A pseudodifferential calculus for maximally hypoelliptic operators and the Helffer-Nourrigat conjecture

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Spring 2022 Workshop on NCG and Geometric Analysis
https://www.craft.do/s/ax0N4GrNp6qdxE

Maximally hypoelliptic operators

A linear diferential operator D with order k on a manifold M is

- hypoelliptic if

$$singsupp(u) \subseteq singsupp(Du)$$

  for every distribution u on M.
- maximally hypoelliptic if

$$Du \in H^s(M) \Rightarrow u \in H^{s-k}(M)$$

  for any $s \in \mathbb{R}$ and every distrubution u on M.
- Sobolev embedding lemma: max hypoelliptic $\Rightarrow$ hypoelliptic

Proposition

D elliptic $\Rightarrow$ D maximally hypoelliptic.

Kolmogorov operator on $\mathbb{R}^2$:

$$D = \partial_x^2 + x\partial_y$$

D hypoelliptic but not elliptic.

Maximally hypoelliptic operators

Ingredients of proof:

- Groupoid psdo calc $\Psi(M) = \Psi(M \times M)$: $\mathrm{Diff}(M) \subset \Psi(M)$
- Exact sequence of principal symbol:

$$0 \to \Psi^{k-1}(M) \to \Psi^{k}(M) \xrightarrow{\sigma_k} C(S^*M) \to 0$$

  So there is parametrix Q for D: $PQ - I, QP - I$ smoothing.
- Pseudodifferential operators as multipliers of $C^*(M \times M)$

$$0 \to K(L^2(M)) \to \overline{\Psi^0(M)} \xrightarrow{\sigma_0} C(S^*M) \to 0$$

- Filtration of Hilbert module $C^*(M \times M)$ by Sobolev spaces $H^s(M)$: Let P any elliptic operator.
  - $s > 0$: $H^s(M) = \mathrm{Dom}(P)$, $\langle a, b\rangle_s = \langle Pa, Pb\rangle + \langle a, b\rangle$.
  - $H^{-s}(M) = \overline{C^*(M \times M)}^{||\cdot||_{-k}}$ with $||\xi||_{-k} = ||(1 + P^*P)^{-1/2}\xi||$.
  - $s > s'$ identity $\iota_{s,s'} : H^s(M) \to H^{s'}(M)$ compact morphism of Hilbert modules.
  - $\Psi^{-\infty}(M) = \bigcap_{s,t} \mathcal{L}(H^s(M), H^t(M))$.

Strategy and challenges I

D non-elliptic differential operator.

$\boxed{\text{Is there some pseudodifferential calculus, in which } \mathrm{D} \text{ is elliptic?}}$

1. - Find the correct (deformation) groupoid for D
   - Build the groupoid psdo calculus Need $\mathrm{D} \in \Psi^{\infty}$.
2. Principal symbol? Exact sequence?

$$0 \to \Psi^{k-1} \to \Psi^{k} \xrightarrow{\sigma_k} \Sigma \to 0$$

3. Groupoid C^*-algebra, exact sequence

$$0 \to C^*(G) \to \overline{\Psi^0} \xrightarrow{\sigma_0} \Sigma \to 0$$

4. Filtration of $C^*(G)$ by Sobolev modules.

Challenges arise from **singularities**:

- Deformation groupoid not smooth.
- Algebra of deformation groupoid not continuous field.
- Principal symbol σ_k does not give exact sequence

Strategy and challenges II: Results

D without singularities: Rockland conjecture/theorem (Helffer-Nourrigat, Melin): Principal symbol invertible in every non-trivial representation $\Rightarrow$ hypoelliptic/Fredholm.

Proof (van Erp, Yuncken): Find appropriate (groupoid) pseudodiferential calculus and construct parametrix in this calculus.

- Deformation groupoid is smooth. $M \times M \times \mathbb{R}_+^\times$ open + dense.
- Deformation groupoid algebra: continuous field.

D with singularities: Helffer-Nourigat conjecture: Enough to check invertibility in a smaller set of representations.

Distributions transverse to a submersion

Let $\pi : N \to M$ (surjective) submersion.

Definition of $\mathcal{E}'_\pi(N)$:

A distribution on N transverse to π is a $C^\infty(M)$-linear map $C^\infty(N) \to C^\infty(M)$.

Example: Projection $\pi : M \times M \to M$. Then

$$\mathcal{E}'(M \times M) = C^\infty(M) \otimes \mathcal{E}'(M)$$

Distributions semi-regular on the first variable are the Schwarz kernels of continuous linear operators $C^\infty(M) \to C^\infty(M)$.

Distributions transverse to a submersion

Let $\xi \in \mathfrak{X}(M)$. How to view ξ as a distribution transverse to π?

- $TM \simeq \ker(d\pi) \simeq \frac{T(M\times M)}{TM}$. So $\xi(p) \in T_{1_p}(M \times M)$.
- So ξ defines linear map

$$C^\infty(M \times M) \to C^\infty(M) \qquad f \mapsto (p \mapsto df_{1_p}(\xi(p)))$$

Likewise, every $\sigma \in \Gamma AG$ is a right-invariant vector field of G, whence $\sigma \in \mathcal{E}'_s(G)$.

Ingredient 2: Classical psdo calc: Debord + Skandalis view

Requirements for a pseudodifferential calculus:

$\boxed{\textbf{Deformation groupoid + action of } \mathbb{R}_+^*\textbf{.}}$

- $P \in \Psi DO^m(M)$ determined by Schwarz kernel

  $k_P \in \mathcal{E}_r'(M\times M) = \{\alpha : C^\infty(M\times M) \to C^\infty(M), C^\infty(M)\text{−linear}\}$

- Action of $\mathbb{R}_+^*$ on $DNC(M) = TM \times \{0\} \coprod M \times M \times \mathbb{R}^*$:
  1. $a_\lambda(x, y, t) = (x, y, \lambda^{-1}t)$ if $(x, y, t) \in M \times M \times \mathbb{R}^*$
  2. $a_\lambda(x, \xi, 0) = (x, \lambda\xi, 0)$ if $\xi \in T_xM$

Theorem (vE-Y): $k \in \mathcal{E}_r'(M \times M)$ is Schwarz kernel of properly supported psdo of order m iff $k = K|_{t=1}$ for some $K \in \mathcal{E}_r'(DNC(M))$ such that $a_{\lambda*}K - \lambda^m K$ is a smooth density for all $\lambda \in \mathbb{R}_+^*$.

Explanations: Homogeneity of Fourier transform (Taylor)

- $A(\mathrm{DNC}(M))$ is vector bundle $\pi : TM \times \mathbb{R}_+ \to M \times \mathbb{R}$
- Every $u \in \mathcal{E}'_\pi(TM \times \mathbb{R}_+)$ concentrated at $\{0\} \times M \times \{0\}$.
- Fourier transform:

$$\mathcal{E}'_\pi(TM \times \mathbb{R}_+) \ni u \mapsto \hat{u} \in C^\infty(T^*M \times \mathbb{R}^+)$$

- Equip T^*M with $\mathbb{R}_+^\times$-action $\hat{\alpha}_\lambda \xi(X) = \xi(\alpha_\lambda(X)) \quad \forall X \in TM$.
- Say $A \in C^\infty((T^*M \times \mathbb{R}^+) \backslash \{0\} \times M \times \{0\})$ is *homogeneous of degree* k if

$$\hat{\alpha}_\lambda^* A = \lambda^k A$$

Proposition (Taylor): homogeneity of Fourier transform

Let u homogeneous of degree k. Put χ cut-off function about $\{0\} \times M \times \{0\}$. There is $A \in C^\infty((T^*M \times \mathbb{R}^+) \backslash \{0\} \times M \times \{0\})$ such that $\hat{u} - (1 - \chi)A$ is of Schwarz class.

Symbol

Full symbols

$S^k(T^*M \times \mathbb{R}_+)$ are k-homogeneous functions in $C^\infty((T^*M \times \mathbb{R}^+)\backslash\{0\} \times M \times \{0\})$.

Principal cosymbol

$$\Sigma^k(M) = ev_0(\{K \in \mathcal{E}'_r(DNC(M)) \quad k - \text{homogeneous}\})$$

(Mod out $C_p^\infty(DNC(M), \Omega_r)$...)

Cosymbol map:

$$0 \to \Psi^{k-1}(M) \to \Psi^k(M) \xrightarrow{\sigma_k} \Sigma^k(M) \to 0$$

Example 1: How to view $\xi \in \mathcal{X}(M)$ as an order-1 psdo?

Enter foliation theory...

Put $t\xi \in \mathcal{X}(M \times \mathbb{R})$

$$(t\xi)(p, s) = (s \cdot \xi(p), 0_s)$$

Then $(M \times \mathbb{R}, t\mathcal{X}(M))$ *almost regular* foliation:

- Holonomy groupoid: $\mathrm{DNC}(M) = TM \times \{0\} \bigcup (M \times M) \times \mathbb{R}^*$.
- Lie algebroid (sections): $t\mathcal{X}(M)$.

Example 1: How to view $\xi \in \mathfrak{X}(M)$ as an order-1 psdo?

- Distribution $\widetilde{\xi} : C^\infty(DNC(M)) \to C^\infty(M \times \mathbb{R})$ supported in $\{0\} \times (M \times M) \times \mathbb{R}$:

$$\langle \widetilde{\xi}, f \rangle (p, s) = \mathcal{L}_{t\xi}(f)(0, p, p, s)$$

- $\mathbb{R}_+^*$-equivariance:

$$\alpha_{\lambda *}(\widetilde{\xi}) = \lambda \widetilde{\xi}$$

- Evaluation at 1:

$$ev_1(\widetilde{\xi}) = \xi$$

- Symbol: evaluation at 0:

$$\sigma_p^1(\xi) = ev_{(p,0)}(\widetilde{\xi}) \text{ mod } C_c^\infty(T_pM)$$

View $\partial_x^2 + x\partial_y$ as a psdo on a deform. gpd?

e.g. Kolmogorov's plane: $M = \mathbb{R}^2 \quad P = X^2 + Y$

$$X = \partial_x, \qquad Y = x\partial_y \qquad [X, Y] = \partial_y$$

Order dictates singular Lie filtration:

$$\mathcal{D}^1 = \langle X \rangle \subseteq \mathcal{D}^2 = \langle X, Y \rangle \subseteq \mathcal{D}^3 = \mathfrak{X}(\mathbb{R}^2)$$

Get singular "adiabatic" foliation on $M \times \mathbb{R}$:

$$a\mathcal{D} = t\widetilde{\mathcal{D}^1} + t^2\widetilde{\mathcal{D}^2} + t^3\widetilde{\mathcal{D}^3}$$

$\mathbb{R}_*^+$-action: $\alpha_\lambda(t^i\mathcal{D}^i) = (\lambda^i t^i)\mathcal{D}^i$

Localizations ($C^\infty(M)$-modules):

$$a\mathcal{D}|_{t\neq 0} = \mathfrak{X}(M), \quad a\mathcal{D}|_{t=0} = \mathfrak{gr}(\mathcal{D}) = \oplus_{i=1}^3 \frac{\mathcal{D}^i}{\mathcal{D}^{i-1}}$$

View $\partial_x^2 + x\partial_y$ as a psdo on a deform. gpd?

Tangent groupoid = holonomy groupoid of $(M \times \mathbb{R}, a\mathcal{D})$:

$$\mathcal{H}(a\mathcal{D}) = \left(\bigcup_{p \in M} \mathfrak{gr}(\mathcal{D})_p\right) \times \{0\} \bigcup (M \times M) \times \mathbb{R}^*$$

where $\mathfrak{gr}(\mathcal{D})_p = \frac{\mathfrak{gr}(\mathcal{D})}{I_p \mathfrak{gr}(\mathcal{D})}$ nilpotent Lie algebra. Its group is:

$$\mathbb{G}\mathfrak{r}(\mathcal{D})_{(x,y)} = \begin{cases} \mathbb{R} \oplus \mathbb{R} \oplus 0, x \neq 0 \\ H^3, x = 0 \end{cases}$$

Singular Lie filtration

Singular Lie filtration $\mathcal{D}^\bullet$:

$$\mathcal{D}^1 \subseteq \mathcal{D}^2 \subseteq \ldots \subseteq \mathcal{D}^{top} = \mathcal{X}(M)$$

- $\mathcal{D}^i$ locally finitely generated $C^\infty(M)$-submodule of $\mathcal{X}_c(M)$
- $[\mathcal{D}^i, \mathcal{D}^j] \subseteq \mathcal{D}^{i+j}$

$\boxed{(M, \mathcal{D}^\bullet) \rightsquigarrow (M \times \mathbb{R}, \mathfrak{a}\mathcal{D} = t\mathcal{D}^1 + \ldots + t^{top}\mathcal{D}^{top}) \text{ singular foliation}}$

Adiabatic foliation $\mathfrak{a}\mathcal{D}$:

1. $\mathcal{H}(\mathfrak{a}\mathcal{D}) = \left(\bigcup_{p \in M} \mathfrak{gr}(\mathcal{D})_p\right) \times \{0\} \bigcup (M \times M) \times \mathbb{R}^*$
2. $\bigcup_{p \in M} \mathfrak{gr}(\mathcal{D})_p$ singular "bundle" of nilpotent Lie algebras.
3. $C^*(\mathfrak{a}\mathcal{D})$: a $C_0(\mathbb{R})$-C^*-algebra.

Differential operators of the filtration

Given $\mathcal{D}^\bullet$, consider smallest filtration:

$$0 \subseteq C^\infty(M) \subseteq \mathrm{Diff}_{\mathcal{D}^1}(M) \subseteq \ldots \subseteq \mathrm{Diff}_{\mathcal{D}^{N-1}}(M) \subseteq \mathrm{Diff}(M)$$

such that:

- $\mathcal{D}^i \subseteq \mathrm{Diff}_{\mathcal{D}^i}(M)$
- $\mathrm{Diff}_{\mathcal{D}^i}(M)\mathrm{Diff}_{\mathcal{D}^j}(M) \subseteq \mathrm{Diff}_{\mathcal{D}^{i+j}}(M)$

Formal symbols: $\Sigma^i = \frac{\mathrm{Diff}_{\mathcal{D}^i}(M)}{\mathrm{Diff}_{\mathcal{D}^{i-1}}(M)}$. ($C^\infty(M)$-module.)

Symbol map for every $p \in M$:

$$\mathrm{Diff}_{\mathcal{D}^i}(M) \xrightarrow{\sigma_p^i} \frac{\mathrm{Diff}_{\mathcal{D}^i}(M)}{\mathrm{Diff}_{\mathcal{D}^{i-1}}(M) + I_p \mathrm{Diff}_{\mathcal{D}^i}(M)}$$

Differential operators of the filtration

Example: $M = \mathbb{R}$, $\mathcal{D}^1 = \langle x^2\partial_x \rangle$, $\mathcal{D}^2 = \langle \partial_x \rangle$. Take $P = x\partial_x$.

$$\sigma_p^2 : \mathrm{Diff}(\mathbb{R}) \to \frac{\mathrm{Diff}(\mathbb{R})}{\mathrm{Diff}_{\langle x^2\partial_x \rangle}(\mathbb{R}) + I_p \mathrm{Diff}(\mathbb{R})}$$

- P lives in $I_0\mathrm{Diff}(\mathbb{R})$, so $\sigma_0^2(P) = 0$.
- About $p \neq 0$ we can divide by x and x^2, so rhs vanishes.

Conclusion: $\sigma_p^2(x\partial_x) = 0$ for every p, but $x\partial_x \notin \mathrm{Diff}_{\mathcal{D}^1}(M)$.

Note:

$$\mathbb{Gr}(\mathcal{D})_p = \begin{cases} \mathbb{R} \oplus 0, p \neq 0 \\ \mathbb{R} \oplus \mathbb{R}, p = 0 \end{cases}$$

Principal symbol PrSymbPsi

Let $D \in \mathrm{Dif}_{\mathcal{D}^k}(M)$, presented as a sum of monomilas $X_1 \dots X_s$ with $X_i \in \mathcal{D}^{\alpha_i}$, where $\sum_{i=1}^s \alpha_i \leqslant k$.
Take π unitary irrep of $\mathfrak{gr}(\mathcal{D})_p$ and put

$$\sigma^k(D,p,\pi) = \sum \pi([X_1]_p) \dots \pi([X_s]_p)$$

where:

- $\sum$ means we sum only over monomials with $\sum_{i=1}^k \alpha_i = k$.
- $[X_i]_p \in \frac{\mathcal{D}^{\alpha_i}}{\mathcal{D}^{\alpha_i - 1} + I_p \mathcal{D}^{\alpha_i}}$

Fact: For arbitrary π, $\sigma^k(D,p,\pi)$ depends on choice of presentation for D.

The issue with the order of P in the filtration

Surjective map: $\mathcal{U}(\mathfrak{gr}(\mathcal{D})) \to \oplus_i \Sigma^i$. Localization at p:

$$\mathcal{U}(\mathfrak{gr}(\mathcal{D})_p) \longrightarrow \oplus_i \frac{\mathrm{Diff}_{\mathcal{D}^i}(M)}{\mathrm{Diff}_{\mathcal{D}^{i-1}}(M) + I_p \mathrm{Diff}_{\mathcal{D}^i}(M)}$$

Question: Does principal symbol at p live in $U(\mathfrak{gr}(\mathcal{D})_p)$?

Another example: $X = x^2\partial_x$, $Y = x\partial_x$, $Z = \partial_x$. Filtration $\mathcal{D}^\bullet$:

$$\langle X \rangle \subseteq \langle Y \rangle \subseteq \langle Z \rangle$$

Put $P = XZ - Y^2 = x^2\partial_x^2 - (x\partial_x)^2$. Order:

- In $\mathcal{D}^\bullet$, $\mathrm{ord}(X) = 1, \mathrm{ord}(Z) = 3, \mathrm{ord}(Y) = 2$, so $\mathrm{order}(P) = 4$.
- Calculation: $P = -Y$. So $\mathrm{order}(P) = 2$.
- But the group at zero is $\mathbb{R}^3$!

The issue with the order of P in the filtration

Conclusion: Natural surjection not injective:

$$\mathcal{U}(\mathfrak{gr}(\mathcal{D})_p) \to \oplus_i \frac{\mathrm{Diff}^i_{\mathcal{D}}(M)}{\mathrm{Diff}^{i-1}_{\mathcal{D}}(M) + I_p \mathrm{Diff}^i_{\mathcal{D}}(M)}$$

Reason: Singularities! Isomorphism when $\mathcal{D}^\bullet$ constant rank.

ker of this map: "Characteristic" ideal of representations (Helffer-Nourigat ideal).

The HN ideal as a limit set

$$\mathfrak{a}\mathcal{D}^* = (T^*M \times \mathbb{R}_+^\times) \coprod (\mathfrak{gr}(\mathcal{D}) \times \{0\})$$

Locally compact space with weakest topology making these maps continuous:

- projection $\mathfrak{a}\mathcal{D}^* \to M \times \mathbb{R}_+$;
- For every $X \in \mathcal{D}^i$, the maps $\begin{aligned} (\xi, p, t) &\mapsto t^i \xi(X(p)) \\ (\xi, p, 0) &\mapsto \xi([X]_p) \end{aligned}$

Fact: $T^*M \times \mathbb{R}_+^\times$ not dense in $\mathfrak{a}\mathcal{D}^*$. HN ideal is the set of limits:

$$\boxed{\mathcal{T}^*\mathcal{D}_p = \{\xi \in \mathfrak{gr}(\mathcal{D})_p^* : (\xi, 0) \in \overline{T^*M \times \mathbb{R}_+^\times}\}}$$

Theorem

1. $\mathcal{T}^*\mathcal{D}_p$ closed by coadjoint action of $\mathbb{G}\mathfrak{r}(\mathcal{D})_p$.
2. For any $\xi \in \mathcal{T}^*\mathcal{D}_p$, $\sigma^k(D, p, \pi_\xi)$ is well defined.

(π_ξ corresponds to ξ by orbit method.)

Examples

1. Kolmogorov operator: $\mathcal{T}^*\mathcal{D}_p = \mathfrak{gr}(\mathcal{D})_p$ at every $p \in \mathbb{R}^2$.

2. $\mathcal{D}^\bullet : \langle x^2\partial_x\rangle \subseteq \langle x\partial_x\rangle \subseteq \langle \partial_x\rangle$ Then

$$\mathfrak{gr}(\mathcal{D})_p = \begin{cases} \mathbb{R}[\partial_x]_p \oplus 0 \oplus 0 & \text{if } p \neq 0 \\ \mathbb{R}[x^2\partial_x]_p \oplus \mathbb{R}[x\partial_x]_p \oplus \mathbb{R}[\partial_x]_p & \text{if } p = 0 \end{cases}$$

We find

$$\mathcal{T}^*\mathcal{D}_p = \begin{cases} \mathbb{R} & \text{if } p \neq 0 \\ \{(\xi_1, \xi_2, \xi_3) : \xi_1\xi_3 = \xi_2^2\} & \text{if } p = 0 \end{cases}$$

The C^*-algebra of the adiabatic foliation I

$$0 \to K(L^2(M)) \otimes C_0(\mathbb{R}_+^\times) \to C^*(\mathfrak{a}\mathcal{D}) \to C^*(\mathbb{G}\mathfrak{r}(\mathcal{D})) \to 0$$

- $C^*(\mathfrak{a}\mathcal{D})$ is a $C_0(\mathbb{R}_+)$-C^*-algebra.
- $C^*(\mathbb{G}\mathfrak{r}(\mathcal{D}))$ is a $C_0(M)$-C^*-algebra.
- Fiber at $p \in M$: $C^*(\mathbb{G}\mathfrak{r}(\mathcal{D})_p)$
- Spectrum: $C^*\widehat{(\mathbb{G}\mathfrak{r}(\mathcal{D}))} = \coprod_{p\in M} \widehat{\mathbb{G}\mathfrak{r}(\mathcal{D})}_p$ (quotient of $\coprod_{p\in M} \mathfrak{gr}(\mathcal{D})^*_p$ by coadjoint action).

But $C^*(\mathfrak{a}\mathcal{D})$ not continuous field of C^*-algebras!

The C^*-algebra of the adiabatic foliation II

Closed $*$-ideal

$$J = \{\alpha \in C^*(\mathfrak{a}\mathcal{D}) : \alpha_t = 0 \quad \forall t \in \mathbb{R}_+^\times\}$$

- J concentrated at $t = 0$, maps injectively to closed ideal J_0 of $C^*(\mathbb{G}\mathfrak{r}(\mathcal{D}))$.
- Put $C_z^*(\mathfrak{a}\mathcal{D}) := C^*(\mathfrak{a}\mathcal{D})/J$.
- Put $C^*T\mathcal{D}$ the 0-fiber of $C_z^*(\mathfrak{a}\mathcal{D})$. Namely $C^*(\mathbb{G}\mathfrak{r}(\mathcal{D}))/J_0$.
  - $0 \to K(L^2(M)) \otimes C_0(\mathbb{R}_+^\times) \to C_z^*(\mathfrak{a}\mathcal{D}) \to C^*T\mathcal{D} \to 0$

Definition

$$\mathcal{T}^*_{ana}\mathcal{D}_p := \{\pi \in \widehat{\mathbb{G}\mathfrak{r}(\mathcal{D})} : J_0 \subseteq \ker \pi\}$$

Theorem (Mohsen)

$$\mathcal{T}^*_{ana}\mathcal{D}_p = \mathcal{T}^*\mathcal{D}_p$$

Our goal

Construct pseudo-differential calculus $\Psi(\mathcal{D}^\bullet)$ such that:

1. There is an algebra homomorphism $\mathrm{Diff}_{\mathcal{D}^\bullet}(M) \to \Psi(\mathcal{D}^\bullet)$.
2. Let $P \in \mathrm{Diff}_{\mathcal{D}^i}(M)$. If, at each $p \in M$, $\sigma^i_p(P)$ is invertible in every representation of $\mathcal{T}^*\mathcal{D}$, then there is a parametrix of P.

Building the psdo calculus: Adiabatic bisubmersions

$(M, \mathcal{D}^\bullet)$ sing. Lie filtration $\rightsquigarrow \begin{cases} (M \times \mathbb{R}, \mathfrak{a}\mathcal{D}) \text{ sing. foliation} \\ \text{and } \mathbb{R}_+^* \text{ action} \end{cases}$

Fiber of $\mathfrak{a}\mathcal{D}$ at p : $\quad V = \oplus_{i=1}^{top} V^i$. (Models $\mathfrak{gr}(\mathcal{D})_p$.)

- V graded Lie algebra.
- $\mathbb{R}_+^*$-action: $\alpha_\lambda(\sum_{i=1}^{top} v_i) = \sum_{i-1}^{top} \lambda^i v_i$.
- $\sharp : V \to \mathcal{X}_c(M)$ such that:
  1. For every i, $\sharp(V^i) \subseteq \mathcal{D}^i$
  2. $\sharp(\oplus_{k=1}^{i} V^k)$ generate $\mathcal{D}^i$ about p.
- $\mathbb{R}_+^*$-action on $V \times M \times \mathbb{R}$: $\quad \lambda \cdot (X, x, t) = (\alpha_\lambda(X), x, \frac{t}{\lambda})$.

Building the psdo calculus: Adiabatic bisubmersions

$$\mathbf{s}, \mathbf{r} : V \times M \times \mathbb{R} \to M \times \mathbb{R} \qquad \mathbf{r}(X, x, t) = \exp(\sharp(\alpha_t(X))(x), t)$$

$$\boxed{\mathbb{U} = \{(X, x, t) : ||\alpha_t(X)|| \leqslant 1, x \in U\}}$$

- $\mathbb{U}$ is $\mathbb{R}_+^*$-invariant.
- $\mathbf{s}, \mathbf{r} : \mathcal{U} \to M$ are $\mathbb{R}_+^*$-equivariant.
- **Invariance by diffeomorphisms:** Let $(V, \sharp)$ and $(W, \sharp)$ graded bases at p and $\mathbb{U}$, $\mathbb{U}'$ their bisubmersions. There exists $\mathbb{R}_+^*$-equivariant morphism $\phi : \mathbb{U} \to \mathbb{U}'$.
- **Composition:** There is a morphism $\mathbb{U} \times_{\mathbf{r},\mathbf{s}} \mathbb{U} \to \mathbb{U}$ realising the Baker-Campbell-Hausdorff formula over zero (group law).
  - $\boxed{ev_1 : \mathcal{E}_\mathbf{s}'(\mathbb{U}) \to \mathcal{E}_\mathbf{s}'(M \times M) \quad u \mapsto (ev_1 \circ q_{\mathbb{U}})_*(u)}$
- $\boxed{ev_{(p,0)} : \mathcal{E}_\mathbf{s}'(\mathbb{U}) \to \mathcal{E}_\mathbf{s}'(\mathfrak{gr}(\mathcal{D})_p) \quad u \mapsto (ev_{(p,0)} \circ q_{\mathbb{U}})_*(u)}$

The space $\Psi(M, \mathcal{D}^\bullet)$: "Image of ev_1"

Action of $\mathbb{R}_+^\times$ on $\mathcal{E}'_s(\mathbb{U})$:

$$\langle \alpha_{\lambda *} u, f \rangle = \alpha^*_{\lambda^{-1}} \langle u, f \circ \alpha_\lambda \rangle$$

Let $k \in \mathbb{C}$. Define $\mathcal{E}'^k_s(\mathbb{U})$ the properly supported $u \in \mathcal{E}'_s(\mathbb{U})$ such that for any $\lambda \in \mathbb{R}_+^\times$

$$\alpha_{\lambda *} u - \lambda^k u \in C^\infty_p(\mathbb{U})$$

(u supported on $\{0\} \times U \times \mathbb{R}_+$.)

Definition

$\Psi^k(M, \mathcal{D}^\bullet) = \{P \in \mathcal{E}'_s(M \times M)$ properly supported$\}$ such that:

1. $singsupp(P) \subseteq M$.
2. For every $(V, \sharp, \mathbb{U}, U)$, $f \in C^\infty_c(ev_1(\mathbb{U}|_1))$, there is a lift $u \in \mathcal{E}'^k_s(\mathbb{U})$ such that $fP = ev_{1*}(u)$.

The algebra $\Psi(M, \mathcal{D}^\bullet)$

- $C_p^\infty(M \times M) \subseteq \Psi^k(M, \mathcal{D}^\bullet)$ for any k.
- $\mathrm{Diff}_{\mathcal{D}}^k(M) \subseteq \Psi^k(M, \mathcal{D}^\bullet)$.
- Full symbol, adjoints...
- $P_i \in \Psi^{k_i}(M, \mathcal{D}^\bullet)$ $(i = 1, 2)$ then $P_1 \star P_2 \in \Psi^{k_1+k_2}(M, \mathcal{D}^\bullet)$ defined using $\mathbb{U} \times_{\mathbf{r},\mathbf{s}} \mathbb{U} \to \mathbb{U}$ which satisfies BCH formula at 0.
- $\Psi^k(M, \mathcal{D}^\bullet) \subseteq \Psi^{k+1}(M, \mathcal{D}^\bullet)$.
- $\Psi^\infty(M, \mathcal{D}^\bullet) \subseteq C_p^\infty(M \times M)$.

Local nature

Proposition: $P \in \Psi^k(M, \mathcal{D}^\bullet)$ is quite local:

Let $P \in \mathcal{E}'_s(M \times M)$ with $singsupp(P) \subset M$. Then $P \in \Psi^k(M, \mathcal{D}^\bullet)$ iff every $p \in M$ has neighborhood W such that $P|_W = ev_{1,*}(u)|_W$ with $u \in \mathcal{E}'^k_s(M \times M)$,

Corollary

Suppose each $\mathcal{D}_i$ is generated by a finite family of vector fields. Then: $\Psi^k(M, \mathcal{D}^\bullet) = ev_{1,*}(\mathcal{E}'^k_s(\mathbb{U})) + C^\infty_p((M \times M) \backslash \Delta_M)$

Let M compact. Then $\mathbb{U} = G \times M$, where G nilpotent.

The principal symbol I : Fourier transform

Let $u \in \mathcal{E}'^{k}_{s}(G \times M)$, $\widehat{u} \in C^{\infty}(\mathfrak{g}^* \times M)$ its Fourier transform.

Replace $\widehat{u}$ with $B \in C^{\infty}((\mathfrak{g}^* \times M) \backslash (\{0\} \times M))$ such that:

$$\widehat{\alpha}_{\lambda}^{*} B = \lambda^{k} B \quad \forall \lambda \in \mathbb{R}_{+}^{\times}$$

- $\mathring{C}^*(G \times M)$: intersection of kernels of trivial rep $C^*(G) \to \mathbb{C}$.
- $\mathcal{S}_0(G)$ Schwarz functions s.t. $\widehat{f}$ flat. Dense subalg. of $\mathring{C}^*(G)$.
- (Christ et al): B defines by convolution $\check{B} : \mathcal{S}_0(G) \to \mathcal{S}_0(G)$ linear + continuous. Extends to $G \times M$.
- Whence $\check{B}(\cdot, x)$ gives unbounded multiplier of $\mathring{C}^*(G \times M)$. (Bounded for $k = 0$.)
- Put $\Sigma^*(G \times M)$ the C^*-subalgebra of $M(\mathring{C}^*(G \times M))$ generated by these multipliers.
- Eske Ewert (2021): Spectrum is $(\widehat{G} \backslash \{\widehat{1}_G\}) / \mathbb{R}_{+}^{\times}$.

The principal symbol II

For $u \in \mathcal{E}'^k_s(G \times M)$, put $\sigma^k(u, x)$ the unbounded multiplier of $\mathring{C}^*(G)$ defined by $\check{B}(\cdot, x)$. Let $\pi \in \widehat{G} \backslash \{\widehat{1}_G\}$.

- π gives irrep of $\mathring{C}^*(G)$ and $M(\mathring{C}^*(G))$.
- Put $\sigma^k(u, x, \pi) = \pi(\sigma^k(u, x))$.

Let $P \in \Psi^k(M, \mathcal{D}^\bullet)$ with global lift $u \in \mathcal{E}'^k_s(G \times M)$. Define

$$\sigma^k(P, x, \pi) = \sigma^k(ev_{0*}(u), x, \pi)$$

- For differential operators, boils down to PrSymbDiff.
- Quotient map $\sharp_x : G \to \mathbb{G}\mathfrak{r}(\mathcal{D})_x$.
- So $\widehat{\mathbb{G}\mathfrak{r}(\mathcal{D})_x} \subseteq \widehat{G}$.

Theorem

If $\pi \in \mathcal{T}^*\mathcal{D}_x \backslash \{\widehat{1}_{\mathbb{G}\mathfrak{r}(\mathcal{D})_x}\}$ then $\sigma^k(P, x, \pi)$ is well defined.

Exact sequence at zero order

Proposition

Every $P \in \Psi^0(M, \mathcal{D}^\bullet)$ defines a multiplier of $\mathring{C}^*_z \mathfrak{a}\mathcal{D}$.

Put $\Sigma^* \mathcal{T}^* \mathcal{D} = \frac{\overline{\Psi^0(M, \mathcal{D}^\bullet)}}{\mathcal{K}(L^2(M))}$. It is a $C(M) - C^*$-algebra.

$$0 \to \mathcal{K}(L^2(M)) \to \overline{\Psi^0(M, \mathcal{D}^\bullet)} \xrightarrow{\sigma^0} \Sigma^* \mathcal{T}^* \mathcal{D} \to 0$$

Theorem

1. $\Sigma^* \mathcal{T}^* \mathcal{D}_x$ identifies naturally with quotient of $\Sigma^* \mathbb{G}\mathfrak{r}(\mathcal{D})_x$ corresponding to the closed set of representations $\mathcal{T}^* \mathcal{D}_x \setminus \{\widehat{1}_{\mathbb{G}\mathfrak{r}(\mathcal{D})_x}\} / \mathbb{R}_+^\times$.

2. For $\pi \in \mathcal{T}^* \mathcal{D}_x \setminus \{\widehat{1}_{\mathbb{G}\mathfrak{r}(\mathcal{D})_x}\}$ we have

$$\pi(\sigma^0(P, x)) = \sigma^0(P, x, \pi)$$

Sobolev scale

Proposition (Christ et al)

There is family $\{P_k\}_{k\in\mathbb{C}}$ of operators in $\Psi^k(M, \mathcal{D}^\bullet)$ s.t. for all k, k'

1. P_k has a (global) lift u such that $\sigma^k(u, x, \pi)$ is injective for every non-trivial irrep.
2. $P_k * P_{k'} - P_{k+k'} \in \Psi^{k+k'-1}(M, \mathcal{D}^\bullet)$.
3. $P_k - P_k^* \in \Psi^{k-1}(M, \mathcal{D}^\bullet)$.

Get filtration of $\mathcal{K}(L^2(M))$-C*-modules:

$$\dots H^1_{\mathcal{D}^\bullet}(M) \subseteq \mathcal{K}(L^2(M)) \subseteq H^{-1}_{\mathcal{D}^\bullet}(M) \subseteq \dots$$

- If $k > 0$, $H^k_{\mathcal{D}^\bullet}(M) \subseteq \mathcal{K}(L^2(M))$ is the domain of $\overline{P}_k$.

  (Is $K(L^2(M))$-C*-module when identified with graph of $\overline{P}_k$.)
- If $k < 0$ put $H^k_{\mathcal{D}^\bullet}(M) = \mathcal{K}(H^{-k}_{\mathcal{D}^\bullet}(M), \mathcal{K}(L^2(M)))$.
- If $s \in \mathbb{N}$, $H^{sN}(M) \subseteq H^{sN}_{\mathcal{D}^\bullet}(M) \subseteq H^s(M)$.

Affirmative answer to Helfer-Nourrigat conjecture

Theorem (A-Mohsen-Yuncken)

Let $P \in \Psi^0(M, \mathcal{D}^\bullet)$. The following are equivalent.

1. $\sigma^0(P, x)$ left invertible for every $x \in M$.
2. $\sigma^0(P, x, \pi)$ left invertible (injective) for every $x \in M$ and $\pi \in \mathcal{T}^*\mathcal{D}_x \setminus \{\widehat{1}_{\mathbb{G}\mathfrak{r}(\mathcal{D})_x}\}$.
3. Bounded extension $P : H^s_{\mathcal{D}^\bullet}(M) \to H^s_{\mathcal{D}^\bullet}(M)$ left invertible mod compact operators, for all $s \in \mathbb{R}$.
4. For every $r \in \mathbb{N}$, there is $Q \in \Psi^0(M, \mathcal{D}^\bullet)$ such that $Q * P - \mathrm{id} \in \Psi^{-r}(M, \mathcal{D}^\bullet)$.
5. For all $s \in \mathbb{R}$ and any distribution u on M,

$$Pu \in H^s_{\mathcal{D}^\bullet}(M) \Rightarrow u \in H^s_{\mathcal{D}^\bullet}(M)$$

If $P \in \Psi^k(M, \mathcal{D}^\bullet)$, apply the theorem to $P * P_{-k}$.

Application 1: Hoermander's theorem and beyond

Proposition

Let $\mathfrak{g}$ nilpotent and $\{x_0, x_1, \ldots, x_k\}$ generating family. Let $\alpha_1, \ldots, \alpha_k \in \mathbb{N}$ even and $\alpha_0 \in \mathbb{N}$ odd. Then, for any non-trivial irrep π of G, $\pi(x_0^{\alpha_0} + \sum_{i=1}^{k} (-1)^{\frac{a_i}{2}} x_i^{\alpha_i})$ is injective.

Proof Let $v \in \ker$. The operator $\pi(x_0^{\alpha_0} + \sum_{i=1}^{k} (-1)^{\frac{a_i}{2}} x_i^{\alpha_i})$ is positive and $\pi(x_0^{\alpha_0})$ self-adjoint. Whence $v \in \ker(\pi(x_i))$ for every i. This means $v \in \ker(\pi(\mathfrak{g}))$, so $v = 0$.

Corollary (Hoermander's theorem and beyond)

Let $X_0, X_1, \ldots, X_k$ real vector fields, bracket-generating. The operator $X_0^{\alpha_0} + \sum_{i=1}^{k} (-1)^{\frac{a_i}{2}} X_i^{\alpha_i})$ is maximally hypoelliptic.

Proof Define $\mathcal{D}^\bullet$, declaring X_i to have order $\frac{LCM(\alpha_0, \ldots, \alpha_k)}{\alpha_i}$. Then $\mathfrak{gr}(\mathcal{D})_p$ is generated by $\{[X_i]_p\}_{i=1,\ldots,k}$ at every $p \in M$.

Application 2: Kohn's theorem and beyond (complex vector fields)

Let X_0 real vector field and $X_1, \dots, X_k$ complex. Assume $TM \otimes \mathbb{C}$ generated by iterated brackets of length $\leqslant N$. Define $\mathcal{D}^\bullet$ with:

- X_0 has order 2.
- $\mathrm{Re}(X_i), \mathrm{Im}(X_i)$ have order 1.

Fact: $[X_i]_p \in \mathfrak{gr}_p \otimes \mathbb{C}$ do not always generate the whole Lie algebra. If so, P is hypoelliptic:

$$P = \sum_{i=1}^{k} (X_i^* X_i)^\alpha + X_0^\alpha, \quad \alpha \text{ odd}$$

Proposition

Let $\mathfrak{g}$ graded + nilpotent of depth N,
$x_0 \in \mathfrak{g}_2, x_i \in \mathfrak{g}_1 \otimes \mathbb{C}, i = 1, \dots, k$. Put $\mathfrak{h} \subseteq \mathfrak{g} \otimes \mathbb{C}$ subalgebra they generate. Suppose $\mathfrak{g}$ over $\mathbb{R}$ is generated by $\{\mathrm{Re}(x), \mathrm{Im}(x) : x \in \mathfrak{h}\}$.
Then:
For any non-trivial irrep π, $\alpha \in \mathbb{N}$ odd, $\pi(P)$ is injective.

Thank you!

Ευχαριστὼ!