shrinking context, students work with quantities corresponding to side measures of two geometrical figures: an initial figure, which is to be enlarged or shrunk, and the final figure. The respective sides of the initial and the final figure are considered as belonging to the same measure space. The construction of a similar model of a rectangle, for instance, requires students to use quotients of the same measure space (i.e. ratios of measures within each geometrical figure which have to be equal, defining *scalar strategies*), or quotients of different measure spaces (i.e. ratios of measures across the two geometrical figures, defining *functional strategies*). If the initial rectangle has length $=50$ and height $=30$, and the final one has length $=40$, then, to find the height of the final rectangle, we need to multiply 30 with the quotient 40/50 (*scalar operator*) or to multiply 40 with the quotient 30/50 (*functional operator*). The problem can also be solved via algebraic means through the proportion $50/30=40/x$, known as the 'equationising' proportion approach

(Post, Behr and Lesh 1988). If the final length is x, then the final height is 30x/50. This expression provides an enlarging-shrinking model of the initial rectangle (i.e. for any value of final length x). The focus in this approach is on identifying an appropriate algebraic relationship. This approach was the one mainly employed by the students in the present study.

From an epistemological point of view, in the context of the present study, we identified the following further elements in the concept of ratio and proportion evident in the process of enlarging-shrinking geometrical figures:

- a potential for identifying a mutual dependence between two comparable quantities (interdependency);
- a potential for identifying certain kinds of dependence (e.g. additive, multiplicative) between two comparable quantities (correlation);
- a potential for identifying a multiplicative functional relationship between two comparable quantities (co-variation).

By focusing on ratio and proportion in the geometrical enlargement context, we also gained insights about students' understandings and difficulties from relevant studies in the school or laboratory setting.

Research reporting school or laboratory studies of students' understandings of and difficulties with the concept of enlargement-shrinking has shown that, regardless of context or numerical type of scalar/functional operators (e.g. integers, fractions) (Kuchemann 1989), students can hardly identify a ratio relationship and they employ mostly additive strategies in enlargement settings (Hart 1981, 1984; Clarkson 1989).

Some explanations of these poor levels of performance have highlighted four areas of difficulty:

- students often enlarge the sides of the original figure by addition (i.e. *replicative* growth as opposed to multiplicative growth, Kaput and West 1994) but still producing the same kind of figure (e.g. a rectangle);
- the geometrical and the arithmetic/algebraic aspects appear rather fragmented while solving enlargement tasks, i.e. most students ignore that the resulting enlargement should be the same shape as the original because of being so intensely focused on the method to be used and the arithmetic calculations (Hart 1981);
- the use of functional strategies requires students to find quotients across measures, producing a new quantity that has no direct relationship with the original quantities but rather express a relationship between the two (i.e. the quotient length/height is an absolute number) (Vergnaud 1980);
- the dynamic nature of the enlarging-shrinking process cannot be represented with the use of static representational means (Psycharis and Kynigos 2004).

Enlarging-shrinking geometrical constructions with computational tools

Over the years researchers have explored the conjecture that particular computer-based pedagogical settings that involve linked visual, numerical and symbolic representations of geometrical objects might provide students opportunities to develop a sophisticated comprehension of proportional ideas in the context of geometrical enlargement (Hoyles, Noss and Sutherland 1989; Hoyles and Noss

1989). In particular, Hoyles and Noss discuss the qualitatively different kinds of work facilitated within such a computational setting in enlargement tasks by focusing on the child-tool interaction, which

> is built on the synthesis between the child's need to formalise the relationship algebraically (i.e. to type a programme) and to receive confirmation of intuitions (i.e. to perceive the intended geometrical effect on the screen). (1989, 65).

In resonance with this idea, in this study we aim to highlight the role of dynamic manipulation in enlarging-shrinking geometrical figures, a facility which was not available to students in the previous research studies.

Over the last decade, manipulation of graphically represented geometrical objects has attracted the interest of mathematics educators as a means to make mathematical objects more concrete and 'tangible' for students. Research has mainly been concerned with dynamic geometry software (DGS) environments. In these environments, manipulation can be characterised as dynamic since it is realised through dragging actions, offering the ability to change the constructed figures by modifying some of their features while complying with specific mathematical rules. Some researchers have considered dragging as an instrument of mediation between the perceptual level of figures on the screen and the conceptual control on them (Hölzl 1996), while others have confirmed its crucial role in supporting students towards developing deductive explanations when they encounter unexpected graphical results (Hadas, Hershkowitz and Schwarz 2000).

In our approach dragging on sliders – which represent variables involved in a Logo procedure – produces DGS-style continuous change in the figure constructed by the respective procedure. Dragging thus appears as a coherent part of a computer environment which integrates symbolic and graphical representations of geometrical objects. In our view, it is by acting on different types of representations as well as exploring the connections between them that the learners engage in developing meanings as abstractions emerging in activity (i.e. *situated abstractions*, see Noss and Hoyles 1996). We also paid particular attention to the instrumental aspect of the activity, namely the process through which the tool was appropriated and integrated into the students' practice and eventually transformed into an *instrument* (Guin and Trouche 1999). This seemed appropriate in the present study which aimed to engage students in interacting with different types of interconnected computational representations in order to identify proportional relationships. In that sense we took into account the different instruments created by the students through the development of relevant *schemes* of tool use. Our approach to the notion of scheme initially follows recent elaborations on the Piagetian theory (as previously elaborated by Vergnaud) in the case of technological tools for mathematics. According to these elaborations, a scheme is considered as a mental structure which involves the technical skills for using the tool in an efficient way to complete specific tasks, the development of strategies, and the mathematical concepts that underpin these strategies (Drijvers and Trouche 2008). In this view, schemes indicate the dialectic relationship between activity and implicit mathematical knowledge that a learner operationalises when using the tool purposefully to carry out a task (Artigue 2002).

From a constructionist theoretical perspective in the present study we considered schemes not as ready made mental structures that students come to discover or recognise ("out there") in some way. Rather, we considered schemes as emerging in students' active construction of meanings which are contingent on the available computational tools in their specific context of use. Our research aim in this study was to explore how students used the dynamic manipulation feature of the environment (we call this *variation tool*) during their ongoing experimentation with enlarging-shrinking geometrical figures. We were interested in the students' experimentation at two levels. First, we were interested to raise students' awareness of ratio and proportion as they were working within the respective conceptual field. Our aim was to explore further, through specially designed tasks, and within an appropriately developed computational medium, if and how enlarging-shrinking geometrical figures could be approached by the students in meaningful ways. Secondly, we were interested in shedding light on what dragging entails for mathematical insight in the context of enlarging-shrinking geometrical figures. For that reason we chose to investigate the dragging schemes developed by the students by focusing on the students' construction of meanings for particular aspects of ratio and proportion (i.e. interdependency, correlation and co-variation).

The computer environment

Description

Turtleworlds is a microworld designed to integrate formal mathematical notation with dynamic manipulation of variable values (Kynigos 2004). This is achieved by means of writing variable procedures to construct geometrical figures and dragging on sliders representing variables to observe continuous change in the figures. The main part of Turtleworlds consists of the following three interrelated components:

- *Logo Editor.* This is the symbolic Logo-like programming interface. The user can write, run and edit Logo procedures to 'drive' a 'turtle' (an entity defined by its position and heading) to construct geometrical figures.
- *Canvas.* This is the graphic representation modality of the software in which geometrical figures are designed as results of commands or procedures executed in Logo Editor.
- *Variation tool.* This is the dynamic manipulation feature of the computer environment, providing 'number line'-like representations of variables in the form of sliders. The user can drag a specific pointer along a number line – corresponding to a specific variable – so as to manipulate the geometrical objects, either segments or turns, represented with the use of this variable.

The variation tool appears through the following steps: (1) A Logo procedure with at least one variable has to be executed using specific values for each one of the variables involved in the procedure. (2) Clicking on the turtle's trace that appears on the Canvas after executing the Logo procedure causes the variation tool to activate. For each one of the variables involved in the Logo procedure the variation tool provides a separate slider (see the bottom of Figure 1). For instance, the procedure 'letterA' in Figure 1 creates a model of the letter 'A' with three variables: the variable :x represents the whole length of the 'tilted' segments, the variable :y represents the length on the 'tilted' segments from the base to the edges of the horizontal segment, and the variable :z represents the horizontal segment. The variation tool in Figure 1 appeared after executing the procedure 'letter A' with the values 75, 30 and 37 (corresponding to the

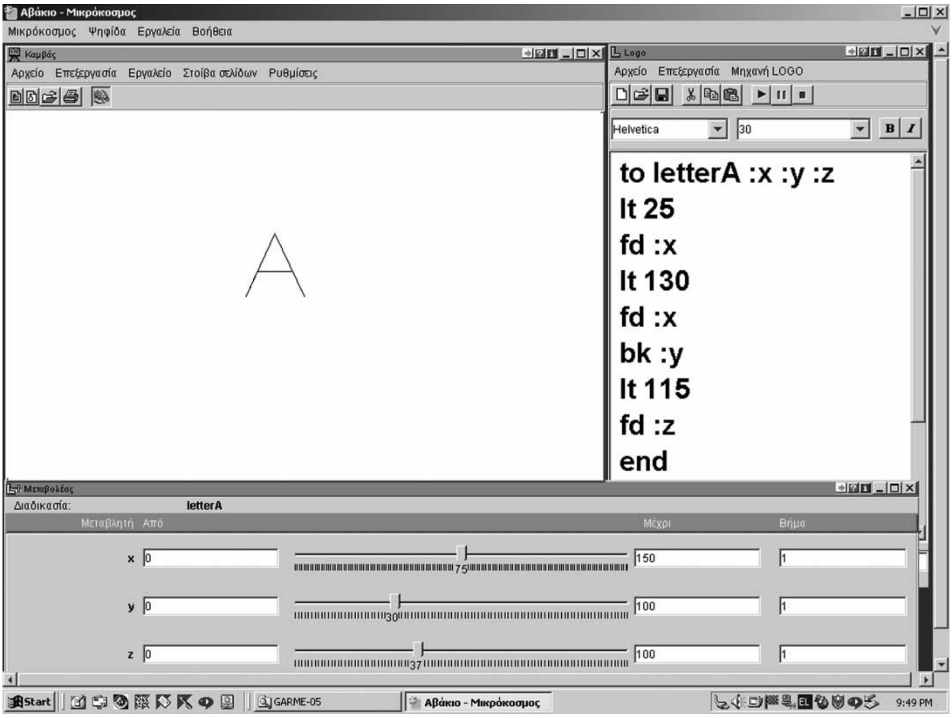

Figure 1. A model of 'A' with three variables.

variables :x, :y and :z respectively) and clicking on the resultant model of 'A' which was visualised on the screen. The dragging of a slider results in a continual reshaping of the figure according to the corresponding variable value. The user can change in each slider the initial value, the end value and the step of the variation (these numbers are shown in Figure 1 in the small boxes beside the sliders).

The novel character of the dynamic manipulation in Turtleworlds is mainly due to the fact that what is manipulated is not the figure itself – as for example in DGS – but the value of a variable of a procedure by dragging on the sliders provided on the variation tool. Dragging thus affects both the graphics and the symbolic expression through which it has been defined, combining in that sense these two kinds of representations which appear rather static in most other enlarging geometrical settings. The graphics, the variation tool and the Logo editor are all available on the screen at all times.

Turtleworlds as a microworld for the study of ratio and proportion in enlarging-shrinking geometrical constructions

Variables and functional relations can be used for the expression of the lengths of the segments of a geometrical figure. This is at the core of the idea of enlarging-shrinking it if appropriate proportional relations with the same independent variable are built within the Logo code. For instance, the procedure for drawing the enlarging-shrinking model of a letter with one variable can be derived through the functional relationships of the only variable to the ratios of the sides of a fixed model of the letter. For enlarging-shrinking the fixed model of 'A' visualised in Figure 1, we can consider :x as independent variable and express variables :y and :z through the functional relationships (:x)·30/75 and (:x)·37/75 respectively. With this perspective, some specific features of the provided representations in Turtleworlds may offer students opportunities to further appreciate their use in the construction of enlarging-shrinking geometrical figures.

In relation to the Logo Editor, the syntax of the Logo code can be considered as a means to express and reflect upon mathematical objects and relationships underlying the construction of proportional geometrical figures. Logo procedures are not only dynamic in the sense that basic elements of a geometrical construction can be represented with the use of variables and relations, but they can also be considered as a means for further reflection since they can be modified by the user at different stages of exploration.

Instead of viewing only the end-result of a construction process, the variation tool in Turtleworlds allows students to capture the change of a graphical object as particular strategies unfold. We found this to be particularly useful in studying proportional relations in the geometrical context since the result of the process, an enlarged-shrunk geometrical figure, can only be verified by checking what happens to a figure when the respective Logo procedure is executed. The graphical representational modality in the present study was thus aimed at approaching the possibility of visualising the geometrical construction during its elaboration by students. The visualisation of failures and mistakes, in particular, was expected to challenge students to validate the graphical outcome each time, and question/modify their strategies accordingly.

The variation tool provides to the user opportunities to manipulate geometrical objects dynamically and explore these objects and their relationships. For instance, the inclusion of an additive relationship in a procedure would result in a 'distorted' figure. Thus, employment of the variation tool provides opportunities for the students to take a reflective view on the enlarging-shrinking process and be directly engaged with its dynamic character. This engagement often consists of confirming or dismissing conjectures derived from the use of the tool in conjunction with the other representations (e.g. by adding or erasing commands or variables, by expressing or modifying functional relations between variables etc.).

Methodology

Our research approach is informed by the 'design experiment' approach (Cobb et al. 2003). This approach aims to address the complexity of domain-specific learning processes by specifying "successive patterns in students' reasoning together with the substantiated means by which the emergence of those successive patterns can be supported." (ibid. 9).

Task design

The constructionist theoretical framework that underlies our study describes designing a task that offers a context for investigating students' encounters with powerful mathematical ideas by providing them with opportunities for working with concepts before being introduced to them formally (Papert 1996). In contrast with conventional approaches, in such tasks understanding is expected to emerge through activity within the classroom during which the students construct a *purpose* for their activity. They engage with mathematical ideas to produce a meaningful outcome, and simultaneously appreciate the *utility* of the respective mathematical ideas, the why and how these ideas are useful (Ainley, Pratt and Hansen 2006).

The task – called *Dynamic Alphabet* – was designed to engage a class in constructing enlarging-shrinking models of all the capital letters (i.e. of variable sizes) so that they could be used for writing words and phrases. Students were informed that the letters would be used, by themselves and by other students in a subsequent classroom activity, to construct 'dynamic posters' in which particular words or phrases had to change size in the same way according to their height when written next to other. Moving the slider of the variation tool in this case would result in the visualisation of the letter as an enlarging-shrinking geometrical figure. In formal mathematical terms this means that each letter procedure had to contain only one variable, so all of its varying lengths would be expressed with appropriate functional relationships according to this variable. Students, however, were simply asked to design letters having the same height, as if they were designed between two lines of a note book. We intended to see if and how students might come to 'translate' these constraints in formal mathematical notation through their interaction with the available tools.

The *Dynamic Alphabet* task was designed to enhance students' purposeful engagement in the process of constructing enlarging-shrinking models of capital letters. At one level, the purposeful nature of the task is related to the fact that letters can be considered as providing an example in which the similarity of the different sizes of the same letter is a basic rule underlying any font (e.g. in a word processor). At a second level, purpose is generated by the request to construct letters which remain 'robust' under the enlarging-shrinking dragging on the only slider of the variation tool. The fact that the enlarging-shrinking letters would subsequently be used by other students further provokes a shift in purpose to identifying a general 'method' to prevent distortions (e.g. caused by the use of non-proportional relations). Our general aim was to utilise the functionalities of the computer environment and the feedback it can provide, so as to prompt students to construct relationships and figures according to the rules of proportionality. We stress that these rules were not initially explicit to the students, the aim being that they experience visually-based cognitive conflict, particularly when using additive strategies.

Context, participants and implementation

The study was carried out in a Greek secondary school with two 7th grade mixed-ability classes (first grade of the secondary level) with 26 13-year-old students in each class. One researcher, the first author, worked in close collaboration with two mathematics teachers as a participant observer for a period of about two months before the implementation of the classroom activities. Collaboration consisted of weekly meetings during which the *Dynamic Alphabet* task implementation was configured (e.g. in terms of the mathematical content involved, time restrictions etc.). Arrangements were made for the research to take place before the concepts of shrinking and enlarging geometrical figures were officially introduced to the students. This was done in order to avoid students' attempts to reproduce taught methods and algorithms and, also, in order to enhance the exploratory potential of their engagement with the task.

At the time of the study, the textbook content that the students had covered included chapters on arithmetic (e.g. a review of basic arithmetic operations, fractions, percentages). They had also had some experience with traditional Logo constructions, including variable procedures. At the beginning of the study the students were introduced to the use of the variation tool. We divided the implementation of the task into two successive phases: in the first phase each group of students was asked to construct two letters; in the second phase the groups were asked to interchange their constructions so as to check and correct other students' work. Problems stemming from the use of different variables by different groups were left as a point of interaction among students and teachers.

Data collection

Each of the two classes had a total of 16 teaching sessions – 2 sessions per week – with the participating teachers over two months. During the implementation another researcher participated in the classroom activity, mainly in order to handle the observation recording equipment (two video-cameras and two microphones). We focused on one group of students in each class (focus groups) and on the classroom as a whole, recording overall classroom activity. On the basis of previous assessments, the ability of the focus groups students had been reported as average by their teachers. During data collection one camera and one wired microphone were on the groups of students who were our focus. A second wireless microphone was attached to the teachers, capturing their interactions with all groups of students. The co-researcher was occasionally moving the second camera to capture the overall classroom activity as well as interesting interactions in student groups other that the focus group. As participant observer, the first researcher intervened, more or less assuming the role of a teacher, by posing questions, encouraging students to explain their ideas and strategies and asking for refinement and revision when appropriate. Background data (observation notes, students' electronic and written work) was also collected. Verbatim transcriptions of all audio and video recordings were produced for the analysis.

Distortion of geometrical figures as the framework of the analysis

The first level of analysis involved reviewing video and audio recordings in conjunction with observation notes and students' work so as to produce extended narrative accounts of the work of all groups throughout the teaching sequence. The accounts revolved primarily around two axes: evolution of classroom mathematical practice and tool use by individual students. Constant comparative analysis of these accounts (Strauss and Corbin 1998) revealed the significance of the students' use of the variation tool, and in particular it highlighted different levels in the students' – mostly unsuccessful – attempts to identify the role of variables and their specific values in order to express proportional relationships in formal notation. The fact that these attempts were unsuccessful was highlighted by the graphical distortions in the figures. The visualised distortions, which usually appeared as a result of dragging on the variation tool, challenged students to engage in corrective processes.

The second level of analysis focused on the students' use of the variation tool to correct graphical distortions on their geometrical constructions. This corrective action emerged in their attempts to determine relations among the relevant quantities or to confirm – through trial and error – that the enlarging-shrinking mechanism actually 'works'. We adopted an analytic stance, integrating conditions (*why* the geometrical figures were distorted) with interactions (*how* students responded to the distortions in the figures, i.e. what were the subsequent actions taken by the students) (Strauss and Corbin 1998). We 'read' each dragging on the variation tool as an incident directly linked to 'before' (cause) and 'after' (result). The unit of analysis was the episode, defined as an extract of actions and interactions developed in a continuous period of time around a particular issue. A first strand of analysis involved extraction of episodes based on the following criteria: (a) the 'initial motive' of the dragging, which mostly concerned distortions to the figural representations, (b) the students' 'focal point' while dragging, which we identified in what they said and did and (c) the 'chain of proportional meanings', which accompanied the students' actions while or after dragging. A second strand of analysis focused on the identification of patterns in the episodes, categories for which emerged through scrutiny of the data. The iterative definition and refinement of these categories revealed four distinct Dynamic Manipulation Schemes (DMS) which describe students' normative actions using the variation tool to identify and understand proportional relationships throughout their experimentation.

Normalising

We use the term *normalising* to describe activity during which the students, not accepting the shape of the letter under construction, attempted to modify this shape towards an acceptable shape. Early in their work most of the students constructed a model of their letter – which we refer to as *the original pattern* – sometimes without using any variables. In subsequent phases of their exploration, students experimented with the use of variables for all of its segments, to change it proportionally, until they built their final model with one variable. The dragging on the one and only slider of the variation tool either confirmed or refuted the proportional enlarging-shrinking of the geometrical figure for all values of the respective variable.

Dragging on the variation tool thus emerged as an inevitable part of a student's actions and is at the heart of our account of the following four DMSs:

i. *Reconnaissance DMS*, for recognising the interdependence of variables;
ii. *Correlation DMS*, characterising students' attempts to correlate variables;
iii. *Testing DMS*, characterising students' control of the figure and the specification of multiplicative relations;
iv. *Verification DMS*, indicating students' engagement in verifying proportional strategies.

It is important to note that schemes, as mental constructs, cannot be directly observed. Since our observations were limited to what students did and said, the above DMSs are merely our own reconstructions of student actions and utterances (for a discussion on similar issues see also Drijvers and Trouche 2008).

Reconnaissance DMS: visual manipulation of the interdependence of variables

During the early phases of the experimentation the students seemed fascinated by the visual impact of moving the sliders of the variation tool at random. They observed the continuing changes in the corresponding geometrical objects on the screen which were mainly related to the creation of distortions to the figure under construction. Their respective normalising actions were associated with noticing the interdependence of the variable lengths of the geometrical figure under construction. This is what we call *the reconnaissance DMS* of the variation tool.

In the following extract Yiannis and Nicolas (focus group – Class A1) have created a model of the letter 'A', using three variables as shown in Figure 1. Early in their work they had constructed the displayed figure as the original pattern of the letter without using any variables. In the next stages of their exploration, they tried to change it proportionally. The three sliders were set to the values of the original pattern as displayed at the bottom of the screen (:x = 75, :y = 30 and :z = 37) when Yiannis started to move the slider of :x for the first time. The figure was distorted (Figure 2).

Yiannis: [*While moving slider :x he is showing the 'tilted' segments of 'A'*] These segments are increasing.

[*The figure is distorted*]

Nicolas: [*To Yiannis*] Change it [*i.e. the slider :x*] to 150.

[*Yiannis puts the slider :x to the value 150. The figure is still distorted*]

Yiannis: Ok, but this little line [*i.e. the horizontal line*] is now too low. I also have to increase this one [i.e. the :y variable].

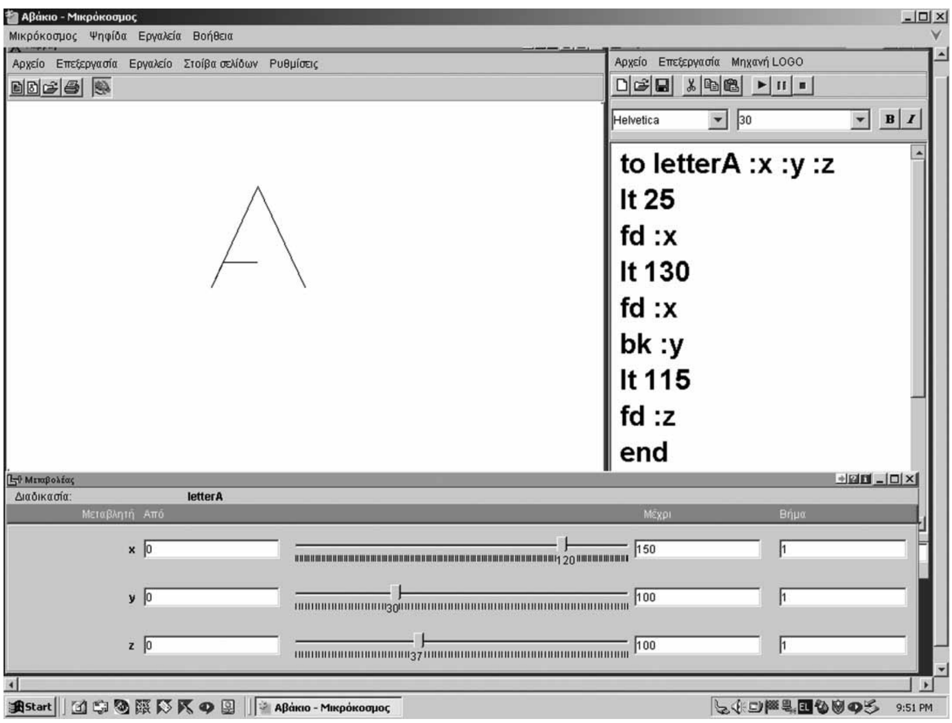

Figure 2. The distortion of 'A'.

[*Yiannis drags the slider :y to a higher value, thus pulling up the horizontal segment. Then he drags the slider :z until the horizontal segment fits the right 'tilted' segment*].

The distortion of the figure in this case led students to move all the other sliders of the variation tool to higher values so as to design an acceptable – bigger than the initial one – model of letter 'A' which, in turn, emerged from the need to 'close' the shape. Although Nicolas' suggestion of 150 being twice as much as the initial value of :x may be an indication of a proportional prediction for the values of the other variables, Yiannis did not change them proportionally. In this phase students seemed to give priority to completing the figure, motivated by the visual outcome on the screen and not paying attention to any relationship between the selected values. Thus, the resulting figure was not similar to the initial one. However, at an intuitive level, the students certainly started to suggest, in their articulations about the figure, the interdependence of the variable lengths involved.

Correlation DMS: graphical and numerical control of the similarity ratio

Evidence of what we call the *correlation DMS* at first seemed to be another instance of the *reconnaissance DMS* emerging during student's transition from the construction of the original pattern to the dynamically changing constructions with the use of variables. However, further consideration showed that students were not simply using the variation tool to complete the shape of a letter in response to the visual feedback, as in the *reconnaissance DMS* instances. Rather, there was a partnership evolving, with the variation tool assigned a defined role in their attempts to identify the kind of dependence between the comparable lengths and distinguish some relations or invariant properties, so that they could create similar models of their original letters in different sizes.

In a 'P' construction (Group 9 – Class A2, Vassilis and Costas), the correlation dragging of the two sliders took its meaning via the equivalence of the ratios of the two variables involved in the construction. In the original pattern (:x =400, :y =2), the students considered that the semicircle coincided with the middle point of the vertical segment. Experimenting to construct similar 'P' models of different sizes, Vassilis had the idea of setting, as end value for each slider, the corresponding values in the original pattern. Initial trial and error use of the variation tool confirmed that the figure was distorted during unsystematic dragging. He then constructed a (similar) figure of 'P' so as to preserve the property 'intersection in the middle' by dragging the two sliders to half of the values in the original pattern (:x =200, :y =1) that corresponded to their middle points (see in Figure 3 the position of the pointer in each slider).

Vassilis: When this [*i.e. slider :x*] is set to 200, this means it is in the middle. Because it is a half of 400.
Researcher: And how do you know that the semicircle is in the middle?
Vassilis: We'll also place this [*i.e. the slider :y*] in the middle. It starts from 0, until 2. So we'll place it exactly in 1.

The researcher's remark about whether the 'intersection in the middle' is preserved in different-sized models of the letter triggered students' actions to normalise the visual outcome, aiming at constructing other models of 'P' in different sizes with the same

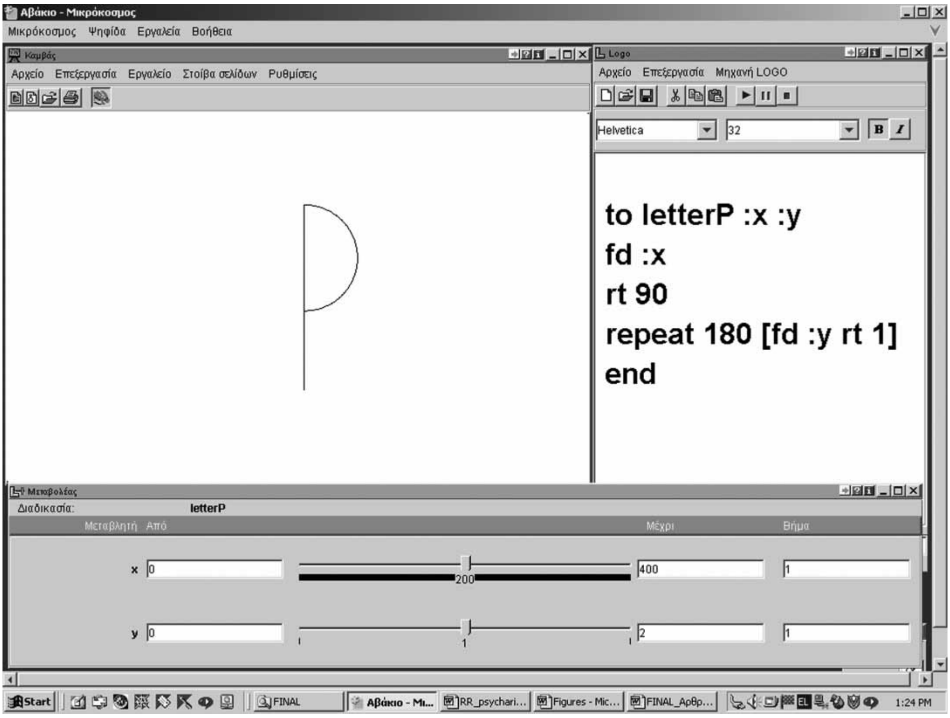

Figure 3. 'P' with two variables.

property. Vassilis appeared to use a sense of proportionality (scalar proportional correlation among the two variables) to express how this specific geometrical property would be preserved in other models of 'P'. The evolution of this idea to predict a functional relationship which can be used as a norm for enlarging-shrinking the figures will be discussed in the following paragraphs. Here we focus on the meanings generated by Vassilis *through* the use of the tool. Vassilis' manipulation of the variation tool reflects the purposeful way in which the computational setting provided a *web of structures* (Noss and Hoyles 1996), which students could exploit in shaping the available resources to satisfy an emerging (scalar) proportional rule, i.e. the appreciation of a scalar correlation within the arithmetic values of each variable represented by the two sliders.

Testing DMS: articulating multiplicative relationships between variables

The *testing DMS* incidents were characterised by qualitative transformations in students' activity, involving the expression of functional relationships in formal notation. The students then tested whether the enlarging-shrinking mechanism actually 'works'.

In an 'N' construction, trials of dragging on the variation tool led the students to articulate in formal notation and further elaborate the multiplicative relation between the 'tilted' and the vertical side of the construction. Initially, Christina and Alexia (focus group – Class A2) realised that the use of an additive algebraic expression constituted an erroneous strategy, as confirmed by the graphical distortion of the figure for certain arithmetic variable values. Without pursuing this further, the students attempted to test the multiplicative correlation, which emerged as a 'translation' into formal notation of the situated abstraction "the tilted one is one and a half times the other", posited by Alexia. Alexia based her prediction on two specific numerical values of the variables :r and :y which were initially used by the students to symbolise the two construction lengths. The students had constructed a model of 'N' for the values :r = 100 and :y = 145, and the correlation of these values led Alexia to express the tilted length with the functional relationship 1.5·(:r). This seemed to have created a basis for the students to further elaborate this relationship according to the graphical feedback resulting from the use of the variation tool.

Dragging the only slider :r, students realised that the side length – equal to 1.5·(:r) – did not exactly coincide with the horizontal line. In response to this, they drew a line at the letter base so as to evaluate the accuracy of their method under the subsequent experimentation (Figure 4).

Alexia:	It is exactly the same, or even worse [*i.e. the distortion of the figure*].
Christina:	Therefore, this [*i.e. the function operator*] is probably not 1.5 times …
Alexia:	Yes, it [i.e. *the function operator*] may be 1.45. [*Christina replaces in the procedure 1.5 by 1.45 and moves the only slider so as to test the new value*].

Dragging on the variation tool here obviously reflects the students' attempt to coordinate the specification of a non-integer function operator with the ongoing normalising of the enlarging-shrinking model of 'N'. At the same time a shift is indicated in their use of the variation tool for validating specific relationships described in the symbolic expression. The abnormality of the graphical outcome led them to use the tool in such a way that dragging, in conjunction with the symbolic notation, helped them to extend the elaboration of the proportional relation between the covariant magnitudes so as to prevent the distortion of the shape. We suggest that the students moved their focal point from the process of replacing specific numerical values on the variation tool to developing criteria for taking control of whichever manipulations in the symbolic expression (e.g. 'approximations') it was useful to perform so as to 'improve' the visual outcome.

Verification DMS: forming and using multiplicative relationships to verify proportional strategies

Incidents of the *verification DMS* concerned the verification that a proportional strategy was appropriate for enlarging-shrinking particular models of letters. In other words, this DMS can be considered as the most sophisticated form of the testing DMSs, as it indicates a more complex use of normalising associated with the implementation of proportional strategies.

The exact calculation of the functional operators involved in the relationship between two variables was the most difficult aspect of the enlarging-shrinking mechanism, especially in cases involving arithmetic values resulting in non-integer ratios. In several cases, forming such relationships was facilitated by the fact that students had already confirmed the validity of multiplicative correlations between variables for enlarging-shrinking particular geometrical figures based on the correlation of specific values of these variables that involved integer quotients.

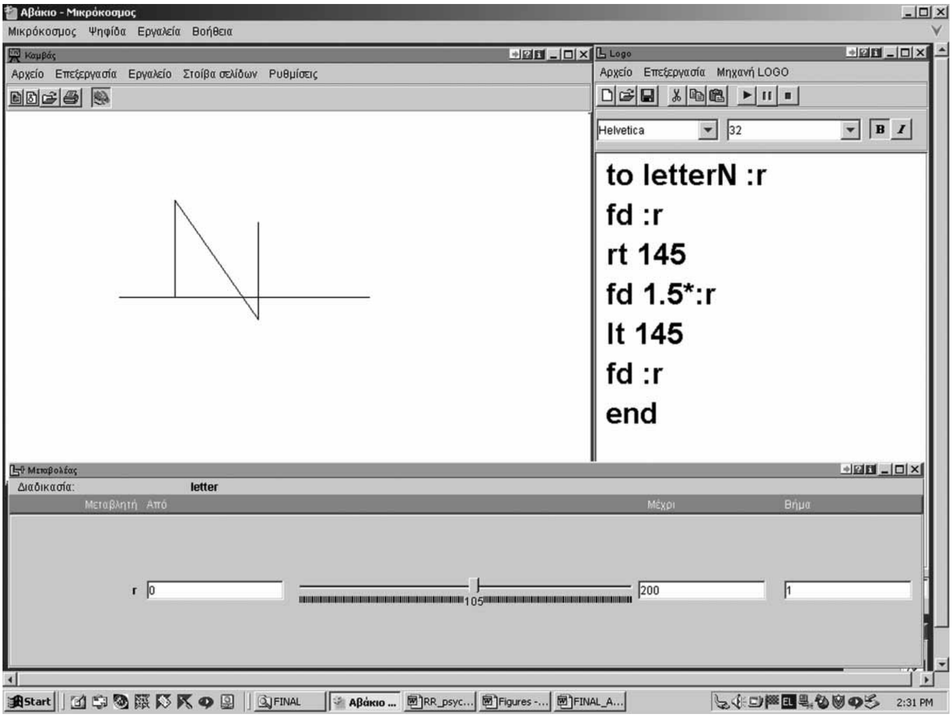

Figure 4. 'N' with one variable.

For the construction of an enlarging-shrinking model of 'B', Alexia and Christina (focus group-A2) chose to employ an already developed multiplicative strategy including integer quotients, that they had applied successfully in the construction of the Greek letter 'Ξ'. In the first phase of the implementation the students had successfully constructed an enlarging-shrinking model of this letter by considering it as 'framed' inside a square. Thus, they used variable :x – corresponding to the height of the letter – for symbolising the equal parallel segments, and the expression (:x)/2 for the middle segment which emerged through the division of the constant values 100 and 50 used for the corresponding segments in the original pattern of the letter. The code of the letter 'B' (Table 1) with two variables was given to Alexia and Christina by another group of students when they exchanged their constructions during the second phase of implementation. These students, based on their procedure for 'B' with two variables (Table 1), had constructed their original pattern for the values :x = 100 and :y = 0.44, but failed to identify the appropriate proportional relation between the two variables for enlarging-shrinking it. For the final enlarging-shrinking model of the letter with one variable, Alexia and Christina substituted variable :y with the expression (:x)/227.3, since they decided to round off the result of the division 100:0.44 = 227.272727272 ...

[*Alexia drags the only slider :x for enlarging and shrinking the letter.*]
Alexia: [*To the researcher*] You see?
Researcher: How did you achieve this?
Christina: We divided 100 by 0.44 and got 227.3.

Table 1. The initial procedure of "B".

```
to letterB :x :y
fd :x
rt 90
repeat 180 [fd :y rt 1]
lt 180
repeat 180 [fd :y rt 1]
end
```

By dragging the only slider, Alexia verifies the successful outcome of the multiplicative construction strategy, implying that it can also be followed in cases including non-integer ratios. In that sense, this specific dragging signals the use of the variation tool as an instrument mediating strategies based on properties and relations rather than on arithmetic values of a particular type. We also see this as a clear indication that the students have appreciated the usefulness of proportional relations for enlarging-shrinking geometrical figures.

Summary and concluding remarks

We have illustrated our analysis in terms of four different DMSs that became evident as students used the variation tool in their ongoing experimentation to avoid graphical abnormalities, and achieve the enlarging-shrinking of geometrical figures by means of relations abstracted, i.e. constructed and expressed within this particular computational setting. From a constructionist perspective, it is important to note that these DMSs do not form a strict developmental sequence followed by all groups in the same way or order. Rather, our results indicate that most of the groups of students had a multiplicity of ways by which they developed part or all of these schemes and accessed different layers of complexity at different times of their engagement with the task. The differences amongst the DMSs were sometimes related to the level of the students and/or the level of observation (as similarly observed by Drijvers and Trouche 2008). For instance, what may first seem a correlation DMS for a particular group of students, may later act as a building block in the development of a higher order scheme such as testing DMS.

Through this research we aim to contribute to existing research on two levels. First, to explore what dragging may entail for mathematical insight in Logo-based environments designed to connect dynamic manipulation of geometrical objects with symbolic notation. This connection needs investigation since, in contrast with the DGS, Logo environments offer possibilities for students to relate the symbolisation to algebraic objects and procedures. Our results indicate that the integrated use of programming and dragging seemed to play a critical role in enhancing students' shift from visual to mathematical practices to determine proportional relations. Secondly, the purposeful nature of the task casts new light on the existing research on proportional reasoning. The described DMSs indicate that when the affordances of specially-designed constructionist microworlds are intimately linked with purposeful tasks, opportunities may be created for students to appreciate how, when and why proportional relations are useful, i.e. the *utility* (Ainley et al. 2006) of proportional relations. In our study, the purpose of *Dynamic Alphabet* to create 'robust' enlarging-shrinking models of letters that would then be used by other students, facilitated students' engagement in extensive corrective processes. The analysis indicated that such processes were accompanied by students' evaluation of the graphical outcome which fed into the conceptual understanding of the relationships involved. Figure distortion, indicating an unsuccessful attempt to achieve proportional enlarging-shrinking, prompted students to express their perceptions of the problem and develop strategies for solving it, as they were eventually constrained by the tools provided to focus on properties and relations expressed in formal mathematical terms, i.e. through appropriate changes in the symbolic code.

Note

1. Turtleworlds is available at: http://etl.ppp.uoa.gr/_content/download/index_download_en.htm

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