Extensions of the disc algebra and of Mergelyan's theorem

Iakovos Androulidakis[a] and Vassili NESTORIDIS[b]

[a]Georg - August Universität Göttingen, Institute des Mathématiques,
Bunsenstrasse 3-5, 37073 Göttingen, Allemagne
Courriel : iakovos@uni-math.gwdg.de
[b]Département des mathématiques, Panepistemiopolis, 157-84, Athènes, Grèce
Courriel : vnestor@math.uoa.gr

Résumé A toute compactification metrizable S du plan complexe, nous associons une extension $A(D,S)$ de l' algebre du disc $A(D)$. Un cas fondamental est le cas $S=\mathbb{C}\cup\{\infty\}$. Nous déterminons l' ensemble de limites uniformes des polynômes sur le disc unité fermé $\overline{D}$, par rapport à la metrique chordale ; ensuite nous étendons cette étude dans le cas géneral.

Abstract *We investigate the uniform limits of the set of polynomials on the closed unit disc* $\overline{D}$ *with respect to the chordal metric* χ. *More generally, we examine analogous questions replacing* $\mathbb{C}\cup\{\infty\}$ *by other metrizable compactifications of* $\mathbb{C}$.

Version française abrégée

Il est connue qu'ils existent beaucoup de compactifications metrizables du plan complexe $\mathbb{C}$; par example $\mathbb{C}\cup\{\infty\}$, le disque de Poincaré, le tore T^2 et toute surface compacte. Soit $S=\mathbb{C}\cup\mathbb{C}^\infty$ une compactification metrizable de $\mathbb{C}$ munie d' une metrique ρ. $\mathbb{C}$ est un ouvert dense dans S. Nous identifions les limites uniformes sur le disque unité fermé $\overline{D}$ des polynômes par rapport à la metrique ρ. Leur ensemble $A(D,S)$ contient des fonctions de deux types. Le type finie sont les fonctions holomorphes $f:D\to\mathbb{C}$ sur le disque unité ouvert D, telles que $\lim\limits_{z\to\zeta, z\in D} f(z)$ existe dans S pour tout $\zeta\in\vartheta D$. Il est facile de voir que toutes ces fonctions sont en réalité limites uniformes des polynômes. Evidemment le type finie contient l'algebre classique $A(D)$ du disque.

Le type infinie contient des fonctions $f:\overline{D}\to\mathbb{C}^\infty$ continnes sur $\overline{D}$; mais il n'est pas en general vraie que toute telle fonction est une limite des polynômes. Dans chaque cas particulier de compactification S on doit charactérizer les fonctions du type infinie appartenant à $A(D,S)$. La class $A(D,S)$ admet une metrique complète naturelle. $A(D)$ avec sa topologie usuelle est une partie ouverte et dense dans $A(D,S)$.

Proposition. Soit $S=\mathbb{C}\cup\mathbb{C}^\infty$ et $S'=\mathbb{C}\cup\mathbb{C}'^\infty$ deux compatifications metrizable du plan $\mathbb{C}$. Alors les suivant sont équivalent.

1) Il existe un homéomorphisme $T : S \to S'$ tel que $T(w) = w$ pour tout $w \in \mathbb{C}$.

2) Il existe un homéomorphisme $F : A(D,S) \to A(D,S')$ tel que $F(P) = P$ pour tout polynôme P.

Question: Soit $L \subseteq \mathbb{C}$ un ensemble compact avec L^c connexe et $f : L \to S$ une fonction continue telle que $f(L^0) \subseteq \mathbb{C}$ et $f|_{L^0}$ est holomorphe. Est il vraie qu'il existe une suite de polynômes P_n telle que $P_n \to f$ uniformement sur L par rapport à la metrique ρ des S ?

La question precedente admet une réponse positive dans le cas particulier $L = \overline{\Omega}$, où Ω est l'intérieur d'une courbe de Jordan. La raison est que la fonction de Riemann $\varphi : D \to \Omega$ s'étend à un homéomorphisme $\varphi : \overline{D} \to \overline{\Omega}$ (Carathéodory [5]). Alors on se ramène dans le cas $L = \overline{D}$, où la répouse est connue et positive.

1 Spherical approximation

A classical approximation theorem in Complex Analysis is Mergelyen's theorem [11] : If $K \subseteq \mathbb{C}$ is a compact set with K^c connected and if $f : K \to \mathbb{C}$ is continuous on K and holomorphic in K^0 (i.e. $f \in A(K)$) and $\varepsilon > 0$ is given, then there exists a polynomial P such that $\sup\limits_{z \in K} |f(z) - P(z)| < \varepsilon$.

The proof of the above theorem is not easy; however, if $K = \overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, then the proof is elementary : Since f is uniformly continuous on $\overline{D}$, there exists r, $0 < r < 1$, so that $|f(z) - f(rz)| < \frac{\varepsilon}{2}$, for all $z \in \overline{D}$. On $|w| \leq r$, the convergence of the Taylor expansion of f towards f is uniform. Thus, there exists $N \in \mathbb{N}$ so that $\left| \sum\limits_{n=0}^{N} \frac{f^{(n)}(0)}{n!}(rz)^n - f(rz) \right| < \frac{\varepsilon}{2}$ for all $z \in \overline{D}$. The triangle inequality implies $|f(z) - P(z)| < \varepsilon$ for all $z \in \overline{D}$, where P is the polynomial $P(z) = \sum\limits_{n=0}^{N} \left[\frac{f^{(n)}(0)}{n!} r^n \right] z^n$, which completes the proof.

In the above consideration we have uniform approximation on $\overline{D}$ (or on K) by polynomials with respect to the Euclidean metric on $\mathbb{R}^2 = \mathbb{C}$. However, if we consider functions taking the value ∞, as it happens very often in Complex Analysis, then the Euclidean metric on $\mathbb{R}^2 = \mathbb{C}$, does not allow us to speak about uniform approximation. But if we consider the one-point compactification $\mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$ and the chordal distance χ on it ([1]), then we can speak about uniform approximation on a compact set (or on several compact sets) of functions taking values in $\mathbb{C} \cup \{\infty\}$. Indeed, since $\mathbb{C} \cup \{\infty\}$ is compact, any other metric giving the same topology is uniformly equivalent to the metric χ. Thus, most of our results remain valid if we replace χ by another equivalent metric on $\mathbb{C} \cup \{\infty\}$.

If we consider Ω a simply connected domain in $\mathbb{C}$, then the uniform limits on compact subsets of Ω of the polynomials with respect to the Euclidean metric on $\mathbb{R}^2 = \mathbb{C}$, are exactly all holomorphic functions f on Ω. If we replace the Euclidean metric by the chordal metric χ, then the limits are the same with an additional limit $f \equiv \infty$. Thus, one gets the impression that there is no essential difference.

Gauthier, Roth and Walsh in [3] compared two approximation theorems on a compact set $K \subset \mathbb{C}$; one with respect to the Euclidean metric and the other with respect to the chordal metric χ. It turned out that both theorems were valid exactly for the same compact sets K. Thus, again one gets the impression that there is no essential difference.

In [6], [7], the uniform limits of the polynomials on $\overline{D}$ (or on K) with respect to the chordal distance χ are investigated and compared with those defined with respect to the Euclidean metric. The limits with respect to the chordal metric are of two types : the infinite type, which contains only one function $f \equiv \infty$ and the finite type. The finite type contains all holomorphic functions $f : D \to \mathbb{C}$ on the open unit disc D, such that, for every $\zeta \in \partial D$ the limit $\lim\limits_{z \to \zeta, z \in D} f(z)$ exists in $\mathbb{C} \cup \{\infty\}$.

All these functions are uniform limits of polynomials on $\overline{D}$ with respect to χ and only these functions. The set of these functions is denoted by $\widetilde{A}(D)$.

The function $\frac{1}{1-z}$ belongs to $\widetilde{A}(D)$ but not to the classical $A(D)$.

In [6], [7], some properties of the elements of $\widetilde{A}(D)$ are investigated. Privalov's theorem implies that either the set $(\partial D) \cap f^{-1}(\infty)$ is a compact with zero length or $f \equiv \infty$. Conversely, every compact set $K \subset \partial D$ with zero length coincides with $f^{-1}(\infty)$ for an element $f \in \widetilde{A}(D)$ of finite type.

A natural question is to search for a characterization of all compact sets $K \subset \partial D$ which are compacts of interpolation for $\widetilde{A}(D)$, that is, for all continuous functions $\varphi : K \to \mathbb{C} \cup \{\infty\}$, is it true that there exists $f \in \widetilde{A}(D)$ such that $f_{|K} = \varphi$. If $K \subset \partial D$ has positive length, then we know that K is not a compact of interpolation for $\widetilde{A}(D)$. We do not know if every compact set $K \subset \partial D$ with zero length is a compact of interpolation of $\widetilde{A}(D)$.

It is also true that for $f, g \in \widetilde{A}(D)$ the quantity $\sup\limits_{z \in \overline{D}} \chi(f(z), g(z))$ is not controlled by $\sup\limits_{\zeta \in \partial D} \chi(f(\zeta), g(\zeta))$. Thus, the maximum principle fails. However, if $f(\zeta) = g(\zeta)$ for all $\zeta \in \partial(D)$ (or for a compact set of positive length in ∂D) for some $f, g \in \widetilde{A}(D)$ then $f \equiv g$.

If f and g belong to $\widetilde{A}(D)$, then the natural distance of f and g is $\chi(f, g) = \sup\limits_{z \in \overline{D}} \chi(f(z), g(z))$. With this metric $\widetilde{A}(D)$ is a complete metric space. The classical disc algebra $A(D)$ is an open and dense subset of $\widetilde{A}(D)$. Furthermore, the relative topology of $A(D)$ from $\widetilde{A}(D)$ coincides with the usual topology of $A(D)$.

Generically for all $f \in \widetilde{A}(D)$ the length of $\{\zeta \in \partial D : f(\zeta) \notin f(D)\}$ is zero ([6], [7]) (see also [13]). We also consider the sets $X = \{f \in \widetilde{A}(D) : f(D) \subset f(\partial(D)\}$ and $Y = \left\{f \in \widetilde{A}(D) : f(\partial D) = \mathbb{C} \cup \{\infty\}\right\}$.

Then X, Y are closed subsets of $\widetilde{A}(D)$ of the first category. The set X is non void, but it is not known if Y is non void or not. If every compact set $K \subset \partial D$ with zero length is a compact of interpolation for $\widetilde{A}(D)$, then we can prove that $Y \neq \emptyset$.

Replacing $\overline{D}$ by other compact sets $L \subset \mathbb{C}$ with L^c connected we arrive to the following.

Question: Let $L \subset \mathbb{C}$ be a compact set with L^c connected, $f : L \to \mathbb{C} \cup \{\infty\}$ a continuous function such that, for every component V of L^0 it holds $f_{|V} \equiv \infty$ or $f(V) \subset \mathbb{C}$ and $f_{|V}$ holomorphic. Is it then true that there exists a sequence of polynomials P_n so that $P_n \to f$ uniformly on L with respect to the metric χ?

We are able to answer in the affirmative the previous question in some particular cases. The general case remains open.

2 Another compactification

In [8], [9] $\mathbb{C}$ is identified with D by the homeomorphism $\mathbb{C} \ni z \to \frac{z}{1+|z|} \in D$. This way $\overline{D}$ becomes an obvious compactification of D endowed with the usual Euclidean metric. This induces a compactification of $\mathbb{C}$, where there are several points at ∞ situated on a circle : $\overline{\mathbb{C}} = \mathbb{C} \cup \mathbb{C}^{\infty}$, $\mathbb{C}^{\infty} = \{\infty e^{i\vartheta} : \vartheta \in [0, 2\pi)\}$. The metric d on $\overline{\mathbb{C}}$ is $d(z,w) = \left|\frac{z}{1+|z|} - \frac{w}{1+|w|}\right|$, for $z, w \in \mathbb{C}$, $d(z, \infty e^{i\vartheta}) = \left|\frac{z}{1+|z|} - e^{i\vartheta}\right|$ for $z \in \mathbb{C}$, $\vartheta \in \mathbb{R}$ and $d(\infty e^{i\varphi}, \infty e^{i\vartheta}) = |e^{i\varphi} - e^{i\vartheta}|$, $\varphi, \vartheta \in \mathbb{R}$.

The set of uniform limits $\overline{A}(D)$ of the polynomials on $\overline{D}$ with respect to d consists of the finite type and the infinite type. The finite type contains all holomorphic functions $f : D \to \mathbb{C}$ such that, for every $\zeta \in \partial D$ the limit $\lim\limits_{z \to \zeta, z \in D} f(z)$ exists in $\overline{\mathbb{C}} = \mathbb{C} \cup \mathbb{C}^{\infty}$.

The infinite type contains all continuous functions $f : \overline{D} \to \mathbb{C}^{\infty}$, $f(z) = \infty e^{i\vartheta(z)}$, where $\vartheta : \overline{D} \to \mathbb{R}$ is continuous on $\overline{D}$ and harmonic in D.

Furthermore, in [8], [9] one finds an investigation of properties of the elements of $\overline{A}(D)$, topological properties of $\overline{A}(D)$ endowed with its natural metric topology, as well as possible extensions of Mergelyan's theorem.

3 The general case

We consider an arbitrary metrizable compactification $S = \mathbb{C} \cup \mathbb{C}^{\infty}$ of $\mathbb{C}$ (then the compact plane $\mathbb{C}$ is an open dense subset of S, because $\mathbb{C}$ is locally compact, [2]). We denote the metric on S by ρ. Two such examples are given in Section 1 and Section 2. However, there are several other compactifications.

i) Let Ω be a bounded simply connected domain in $\mathbb{C}$. By the Riemann mapping theorem Ω is homeomorphic to $\mathbb{C}$. Then $\overline{\Omega}$ endowed with the Euclidean distance in $\mathbb{C}$ induces a metrizable compactification of $\mathbb{C}$.

ii) Let Ω be an unbounded simply connected domain in $\mathbb{C}$. Then the closure of Ω in $\mathbb{C} \cup \{\infty\}$ endowed with the metric χ induces a metrizable compactification of $\mathbb{C}$.

iii) A torus in $\mathbb{R}^3$ endowed with the Euclidean metric from $\mathbb{R}^3$ induces a metrizable compactification of $\mathbb{C}$; because, if we cut the torus along two circles it becomes homeomorphic to an open rectangle which is homeomorphic to $\mathbb{C}$. More generally, any compact surface in $\mathbb{R}^3$ gives a metrizable compactification of $\mathbb{C}$, because it becomes homeomorphic to $\mathbb{C}$ if we cut it accordingly. We refer to [4] for a classification of the compact surfaces.

iv) The real projective space of dimension 2 is a metrizable compactification of $\mathbb{C}$. One model of it is closed unit disc $\overline{D}$, where all the points w, $-w \in \partial D$ are identified. This compactification is metrizable [12].

v) We consider $\Omega_1 = D$; then $\overline{D}$ induces a compactification $\mathbb{C} \cup \mathbb{C}^{\infty}$ of $\mathbb{C}$ where $\mathbb{C} \approx D$ and $\mathbb{C}^{\infty} \approx \partial D$. Next we consider $\Omega_2 = D - [0, 1)$; then $\overline{D} = \overline{\Omega}_2$ induces a compactification $\mathbb{C} \cup \mathbb{C}'^{\infty}$ of $\mathbb{C}$ where $\mathbb{C} \approx D - [0, 1)$ and $\mathbb{C}'^{\infty} \approx \partial D \cup [0, 1)$. These two compactifications will be considered as different for our purposes.

So, we fix a metrizable compactification $S = \mathbb{C} \cup \mathbb{C}^{\infty}$ endowed with a metric ρ and we denote by $A(D, S)$ the set of uniform limits with respect to ρ of all polynomials on $\overline{D}$. Let $\phi : S = \mathbb{C} \cup \mathbb{C}^{\infty} \to \mathbb{C} \cup \{\infty\}$ be the function $\phi(w) = w$ for $w \in \mathbb{C}$ and $\phi(w) = \infty$ for $w \in \mathbb{C}^{\infty}$. It is easily seen that ϕ is continuous. Since S is compact, it follows that ϕ

is uniformly continuous. Thus, if $P_n \to f$ uniformly on $\overline{D}$ with respect to ρ, it follows that $P_n = \phi(P_n) \to \phi(f)$ uniformly on $\overline{D}$ with respect to the metric χ. (Here P_n are polynomials). Combining this with the results of Section 1 we see that $A(D,S)$ consists of two types : the finite type and the infinite type. The finite type contains all functions $f : D \to \mathbb{C}$ holomorphic, such that $\lim_{z\to\zeta, z\in D} f(z)$ exists in S for all $\zeta \in \partial D$. It is easily seen that every such function can be approximated by polynomials. The infinite type contains functions $f : \overline{D} \to \mathbb{C}^\infty$ which are continuous ; but it is not true in general that all such functions can be approximated by polynomials. In the example of Section 2, we see that only some of these functions can be approximated by polynomials. Thus, there is no general result. In each particular case one should investigate which exactly are the functions of the infinite type belonging to $A(D,S)$.

So, we denote by $A^f(D,S)$ and $A^{inf}(D,S)$ the sets of elements of $A(D,S)$ which are of finite type and, respectively, of infinite type. We have $A(D,S) \supset A^f(D,S) \supset A(D)$ and $A^{inf}(D,S) \neq \emptyset$. For $f, g \in A(D,S)$ the natural metric is $\rho(f,g) := \sup_{z\in\overline{D}} \rho(f(z), g(z))$. Thus, $A(D,S)$ becomes a complete metric space. $A(D)$ with its natural topology is an open and dense subset of $A(D,S)$. The set $A^f(D,S)$ is also open and dense in $A(D,S)$.

Proposition 3.1. *Let S and S' two metrizable compactification of $\mathbb{C}$ as above. Then the following are equivalent :*

(1) There exists a homeomorphism

$T : S = \mathbb{C} \cup \mathbb{C}^\infty \to S' = \mathbb{C} \cup \mathbb{C}'^\infty$ *so that $T(w) = w$ for all $w \in \mathbb{C}$.*

(2) There exists a homeomorphism

$F : A(D,S) \to A(D,S')$ *so that for every polynomial P we have $F(P) = P$.*

One wonders if the condition $A^f(D,S) = A^f(D,S')$ implies (1) of the previous proposition.

The answer is negative. Let S be the compactification where $\mathbb{C} \approx D$ and $\mathbb{C}^\infty = \partial D$ (the example in Section 2). Let S' be the example of the real projective space of dimension 2 ; that is, $\mathbb{C} \approx D$ again but $\mathbb{C}'^\infty$ is the quotient of ∂D where all points w and $-w$ in ∂D are identified. Let $f : D \to \mathbb{C}$ holomorphic. Let $\zeta \in \partial D$. Suppose $\lim_{z\to\zeta, z\in D} f(z)$ exists in $\mathbb{C}'^\infty$. Then for a connected open subset V of D accumulating in ζ, the set $f(V)$ is contained in the union of two disjoint connected open sets one accumulating at w and the other at $-w$. Since by the continuity of f, the set $f(V)$ should be connected, it follows that $f(V)$ is either close to w or to $-w$. Thus, $\lim_{z\in\zeta, z\in D} f(z)$ exists in $\mathbb{C}^\infty$. This shows that $A^f(D,S) = A^f(D,S')$ but $S \neq S'$ in the sense of Proposition 3.1. We also note that $A^f(D, \mathbb{C} \cup \{\infty\})$ contain $A^f(D,S)$, for all metrizable compactifications S.

An open question is the following.

Question: Is there a metrizable compactification S of $\mathbb{C}$ so that $A^f(D,S) = A(D)$?

Another question relating to Mergelyan's theorem is the following.

Question: Let $L \subset \mathbb{C}$ be a compact set with L^c connected and $f : L \to S$ continuous, such that $f(L^0) \subset \mathbb{C}$ and $f_{|_{L^0}}$ is holomorphic. Does there exist a sequence of polynomials P_n such that $P_n \to f$ uniformly on L with respect to the metric ρ of S ?

In the special case where L is the closure of a Jordan domain Ω in $\mathbb{C}$, the answer to the above question is affirmative (see also [10]). The reason is that if $\varphi : D \to \Omega$ is a Riemann map then, according to a theorem of Caratheodory ([5]), φ extends to a homeomorphism $\varphi : \overline{D} \to \overline{\Omega}$. Thus, we are reduced to the case $L = \overline{D}$, where the result is known.

Références

[1] L. V. Ahlfors, Complex Analysis, An introduction to the theory of analytic functions of one complex variable (2^{nd}) McGraw-Hill, New York, 1966.

[2] J. Dugundji, Topology, Allyn and Bacon, Inc. Atlantic Avenue, Boston, 4^{th} edition 1968.

[3] P. M. Gauthier, A. Roth and J. L. Walsh, Possibility of uniform rational approximation in Spherical metric, Can. J. Math. Vol. XXVIII, No. 1, 1976, 112-115.

[4] M. W. Hirsch, Differential Topology, Graduate Texts in Mathematics 33, Springer, 1976.

[5] P. Koosis, Introduction to H_p spaces, London Math. Soc. Lecture Note Series 40, Cambridge Univ. Press, Cambridge London, N.Y., New Rochelle, Melbourne, Sydeny. 69-82.

[6] V. Nestoridis, An extendion of the disc algebrea, I, submitted.

[7] V. Nestoridis, An extension of the disc Algebra, Arxiv : 1009.5364.

[8] V. Nestoridis and N. Papadatos, An extention of the disc algebra, II, submitted.

[9] V. Nestoridis and N. Papadatos, Another extension of the disc algebra, Arxiv : 1012.3674.

[10] V. Nestoridis, I. Papadoperakis, A remark on two extensions of the disc algebra and Mergelyan's theorem, arxiv : 1104.0833.

[11] W. Rudin, Real and Complex Analysis, McGraw-Hill N.Y. St. Louis, San Franscisco, Toronto, Sydney.

[12] M. Spivak, A comprehensive introduction to Differential Geometry, Volume 1, Brandeis University, 1970.

[13] D. Stegenga and K. Stephenson, Generic covering properties for spaces of Analytic Functions, Pacific J. Math. Vol. 119, No 6 (1985), 227-243.