The analytic index of elliptic pseudodifferential operators on a singular foliation [1] [2]

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Abstract

In previous papers ([1, 2]) we defined the C^*-algebra and the longitudinal pseudodifferential calculus of any singular foliation $(M, \mathcal{F})$. In the current paper we construct the analytic index of an elliptic operator as a KK-theory element, and prove that this element can be obtained from an "adiabatic foliation" $\mathcal{TF}$ on $M \times \mathbb{R}$, which we introduce here.

Introduction

In the celebrated Atiyah-Singer index theorem [3] of the 60's, the homotopy invariance of the analytic index of an elliptic (pseudo)-differential operator P is used to first show that the index only depends on the K-theory class of the principal symbol of P and then to compute the morphism $\mathrm{ind}_{an} : K(TM) \to \mathbb{Z}$ it defines in terms of the topological index. In [4], the analytic index is already presented as a morphism between more general K-groups. This formulation is pushed one step further in [9], namely to establish the analytic index as a KK-theory element, which allows the generalization of the Atiyah-Singer theorem to leaf spaces of (regular) foliations. Note that having the analytic index as a morphism of K-groups allows one to define the Baum-Connes "assembly" map.

The article at hand aims to describe the analytic index for elliptic pseudodifferential operators along a singular foliation in these terms. It follows the methods introduced and the results achieved in [1] and [2] regarding the study of singular foliations $(M, \mathcal{F})$. Specifically, recall that in these papers we constructed

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- the holonomy groupoid $\mathcal{G}(M,\mathcal{F})$, which is a topological groupoid endowed with a usually ill-behaved (quotient) topology;
- the (full and reduced) C^*-algebra of the foliation $(M,\mathcal{F})$;
- the extension of C^*-algebras associated with 0-order pseudodifferential operators: this is a short exact sequence

$$0 \to C^*(M,\mathcal{F}) \to \Psi(M,\mathcal{F}) \xrightarrow{\sigma} B \to 0 \tag{1.1}$$

  where B is the commutative algebra of 0-order symbols: it is (a quotient of) the algebra of continuous functions on a cosphere "bundle" naturally associated with $\mathcal{F}$.

The key to these constructions is the notion of a bi-submersion, which we are going to use here as well. This is given by a manifold U and two submersions $s,t : U \to M$, each of which lifts the leaves of $\mathcal{F}$ to the fibers of s and t. In a broad sense this may be thought of as a cover of an open subset of the holonomy groupoid.

In the current paper we study the analytic index of elliptic longitudinal pseudodifferential operators, *i.e.* the map which to the symbol class $[\sigma_P] \in K^0(\mathcal{F}^*)$ associates the class in $K_0(C^*(M,\mathcal{F})) = KK(\mathbb{C}, C^*(M,\mathcal{F}))$ of the elliptic operator P itself. This map $K_0(C_0(\mathcal{F}^*)) \to K_0(C^*(M,\mathcal{F}))$ can be directly expressed in terms of the extension of 0-order pseudodifferential operators: indeed, the K-theory of $C_0(\mathcal{F}^*)$ can be identified with the relative K-theory of the morphism $p : C_0(M) \to C_0(S\mathcal{F}^*)$, and a natural commutative diagram gives rise to a map $K_0(\mathcal{F}^*) = K_0(p) \to K_0(\sigma) = K_0(C^*(M,\mathcal{F}))$ which is the analytic index. Moreover, using mapping cones, we construct this morphism as an element $\mathrm{ind}_a \in KK(C_0(\mathcal{F}^*), C^*(M,\mathcal{F}))$.

We then prove that the analytic index can be obtained from a tangent groupoid, in the spirit of [8] and [12], as follows:

- Every foliation $(M,\mathcal{F})$ gives rise to an "adiabatic" foliation $\mathcal{TF}$ on $M \times \mathbb{R}$.
- The holonomy groupoid of $\mathcal{TF}$ is a "deformation groupoid", namely

$$\mathcal{G}_{\mathcal{TF}} = (\bigcup_{x\in M} \mathcal{F}_x) \times \{0\} \cup \mathcal{G}_{\mathcal{F}} \times \mathbb{R}^*.$$

- Restricting $C^*(M\times\mathbb{R},\mathcal{TF})$ to the interval $[0,1]$, we find an extension

$$0 \to C_0((0,1]) \otimes C^*(M,\mathcal{F}) \to C^*(M\times[0,1],\mathcal{TF}) \xrightarrow{\mathrm{ev}_0} C_0(\mathcal{F}^*) \to 0 \tag{3.2}$$

- The morphism ev_0 has a contractible kernel and is therefore a KK-equivalence. We finally establish the equality

$$\mathrm{ind}_a = [\mathrm{ev}_0]^{-1} \otimes [\mathrm{ev}_1].$$

We should stress that in the case of singular foliations, the 'cotangent bundle' $\mathcal{F}^*$ has fibers of non constant dimension and the topology of the groupoids is ill behaved. These facts impose special difficulties in implementing the steps above.

- The construction of the adiabatic foliation, and in particular the exact sequence (3.2) is quite different from the Lie groupoid case: here we have to use the description of the representations of the foliation C^*-algebra established in [1].
- The pseudodifferential calculus in [2] has many new subtleties. An advantage of our mapping cone approach is that we do not need to deal very deeply here with this construction.

The rest of the paper is organized as follows:

- To make the paper self-contained, we start in section 1 with a brief overview of various definitions and results in [1] and [2].
- In section 2 we show how extension (1.1) gives rise to the analytic index. This is obtained as the class of the morphism of mapping cones $C_0(\mathcal{F}^*) = \mathcal{C}_p \to \mathcal{C}_\sigma$ under the KK-equivalence associated with the "excision map" $e : C^*(M, \mathcal{F}) = \ker \sigma \to \mathcal{C}$. Finally we
  a) show, using relative K-theory, that the K-theory map $K_0(C_0(\mathcal{F}^*)) \to K_0(C^*(M, \mathcal{F}))$ associated with ind_a is indeed the (analytic) index map;
  b) briefly explain how this construction is related to the element associated with extension (1.1) in $Ext(C_0(S\mathcal{F}^*), C^*(M, \mathcal{F})) = KK^1(C_0(S\mathcal{F}^*), C^*(M, \mathcal{F}))$.
- In section 3 we introduce the "adiabatic foliation" $(M \times \mathbb{R}, \mathcal{TF})$, whose leaves are $L \times \{\beta\}$ where L is a leaf of $(M, \mathcal{F})$ for $\beta \in \mathbb{R}^*$ and for $\beta = 0$ single points $\{(x, 0)\}$ $(x \in M)$ ([3]). We show that its bi-submersions are deformations to the normal cone of identity sections in bi-submersions of $(M, \mathcal{F})$. From this it follows that the holonomy groupoid of $\mathcal{TF}$ is the deformation groupoid of $\mathcal{G}_\mathcal{F}$ we mentioned before. Finally, we show that $C^*(M \times \mathbb{R}, \mathcal{TF})$ lies in a natural exact sequence, namely the extension (3.2) discussed above.
- In section 4 we examine the extension of 0-order pseudodifferential operators of the adiabatic foliation. We deduce that the analytic index can be obtained from the adiabatic foliation.

1 Singular foliations and C*-algebras

We briefly recall here some facts and constructions from [1, 2].

[3] Recall however that in the case of singular foliations, the partition into leaves doesn't determine the foliation. The precise definition of $\mathcal{TF}$ is given in section 3.

1.1 Foliations

Definition 1.1. a) Let M be a smooth manifold. A *foliation* on M is a locally finitely generated submodule of $C_c^\infty(M;TM)$ stable under Lie brackets.

b) For $x \in M$, put $I_x = \{f \in C^\infty(M) : f(x) = 0\}$. The *fiber* of $\mathcal{F}$ is the quotient $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$. The *tangent space of the leaf* is the image F_x of the evaluation map $\mathrm{ev}_x : \mathcal{F} \to T_xM$.

c) The *cotangent "bundle"* of the foliation $\mathcal{F}$ is the union $\mathcal{F}^* = \coprod_{x\in M} \mathcal{F}_x^*$. It has a natural projection $p : \mathcal{F}^* \to M$ ($(x,\xi) \mapsto x$) and for each $X \in \mathcal{F}$, there is a natural map $q_X : (x,\xi) \mapsto \xi \circ e_x(X)$. We endow $\mathcal{F}^*$ with the weakest topology for which the maps p and q_X are continuous. This makes it a locally compact space (*cf.* [2, §2.2])

Example 1.2. Recall from [1] the foliation defined by the action of $SL(2)$ on $\mathbb{R}^2$: Its leaves are $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$, and $\mathcal{F}_x = F_x = \mathbb{R}^2$ for all $x \neq 0$, while at 0 we have $\mathcal{F}_0 = sl_2(\mathbb{R}) \simeq \mathbb{R}^3$ and $F_0 = 0$. (Notice that $\mathcal{F}_x = F_x$ in the open dense subset $\mathbb{R}^2 \setminus \{0\}$). In [1, ex. 3.7] we showed that the associated holonomy groupoid as a set is $SL(2)\times\{0\}\cup(\mathbb{R}^2\setminus\{0\})\times(\mathbb{R}^2\setminus\{0\})$. Its topology is so bad, that for every $x \in \mathbb{R}^2 \setminus \{0\}$ the sequence $(\frac{x}{n}, \frac{x}{n})$ converges to every point of the stabilizer of x, namely every point of the real line.

If $(M, \mathcal{F})$ is a foliation and $f : M \times L \to M$ is the first projection, every vector field X of M extends to a vector field $X \otimes 1$ on $M \times L$, which is tangent along M. We define the foliation $\mathcal{F} \otimes 1$ to be the submodule of $C_c^\infty(M \times L; T(M \times L))$ which consists of all finite sums $\sum f_i(X_i \otimes 1)$ where $f_i \in C_c^\infty(M \times L)$ and $X_i \in \mathcal{F}$. The pull back foliation $f^{-1}(\mathcal{F})$ is the space of vector fields spanned by the vertical vector fields $(\ker df)$ and $\mathcal{F} \otimes 1$. In this way, we define also the pull-back foliation by a submersion.

1.2 Bi-submersions

The key ingredient in our study of the holonomy of a foliation is the notion of a bi-submersion. This can be thought of as a piece of the holonomy groupoid. Explicitly:

Definition 1.3. A *bi-submersion* of $(M, \mathcal{F})$ is a smooth manifold U endowed with two smooth maps $s, t : U \to M$ which are submersions and satisfy:

a) $s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F})$.

b) $s^{-1}(\mathcal{F}) = C_c^\infty(U; \ker ds) + C_c^\infty(U; \ker dt)$.

If (U, t, s) is a bi-submersion then the dimension of the manifold U is at at least $\dim M + \dim \mathcal{F}_{s(u)}$. We say it is *minimal at* $u \in U$ *if* $\dim U = \dim M + \dim \mathcal{F}_{s(u)}$.

If (U, t_U, s_U) and (V, t_V, s_V) are bi-submersions then (U, s_U, t_U) is a bi-submersion - called the *inverse bi-submersion* and denoted by U^{-1}, as well as (W, s_W, t_W) where $W = U \times_{s_U, t_V} V$, $s_W(u, v) = s_V(v)$ and $t_W(u, v) = t_U(u)$ - called the *composition* of U and V and denoted by $U \circ V$ ([1, Prop. 2.4]).

Definition 1.4 (morphisms of bi-submersions)**.** Let (U_i, t_i, s_i) $(i = 1, 2)$ be bi-submersions. A smooth map $f : U_1 \to U_2$ is a *morphism of bi-submersions* if $s_1 = s_2 \circ f$ and $t_1 = t_2 \circ f$.

A notion which is very important for the pseudodifferential calculus is that of identity bisection.

Definition 1.5. An *identity bisection* of (U, t, s) is a locally closed submanifold V of U such that the restrictions to V of s and t coincide and are étale.

Also, for every bi-submersion (U, t, s) and every $u \in U$, there exists a bi-submersion (U', t', s'), and an element $u' \in U'$ such that U' is minimal at u' and carries at u' the same diffeomorphisms as U at u. It follows that there is a neighborhood W of u in U and a submersion which is a morphism $f : (W, t_{|W}, s_{|W}) \to (U', t', s')$.

1.3 The groupoid of an atlas

Definition 1.6. Let $\mathcal{U} = \big((U_i, t_i, s_i)\big)_{i \in I}$ be a family of bi-submersions. A bi-submersion (U, t, s) is *adapted* to $\mathcal{U}$ if for all $u \in U$ there exists an open subset $U' \subset U$ containing u, an $i \in I$, and a morphism of bi-submersions $U' \to U_i$.

We say that $\mathcal{U}$ is an *atlas* if

a) $\displaystyle\bigcup_{i \in I} s_i(U_i) = M$.

b) The inverse of every element in $\mathcal{U}$ is adapted to $\mathcal{U}$.

c) The composition $U \circ V$ of any two elements in $\mathcal{U}$ is adapted to $\mathcal{U}$.

An atlas $\mathcal{U}' = \{(U'_j, t_j, s_j)\}_{j \in J}$ is adapted to $\mathcal{U}$ if every element of $\mathcal{U}'$ is adapted to $\mathcal{U}$. We say $\mathcal{U}$ and $\mathcal{U}'$ are *equivalent* if they are adapted to each other. There is a *minimal atlas* which is adapted to any other atlas: this is the atlas generated by "identity bi-submersions".

The groupoid $G(\mathcal{U})$ of an atlas $\mathcal{U} = \big((U_i, t_i, s_i)\big)_{i \in I}$ is the quotient of $U = \displaystyle\coprod_{i \in I} U_i$ by the equivalence relation for which $u \in U_i$ is equivalent to $u' \in U_j$ if there is a morphism of bi-submersions $f : W \to U_j$ defined in a neighborhood $W \subset U_i$ of u such that $f(u) = u'$.

1.4 The C*-algebra of a foliation

In [1, §4] we associated to an atlas $\mathcal{U}$ its (full) C^*-algebra $C^*(\mathcal{U})$. To any bi-submersion W adapted to $\mathcal{U}$ we associate a map $Q_W : C_c^\infty(W;\Omega^{1/2}W) \to C^*(\mathcal{U})$, where $\Omega^{1/2}W$ is the bundle of half densities on $\ker ds \times \ker dt$. The image $\bigoplus_{i\in I} Q_{U_i}(C_c^\infty(U_i;\Omega^{1/2}U_i))$ is a dense $*$-subalgebra of $C^*(\mathcal{U})$.

When $\mathcal{U}$ is the minimal atlas this algebra is denoted by $C^*(M,\mathcal{F})$.

In [1, §5] it was shown that the representations of the full C^*-algebra of an atlas on a Hilbert space correspond to representations of the associated groupoid on a Hilbert bundle (desintegration theorem). We are going to use this correspondence in this sequel, so let us recall the explicit definition of these groupoid representations:

Definition 1.7. Let $\mathcal{U} = \{(U_i,t_i,s_i)\}_{i\in I}$ be an atlas of the foliation $(M,\mathcal{F})$. A representation of $G(\mathcal{U})$ is a triple (μ, H, χ) where:

a) μ is a quasi-invariant measure on M. Namely, for every $(U,t,s) \in \mathcal{U}$ and positive Borel sections λ^s of $\Omega^1(\ker ds)$ and λ^t of $\Omega^1(\ker dt)$, the measures $\mu\circ\lambda^s$ and $\mu\circ\lambda^t$ are equivalent.

b) $H = (H_x)_{x\in M}$ is a μ-measurable field of Hilbert spaces over M.

c) $\chi = \{\chi^U\}_{\mathcal{U}}$ is a family of $\mu\circ\lambda$-measurable sections of the field of unitaries $\pi_u^U : H_{s(u)} \to H_{t(u)}$. Moreover χ is a homomorphism defined in $G(\mathcal{U})$. That is to say:

   - if $f : U \to U'$ is a morphism of bi-submersions then $\chi^{U'}_{f(u)} = \chi^U_u$ for almost all $u \in U$, and
   - $\chi^{U\circ V}_{(u,v)} = \chi^U_u \chi^V_v$ for almost all $(u,v) \in U \circ V$, for all bi-submersions U, V adapted to $\mathcal{U}$.

Remark 1.8. We fix an atlas $\mathcal{U}$ for our foliation $(M,\mathcal{F})$, which could as well be the smallest one - the *path homotopy atlas* and write $C^*(M,\mathcal{F})$ instead of $C^*(\mathcal{U})$. Actually, this is not an important issue, since we have a natural morphism $C^*(M,\mathcal{F}) \to C^*(\mathcal{U})$ for any atlas, and the index for $C^*(\mathcal{U})$ is just the push-forward by this morphism of the index for $C^*(M,\mathcal{F})$.

1.5 The extension of pseudodifferential operators of order 0

Let us recall briefly the pseudodifferential calculus we constructed in [2]: Consider a bi-submersion (U,t,s) with an identity bisection V. Let $N \to V$ be the normal bundle of V in U. Given a classical symbol $\alpha \in S^m_{cl,c}(V,N^*;\Omega^1 N)$ we can make sense of a distribution $P_\alpha(x) = (2\pi)^{-n}\int_{N_x^*} \alpha(x,\xi)e^{-i\langle u,\xi\rangle}$. Generalized functions with pseudodifferential singularities in V, are functions which near V are of the form P_a and away from V are smooth functions. Such functions are shown in [2, Thm 3.8] to be multipliers of $C^*(M,\mathcal{F})$. This way we obtain subalgebras $\Psi^m(M,\mathcal{F})$ of $M(C^*(M,\mathcal{F}))$. Regularizing operators are elements of

$\cap_{m\in\mathbb{Z}}\Psi^m(M,\mathcal{F})$, and elliptic operators are elements of $\Psi^m(M,\mathcal{F})$ which are invertible up to regularizing operators.

In fact, here we will use very little information on these operators, namely just the exact sequence of order 0 pseudodifferential calculus. This is an exact sequence of C^*-algebras

$$0 \to C^*(M,\mathcal{F}) \to \Psi(M,\mathcal{F}) \to B(M,\mathcal{F}) \to 0 \tag{1.1}$$

where $\Psi(M,\mathcal{F})$ is the C^*-algebra of the zero-order pseudodifferential operators and B is the commutative C^*-algebra of symbols of order 0. It is a quotient of $C_0(S\mathcal{F}^*)$ (the continuous functions on the cosphere "bundle").

2 The analytic index

The analytic index of elliptic pseudodifferential operators on a Lie groupoid G over M (*cf.* [9], [12], [13]) is a group morphism $K_0(C_0(A^*G)) \to K_0(C^*(G))$. It maps the class $[\sigma_P] \in K_0(C_0(A^*G))$ of the principal symbol of an elliptic pseudodifferential operator P to the index class $[P] \in K_0(C^*(G))$ of P. Note that the index class $[P]$ of P is just the class of P in the group $KK(\mathbb{C}, C^*(G)) \simeq K_0(C^*(G))$.

The exact sequence (1.1) of the pseudodifferential calculus, allows us to extend this construction to the framework of singular foliations. This map comes from an element $\mathrm{ind}_a \in KK(C_0(\mathcal{F}^*), C^*(M,\mathcal{F}))$. Since we wish to identify precisely this element with the one obtained using the "tangent groupoid", we will make precise this construction based on mapping cones and the identifications involved.

Let us also point out that the analytic index for smooth groupoids is sometimes presented as the connecting map associated with the exact sequence (1.1) (or the KK^1 element this exact sequence defines). Our presentation has two minor advantages:

- It is slightly more primitive since the element in $KK^1(C_0(S\mathcal{F}^*), C^*(M,\mathcal{F}))$ is in fact the composition of our ind_a with the KK^1 element corresponding with the obvious extension of $C_0(S\mathcal{F}^*)$ by $C_0(\mathcal{F}^*)$ (the one defined by the fiberwise compactification $\mathcal{F}^* \cup S\mathcal{F}^*$);
- Our element is slightly more tractable and has no sign problems since it only involves homomorphisms of C^*-algebras.

2.1 Requirements for an analytic index map

Let's assume first that M is compact. Let $P \in \Psi(M,\mathcal{F})$ be elliptic (of order 0) acting on sections of a $\mathbb{Z}/2\mathbb{Z}$-graded bundle E. The principal symbol of P is a a continuous family of isomorphisms $\sigma_P(x,\xi) : E_x^{(0)} \to E_x^{(1)}$ ($x \in M$, and $\xi \in \mathcal{F}_x \setminus \{0\}$) which is homogeneous ($\sigma(x,t\xi) = \sigma(x,\xi)$ for $t > 0$).

One may extend this symbol by changing it near the zero section *e.g.* putting $\sigma'(x,\xi) = \varphi(\|\xi\|)\sigma(x,\xi)$ where φ is continuous on $\mathbb{R}_+$, and satisfies $\varphi(0) = 0$ and $\lim\limits_{t\to\infty}\varphi(t) = 1$. This symbol is invertible modulo $C_0(\mathcal{F}^*)$ and thus defines an element $[\sigma_P] \in KK(\mathbb{C}, C_0(\mathcal{F}^*)) = K^0(\mathcal{F}^*)$.

On the other hand, P itself acts on the hilbert $C^*(M,\mathcal{F})$-module associated with E (this can be described as $C(M,E) \otimes_{C(M)} C^*(M,\mathcal{F})$) and is invertible modulo the compact operators and thus defines an element $[P] \in KK(\mathbb{C}, C^*(M,\mathcal{F})) = K_0(C^*(M,\mathcal{F}))$.

The analytic index we define here is an element $\mathrm{ind}_a \in KK(C_0(\mathcal{F}^*), C^*(M,\mathcal{F}))$ which maps $[\sigma_P]$ to $[P]$ *i.e.* satisfies $[\sigma_P] \otimes_{C_0(\mathcal{F}^*)} \mathrm{ind}_a = [P]$.

In the non compact case, the operators and symbols we are concerned with are *trivial at infinity*, *i.e.* $\sigma(x,\xi)$ is independent on ξ outside a compact set in M, and P is a multiplication operator outside a compact set. They still define K-theory elements $[\sigma_P] \in KK(\mathbb{C}, C_0(\mathcal{F}^*)) = K^0(\mathcal{F}^*)$ and $[P] \in KK(\mathbb{C}, C^*(M,\mathcal{F})) = K_0(C^*(M,\mathcal{F}))$.

2.2 Mapping cones and relative K-theory

The index map is quite nicely expressed in terms of relative K-theory and mapping cones. We briefly recall these constructions.

2.2.1 Mapping cones

We briefly recall here some facts about mapping cones and their use in KK-theory (*cf.* [11, 10]). Let $\varphi : A \to B$ a homomorphism of unital C^*-algebras.

- The mapping cone of φ is the C^*-algebra $\mathcal{C}_\varphi = \{(f,a) \in C_0([0,1);B) \times A;\ \varphi(a) = f(0)\}$.
- The cone of the C^*-algebra B is the contractible C^*-algebra $\mathcal{C}_B = \mathcal{C}_{\mathrm{id}_B} = C_0([0,1);B)$. If $\varphi : A \to B$ is onto, we have the exact sequence $0 \to \ker\varphi \xrightarrow{e} \mathcal{C}_\varphi \to \mathcal{C}_B \to 0$, where $e(x) = (0,x)$ for $x \in \ker\varphi$ is called the *excision* map. The 6-term exact sequence gives $K_0(\mathcal{C}_\varphi) = K_0(\ker\varphi)$. If φ admits a completely positive splitting, the element $[e] \in KK(\ker\varphi, \mathcal{C}_\varphi)$ is invertible.
- The "cone" construction is natural, namely a commutative diagram of C^*-algebra homomorphisms
$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "\varphi'"] & B' \end{tikzcd}$$
gives rise to a $*$-homomorphism $\mathcal{C}_\varphi \to \mathcal{C}_{\varphi'}$.
- Let $\varphi : Y \to Z$ be a proper map between locally compact spaces. The mapping cone of φ is
$$C_\varphi = Y \times [0,1) \cup Z/_\sim$$
where the equivalence relation is $(y,0) \sim \varphi(y)$ for all $y \in Y$. Abusing the notation, we write $\varphi : C_0(Z) \to C_0(Y)$ for the induced map. The cone of this φ is the algebra of continuous functions on the mapping cone, *i.e.* $\mathcal{C}_\varphi = C_0(C_\varphi)$.

2.2.2 Relative K-theory

Let $\varphi : A \to B$ be a homomorphism of unital C^*-algebras.

Recall that the group $K_0(\varphi)$ is given by generators and relations:

- Its generators are triples (e^+, e^-, u) where $e^+, e^- \in M_n(A)$ are idempotents and $u \in M_n(B)$ is such that $uv = \varphi(e^+)$ and $vu = \varphi(e^-)$ for some $v \in M_n(B)$.
- Addition is given by direct sums.
- Trivial elements are those triples (e^+, e^-, u) for which $u = \varphi(u_0)$ and $v = \varphi(v_0)$ for some $u_0, v_0 \in M_n(A)$ satisfying $u_0v_0 = e^+$ and $v_0u_0 = e^-$.
- The group $K_0(\varphi)$ is formed as the set of those triples divided by trivial triples, and homotopy - which is given by triples associated with the map $C([0,1]; A) \to C([0,1]; B)$ associated with φ.
- For non unital algebras / morphisms, we just put $K_0(\varphi) = K_0(\tilde{\varphi})$, where $\tilde{\varphi} : \tilde{A} \to \tilde{B}$ is obtained by adjoining units everywhere.

Note that $K_0(A) = K_0(A \to 0) = K_0(\epsilon_A)$ where $\epsilon_A : \tilde{A} \to \mathbb{C}$ is the morphism with kernel A.

If φ is onto, then $K_0(\varphi) = K_0(\ker \varphi)$. More precisely, the map $K_0(\epsilon_{\ker \varphi}) \to K_0(\varphi)$ induced by the commuting diagram

$$\begin{array}{ccc} \widetilde{\ker \varphi} & \xrightarrow{\epsilon} & \mathbb{C} \\ \downarrow & & \downarrow \\ \tilde{A} & \xrightarrow{\varphi} & \tilde{B} \end{array}$$

is an isomorphism. The inverse of this isomorphism is the *index map* of the exact sequence: Let (e^+, e^-, u) be a generator for $K_0(\varphi)$; let $w \in M_n(A)$ be such $\varphi(w) = u$; the image of (e^+, e^-, u) is the class of $(e_+(\ker \varphi)^n, e_-(\ker \varphi)^n, e_-we_+)$ in $KK(\mathbb{C}, \ker \varphi) = K_0(\ker \varphi)$.

Finally, there is a natural isomorphism $K_0(\varphi) \to K_0(\mathcal{C}_\varphi)$ which in case φ is onto, is compatible with the identifications of $K_0(\varphi)$ and of $K_0(\mathcal{C}_\varphi)$ with $K_0(\ker \varphi)$.

2.3 Construction of the analytic index

2.3.1 The analytic index as a morphism

Let us now come to the case of a (singular) foliation. Locally, $\mathcal{F}^*$ is a closed subspace of (the total space of) a vector bundle. We may choose a metric on this bundle; this will fix continuously a euclidian metric on each $\mathcal{F}_x^*$. This can even be done globally using partitions of the identity. Let then $S\mathcal{F}^*$ be the sphere "bundle" of $\mathcal{F}^*$, *i.e.* the space of half lines in $\mathcal{F}^*$ identified with the space of vectors of norm 1 in $\mathcal{F}^*$.

We obviously have:

Proposition 2.1. *Let $(M, \mathcal{F})$ be a foliation. The cone of the projection $p : S\mathcal{F}^* \to M$ is canonically isomorphic with $\mathcal{F}^*$.* □

Every function $f \in C_0(M)$ can be considered as the zero-order longitudinal pseudodifferential operator $m(f)$ which acts by multiplication on the algebra $\mathcal{A}_{\mathcal{U}}$. Its principal symbol is constant on covectors $\sigma(m(f))(x, \xi) = f(x)$ for $x \in M$ and $\xi \in \mathcal{F}_x^*$. In other words, we have a commutative diagram

$$\begin{tikzcd} & C_0(M) \arrow[r, "p"] \arrow[d, "m"'] & C_0(S\mathcal{F}^*) \arrow[d, "q"] & \\ 0 \to C^*(M, \mathcal{F}) \arrow[r] & \Psi(M, \mathcal{F}) \arrow[r, "\sigma"] & B(M, \mathcal{F}) \arrow[r] & 0 \end{tikzcd} \tag{2.1}$$

Proposition 2.2. *The morphism $\gamma : K_0(C_0(\mathcal{F}^*)) \simeq K_0(p) \to K_0(\sigma) \simeq K_0(C^*(M, F))$ is the analytic index map: the image of the class of an elliptic symbol $[\sigma_P] \in K_0(C_0(\mathcal{F}^*))$ is the class of the associated operator $[P] \in K_0(C^*(M, F))$.*

Proof. A symbol of an elliptic operator of order 0 is given by two bundles $E_\pm$ and an isomorphism α of the pull-back bundles $p^*(E_\pm)$, where $p : S^*\mathcal{F} \to M$ is the projection. It therefore defines an element of $K_0(p) \simeq K_0(C_0(\mathcal{F}^*))$. The pseudodifferential operator P_α with symbol α satisfies $\sigma(P_\alpha) = \alpha$ and is therefore the class of the image of $(E_\pm, \alpha)$ in $K_0(\sigma)$ under the isomorphism $K_0(\sigma) \simeq K_0(\ker \sigma)$ described above. □

2.3.2 The analytic index as a KK-element

Denote by $\varphi : C_0(\mathcal{F}^*) \simeq \mathcal{C}_p \to \mathcal{C}_\sigma$ the homomorphism induced to the commuting square

$$\begin{tikzcd} C_0(M) \arrow[r, "p"] \arrow[d, "m"'] & C_0(S\mathcal{F}^*) \arrow[d, "q"] \\ \Psi(M, \mathcal{F}) \arrow[r, "\sigma"] & B(M, \mathcal{F}) \end{tikzcd}$$

Denote by $e : C^*(M, \mathcal{F}) \to \mathcal{C}_\sigma$ the "excision" map associated with the exact sequence (1.1). The compatibility of relative K-theory with the mapping cone construction gives rise to a commuting diagram:

$$\begin{tikzcd} K_0(p) \arrow[r, "\sim"] \arrow[d] & K_0(\mathcal{C}_p) \arrow[d, "\varphi"] \\ K_0(\sigma) \arrow[r, "\sim"] & K_0(\mathcal{C}_\sigma) \end{tikzcd}$$

It follows that the index map is just the composition $\gamma = e_*^{-1} \circ \varphi_*$.

Now the excision map e is a KK-equivalence since σ admits a completely positive cross-section (as the algebra $B(M, \mathcal{F})$ is abelian).

Definition 2.3. The analytic index is the element

$$\mathrm{ind}_a = [\varphi] \otimes_{\mathcal{C}_\sigma} [e]^{-1} \in KK(C_0(\mathcal{F}^*), C^*(M, \mathcal{F})).$$

Corollary 2.4. *The K-theory morphism associated with* ind_a *is the analytic index.*

Proof. We just saw that $\gamma = \cdot \otimes \mathrm{ind}_a$. The result follows from prop. 2.2. $\square$

Remark 2.5 (The element of KK^1 associated with the extension (1.1))**.** Let $0 \to J \to A \xrightarrow{p} A/J \to 0$ be an exact sequence of C^*-algebras. Assume that the morphism p admits a completely positive section. Consider the morphisms $e : J \to \mathcal{C}_p$ and $j : C_0((0,1); A/J) \to \mathcal{C}_p$ given by $e(x) = (0, x)$ and $j(f) = (f, 0)$. Recall that the element of $KK_1(A/J, J) = KK(C_0((0,1); A/J), J)$ associated with the exact sequence above is the composition $[j] \otimes_{\mathcal{C}_p} [e]^{-1}$.

It follows that the element of $KK^1(C_0(S\mathcal{F}^*), C^*(M, \mathcal{F}))$ associated with the exact sequence (1.1) is just $j^*(\mathrm{ind}_a)$ where $j : C_0(S\mathcal{F}^* \times (0,1)) \simeq C_0(\mathcal{F}^* \setminus M) \to C_0(\mathcal{F}^*)$ is the inclusion.

In this way, the element ind_a we just constructed is more primitive than the KK^1 element associated with extension (1.1).

3 The tangent groupoid

Recall (*cf.* [12]) that the index of the elliptic pseudodifferential operators associated with a Lie groupoid G over M may also be constructed from a deformation of G: Namely, naturally associated to G is the "adiabatic groupoid", which setwise is $G^T = G \times (0,1] \cup AG \times \{0\}$. This set admits a certain smooth structure making it a Lie groupoid over the manifold $M \times [0,1]$. Its C^*-algebra turns out to be an extension $0 \to C^*(G) \otimes C_0((0,1]) \to C^*(G^T) \xrightarrow{ev_0} C_0(A^*G) \to 0$. Passing to K-theory, this extension gives rise to a morphism $[ev_1] \circ [ev_0]^{-1} \colon K_0(C_0(A^*G)) \to K_0(C^*(G))$. This is exactly the analytic index map. Applying this in the case of the holonomy groupoid $H(F)$ of a regular foliation F, we recover the analytic index map of longitudinal elliptic pseudodifferential operators.

In this section we wish to generalize this construction to singular foliations. We start by constructing the *adiabatic foliation* associated to a foliation $\mathcal{F}$. This is going to be a foliation on $M \times \mathbb{R}$. The holonomy groupoid of this foliation is $\bigcup_{x \in M} \mathcal{F}_x \times \{0\} \cup \mathcal{G} \times \mathbb{R}^*$. It is called the *tangent groupoid.* Its C^*-algebra contains as an ideal $C_0(\mathbb{R}^*) \otimes C^*(M; \mathcal{F})$ with quotient $C_0(\mathcal{F}^*)$.

This tangent groupoid allows to construct an element in $KK(C_0(\mathcal{F}^*), C^*(M; \mathcal{F}))$. As in the case of [12], we will show that this element coincides with the analytic index.

3.1 The "adiabatic foliation"

Let $\lambda : M \times \mathbb{R} \to \mathbb{R}$ be the second projection and $J = \lambda C_c^\infty(M \times \mathbb{R})$ the set of smooth compactly supported functions on $M \times \mathbb{R}$ which vanish on $M \times \{0\}$. Every vector field X of M extends to a vector field in $M \times \mathbb{R}$ tangent along M, which we will denote $X \otimes 1$.

We let $\mathcal{TF}$ to be the submodule of $C_c^\infty(M \times \mathbb{R}; TM \times \mathbb{R})$ generated by $\lambda(\mathcal{F} \otimes 1)$: it is the set of finite sums $\sum f_i(X_i \otimes 1)$ where $f_i \in J$ and $X_i \in \mathcal{F}$.

Proposition 3.1. *$\mathcal{TF}$ is a foliation on $M \times \mathbb{R}$.*

Proof.
- Let U be an open subset of M over which $\mathcal{F}$ is generated by vector fields $X_1, \ldots, X_k$. On $U \times \mathbb{R}$, $\mathcal{TF}$ is generated by the vector fields $\lambda(X_1 \otimes 1), \ldots, \lambda(X_k \otimes 1)$. It follows that $\mathcal{TF}$ is locally finitely generated.

- If $f, g \in J$ and $X, Y \in \mathcal{F}$, we find

  $$[f(X \otimes 1), g(Y \otimes 1)] = f(X \otimes 1)(g)(Y \otimes 1) - g(Y \otimes 1)(f)(X \otimes 1) + fg([X,Y] \otimes 1) \in \mathcal{TF}.$$

  It follows that $\mathcal{TF}$ is integrable. □

Definition 3.2. The foliation $(M \times \mathbb{R}, \mathcal{TF})$ is called the *adiabatic foliation* associated with $\mathcal{F}$.

Remark 3.3. Recall ([1, Def. 1.2]) that associated with a foliation $(M_0, \mathcal{F}_0)$ are two families of vector spaces indexed by M_0: the space tangent to the leaf and the fiber of the module $\mathcal{F}_0$.

The tangent subspace to the leaves at a point (x, β) is $F_x \times \{0\}$ for $\beta \in \mathbb{R}^*$ and $\{(0,0)\}$ if $\beta = 0$. On the other hand, the module $\mathcal{TF}$ is isomorphic (via multiplication by λ to the module $\mathcal{F} \otimes 1$. In particular, these modules have the same fibers. It thus follows that $\mathcal{TF}_{(x,\beta)} \simeq \mathcal{F}_x$ for all $\beta \in \mathbb{R}$.

Also the total space of the cotangent "bundle" is $\mathcal{TF}^* = \mathcal{F}^* \times \mathbb{R}$.

3.2 The holonomy groupoid of the adiabatic foliation

In order to describe a natural family of bi-submersions associated with $\mathcal{TF}$, we will use the classical construction of *deformation to the normal cone.* A complete account of this construction can be found *e.g.* in [6]. We just recall here a few facts about this construction:

- Let U be a smooth manifold and V a (locally closed) smooth submanifold of U. The *deformation to the normal cone* of U along V is a smooth manifold $D(U,V)$ which set-theoretically is $U \times \mathbb{R}^* \cup N \times \{0\}$ where N is the (total space of the) normal bundle of the inclusion $V \subset U$.

- This construction is functorial (*cf.* [6, 3.4]). Namely, if (U,V) and (U',V') are pairs of a manifold and a submanifold, a smooth map $p : (U,V) \to (U',V')$ such that $p(V) \subset V'$ induces a (unique) smooth map $\tilde{p} : D(U,V) \to D(U',V')$ defined by $\tilde{p} = (p, \mathrm{id})$ on $U \times \mathbb{R}^*$ and $\tilde{p}(x,n,0) = (p(x), d_N(p_x), 0)$ for every $(n,0) \in N \times \{0\}$. Here $d_N p_x$ is by definition the map $(N)_x \to (N')_{p(x)}$.

  The map $\tilde{p} : D(U,V) \to D(U',V')$ is a submersion if and only if the map $p : U \to U'$ and its restriction $p_V : V \to V'$ are submersions.

- Let us already notice that there is a smooth map $q : D(U,V) \to U \times \mathbb{R} \,(= D(U,U))$ which is the identity (and a diffeomorphism) on $U \times \mathbb{R}^*$ and such that $q(y,0) = (p(y),0)$ for $y \in N$ where $p : N \to V$ is the bundle projection.

Proposition 3.4. *Let (U,t,s) be a bi-submersion for $\mathcal{F}$ and $V \subset U$ be a closed identity bisection.*

a) *Then $(D(U,V), t\circ q, s \circ q)$ is a bi-submersion for the adiabatic foliation $(M \times \mathbb{R}, \mathcal{TF})$.*

b) *If (U',t',s') is a bi-submersion adapted to (U,t,s) and $V' \subset U'$ is any closed identity bisection of U' such that $s'(V') \subset s(V)$, then $D(U',V')$ is adapted to $D(U,V)$.*

Proof. a) The maps $s \circ q$ and $t \circ q$ are the maps $\tilde{s}, \tilde{t}$ from $D(U,V)$ to $D(M,M) = M \times \mathbb{R}$ associated with the smooth submersions s and t, whose restrictions to V are étale. It follows that they are smooth submersions.

The assertion is local: we may restrict to a small open neighborhood of a given point $u \in U$. The restriction to the open set $U \setminus V$ is easy: we are dealing with the maps $s \times \mathrm{id}_{\mathbb{R}^*}, t \times \mathrm{id}_{\mathbb{R}^*} : (U \setminus V) \times \mathbb{R}^* \to M \times \mathbb{R}^*$ which is easily seen to be a bi-submersion for the product foliation $\mathcal{F} \otimes 1$. Now, the restrictions to the open set on $M \times \mathbb{R}^* \subset M \times \mathbb{R}$ of $\mathcal{TF}$ and $\mathcal{F} \otimes 1$ coincide.

Take now an open neighborhood of a point $v \in V$. We may therefore assume that V is an open subset in M, U is an open subset in $V \times \mathbb{R}^k$, and $s(v,\alpha) = v$. Restricting to an even smaller neighborhood of v if necessary, we may further assume that $t(U) \subset V$ and that t has also a product decomposition. Therefore, $\ker dt$ is spanned by vector fields $(Y_1, \ldots, Y_k)$. Decompose each of these vector fields as $Y_i = (Z_i, Z_i')$, where Z_i is tangent along V and Z_i' is tangent along $\mathbb{R}^k$. The fact that U is a bi-submersion means exactly that the Z_i generate the foliation $\mathcal{F} \otimes 1$ on U.

Now $D(U,V)$ identifies with $\{(v,\alpha,\beta) \in V \times \mathbb{R}^k \times \mathbb{R};\ (v, \beta\alpha) \in U\}$; under this identification, $\tilde{s}(v,\alpha,\beta) = (v,\beta)$ and $\tilde{t}(v,\alpha,\beta) = (t(v,\beta\alpha),\beta)$. It follows that $d\tilde{t}_{(v,\alpha,\beta)}(Z,Z',0) = (dt_{(v,\beta\alpha)}(Z,\beta Z'),0)$ (for Z tangent along V and Z' along $\mathbb{R}^k$), whence $\ker d\tilde{t}$ is spanned by $(\tilde{Y}_1, \ldots, \tilde{Y}_k)$ where $\tilde{Y}_i = (\tilde{Z}_i, \tilde{Z}_i', 0)$, where the vector fields $\tilde{Z}_i$ and $\tilde{Z}_i'$ are defined by $\tilde{Z}_i(v,\alpha,\beta) = \beta Z_i(v,\alpha\beta)$ and $\tilde{Z}_i'(v,\alpha,\beta) = Z_i'(v,\alpha\beta)$. It follows that $\ker d\tilde{s} \oplus \ker d\tilde{t}$ is the set of vector fields $(Z,Z',0)$ where Z' is any section of $\mathbb{R}^k$ (this is $\ker d\tilde{s}$) and Z is in the module spanned by $\tilde{Z}_i$.

This proves that $\tilde{s}^{-1}(\mathcal{F}) = \ker d\tilde{s} \oplus \ker d\tilde{t}$. Exchanging the roles of s and t, we get the equality $\tilde{t}^{-1}(\mathcal{F}) = \ker d\tilde{s} \oplus \ker d\tilde{t}$.

b) We may again consider two cases: the case where V' is empty and the case where we deal with a small neighborhood of $v' \in V'$. It follows from [2, Prop. 2.5], that there is a (local) morphism of bi-submersions mapping V' to V. In both cases, we may assume that we have a morphism of bi-submersions $f : (U',t',s') \to (U,t,s)$ such that $f(V') \subset V$. Then associated to f is a smooth map $\tilde{f} : (\tilde{U}', \tilde{t}', \tilde{s}') \to (\tilde{U}, \tilde{t}, \tilde{s})$ which is a morphism of bi-submersions. ∎

Proposition 3.5. *Let $\mathcal{U} = (U_i, t_i, s_i)_{i\in I}$ be an atlas for $(M, \mathcal{F})$ and let $V_i \subset U_i$ be identity bi-sections* ([4])*. Assume that $\bigcup_{i\in I} s_i(V_i) = M$.*

a) *Then $\widetilde{\mathcal{U}} = (D(U_i, V_i), t_i \circ q_i, s_i \circ q_i)_{i\in I}$ is an atlas for $(M \times \mathbb{R}, \mathcal{TF})$.*

b) *If moreover $\mathcal{U}$ is the path holonomy atlas for $(M, \mathcal{F})$, then $\widetilde{\mathcal{U}}$ is the path holonomy atlas for $(M \times \mathbb{R}, \widetilde{\mathcal{F}})$.*

Proof. a) Since $\tilde{s}(V_i \times \mathbb{R}) = V_i \times \mathbb{R}$, it follows that $\bigcup_{i\in I} \tilde{s}_i(D(U_i; V_i)) = M \times \mathbb{R}$.

Let (U, t, s) be a bi-submersion adapted to $\mathcal{U}$ and $V \subset U$ be a closed identity bisection. It follows from prop. 3.4.b) that $(D(U, V), \tilde{t}, \tilde{s})$ is adapted to $\widetilde{\mathcal{U}}$.

It follows that the inverse $(D(U, V), \tilde{s}, \tilde{t})$ of $(D(U, V), \tilde{t}, \tilde{s})$ is adapted to $\tilde{\mathcal{U}}$ since (U, s, t) is adapted to $\mathcal{U}$.

If (U', t', s') is another bi-submersion adapted to $\mathcal{U}$ and $V' \subset U'$ is a closed identity bisection, one easily identifies the composition $(D(U, V), \tilde{s}, \tilde{t}) \circ (D(U', V'), \tilde{s}', \tilde{t}')$ with the bi-submersion $(D(U \circ U', V \circ V', (t \times \mathrm{id}) \circ q, (s \times \mathrm{id}) \circ q)$.

b) From the above arguments, it follows that if $\mathcal{U}$ is generated by a subfamily $(U_i)_{i\in J}$ such that $\bigcup_{i\in J} s_i(V_i) = M$, then $\tilde{\mathcal{U}}$ is generated by $(D(U_i, V_i))_{i\in J}$. Now, the path holonomy atlas is generated by a family (U_i, s_i, t_i) of bi-submersions with identity bisections V_i such that $s_i(U_i) = t_i(U_i) = s_i(V_i)$ and with connected fibers. Then $(D(U_i, V_i), \tilde{t}_i, \tilde{s}_i)$ satisfies the same properties. It generates the path holonomy atlas for $(M \times \mathbb{R}, \widetilde{\mathcal{F}})$. □

Proposition 3.6. *Let $\mathcal{U}$ be an atlas for $(M, \mathcal{F})$ and $\tilde{\mathcal{U}}$ the corresponding atlas for $(M \times \mathbb{R}, \mathcal{TF})$. The groupoid of the atlas $\tilde{\mathcal{U}}$ naturally identifies with $\bigcup_{x\in M} \mathcal{F}_x \times \{0\} \cup \mathcal{G}(\mathcal{U}) \times \mathbb{R}^*$.*

Proof. Since the equivalence relation defining $\mathcal{G}(\tilde{\mathcal{U}})$ respects the source and target maps, we find by composition with the second projection a well defined map $\tau : \mathcal{G}(\tilde{\mathcal{U}}) \to \mathbb{R}$. Therefore $G(\tilde{\mathcal{U}})$ is the union $\tau^{-1}(\mathbb{R}^*) \bigcup \tau^{-1}(\{0\})$. We conclude by identifying $\tau^{-1}(\mathbb{R}^*)$ with $G(\mathcal{U}) \times \mathbb{R}^*$ and $\tau^{-1}(\{0\})$ with $\bigcup_{x\in M} \mathcal{F}_x$.

a) Let (W, t, s) be a bi-submersion adapted to $\tilde{\mathcal{U}}$. For $\beta \in \mathbb{R}$, put $W_\beta = s^{-1}(M \times \{\beta\})$. For $\beta \neq 0$, by restriction of t, s to W_β we get a bi-submersion $(W_\beta, t_\beta, s_\beta)$ adapted to $\mathcal{U}$. Also if (W', t', s') is adapted to (W, t, s) then the restriction $(W'_\beta, t'_\beta, s'_\beta)$ is adapted to $(W_\beta, t_\beta, s_\beta)$. We have constructed a map $P_{\mathbb{R}^*} : \tau^{-1}(\mathbb{R}^*) \to G(\mathcal{U}) \times \mathbb{R}^*$.

Let (U, t, s) be a bi-submersion adapted to the atlas $\mathcal{U}$. Putting $V = \emptyset$, we find a bi-submersion $(U \times \mathbb{R}^*, t \times \mathrm{id}_{\mathbb{R}^*}, s \times \mathrm{id}_{\mathbb{R}^*})$ adapted to $\tilde{\mathcal{U}}$. Also if (U', t', s') is adapted

[4] Some of the V_i's may be empty

to (U,t,s) then $(U'\times\mathbb{R}^*, t'\times \mathrm{id}_{\mathbb{R}^*}, s'\times \mathrm{id}_{\mathbb{R}^*})$ is adapted to $(U\times\mathbb{R}^*, t\times \mathrm{id}_{\mathbb{R}^*}, s\times \mathrm{id}_{\mathbb{R}^*})$. This way we construct a map $G(\mathcal{U})\times\mathbb{R}^*\to\tau^{-1}(\mathbb{R}^*)$, which is easily seen to be inverse to $P_{\mathbb{R}^*}$.

b) Let also $V\subset U$ be an identity bisection. Assuming that s is injective on V, we identify V with its image in M which is an open subset of M. Consider the map $dt-ds$ which to a vector field $\xi\in C_c^\infty(V;TU)$ associates the vector field $dt(\xi)-ds(\xi)\in C_c^\infty(V;TM)$. By definition of a bi-submersion, its range lies in $\mathcal{F}$. Note that since ds and dt coincide for vectors along V, $(dt-ds)(\xi)$ only depends on the normal part of ξ, *i.e.* its image in $C_c^\infty(V;NV)=C_c^\infty(V;TU/TV)$; we get in this way a map $\Phi: C_c^\infty(V;NV)\to\mathcal{F}$ which is $C^\infty(M)$ linear - *i.e.* a module map. At each point of V, we get a map between the fibers $q_x^V: N_xV\to\mathcal{F}_x$. Again, if (U',t',s') is another bi-submersion carrying the identity at x and V' is an identity bi-section through x, then we have a morphism $j_x: N_xV'\to N_xV$ and it is easily seen that $q_x^{V'}=q_x^V\circ j_x$, so that we constructed a map $G(\tilde{U})_{(x,0)}\to\mathcal{F}_x$.

We have constructed a map $P_0:\tau^{-1}(\{0\})\to\bigcup_{x\in M}\mathcal{F}_x$.

The image of the map Φ is (again by definition of bi-submersions) the space $C_c^\infty(V).\mathcal{F}$ of elements of $\mathcal{F}$ with support in V. It follows that P_0 is onto.

Now, if U is minimal at x, then q_x is injective, and it follows that P_0 is injective. $\square$

3.3 The short exact sequence

As explained above (page 6), from now on we fix an atlas $\mathcal{U}$ for $(M,\mathcal{F})$ and the corresponding atlas $\tilde{\mathcal{U}}$ for $(M\times\mathbb{R},\mathcal{TF})$. All bi-submersions considered here are assumed to be adapted to this atlas. Also what we call $C^*(M,\mathcal{F})$ and $C^*(M\times\mathbb{R},\mathcal{TF})$ are in fact the (full) C^*-algebras associated with these atlases.

Here we construct a short exact sequence of C^*-algebras

$$0\to C_0(\mathbb{R}^*)\otimes C^*(M,\mathcal{F})\xrightarrow{j} C^*(M\times\mathbb{R},\mathcal{TF})\xrightarrow{\pi} C_0(\mathcal{F}^*)\to 0 \tag{3.1}$$

We first identify $C_0(\mathbb{R}^*)\otimes C^*(M,\mathcal{F})$ with an ideal in $C^*(M\times\mathbb{R},\mathcal{TF})$, then construct the homomorphism π, show that it is onto, and finally identify the kernel of π with the image of $C_0(\mathbb{R}^*)\otimes C^*(M,\mathcal{F})$.

3.3.1 Construction of j

The C^*-algebra of the restriction of $\mathcal{TF}$ to $M\times\mathbb{R}^*$ is an ideal J in $C^*(M\times\mathbb{R};\mathcal{TF})$. Now, the restriction of $\mathcal{TF}$ to $M\times\mathbb{R}^*$ coincides with $\mathcal{F}\otimes 1$. Evaluation at each point β of $\mathbb{R}^*$, gives a map: $\mathrm{ev}_\beta: J\to C^*(M,\mathcal{F})$. By density of $C_c^\infty(U)\otimes C_c^\infty(\mathbb{R}^*)$ in $C_c^\infty(U\times\mathbb{R}^*)$ with respect to the L^1-estimate ([1, §4.4]), it follows that for every $x\in J$, the map $\beta\mapsto\mathrm{ev}_\beta(x)$ is continuous. In this way, we constructed a $*$-homomorphism $J\to C_0(\mathbb{R}^*)\otimes C^*(M,\mathcal{F})$. Using again functions in $C_c^\infty(U)\otimes C_c^\infty(\mathbb{R}^*)$, it follows that this map is onto.

To show that it is injective, we have to show that every irreducible representation θ of J factors through $C_0(\mathbb{R}^*) \otimes C^*(M, \mathcal{F})$. But representations of $J = C^*(M \times \mathbb{R}^*; \mathcal{F} \otimes 1)$ were described in [1, §5] and in particular they give rise to a measure on $M \times \mathbb{R}^*$. Denote by $\overline{\theta}$ the extension of θ to the multipliers. Since $C_0(\mathbb{R}^*)$ lives in the center of the multipliers of J, $\overline{\theta}(C_0(\mathbb{R}^*))$ lies in the center of the bi-commutant of θ, and is therefore a scalar. In other words, $\overline{\theta}$ is a character of $C_0(\mathbb{R}^*)$. It follows that there exists $\beta \in \mathbb{R}$ such that this measure is carried by $M \times \{\beta\}$. The representation θ is really a representation of the groupoid $G(\mathcal{U}) \times \mathbb{R}^*$, and since the corresponding measure is carried by $M \times \{\beta\}$ it is in fact a representation of the groupoid $G(\mathcal{U}) \times \{\beta\}$. It follows, that θ is of the form $\theta' \circ \mathrm{ev}_\beta$.

3.3.2 Construction of π.

Let (U, t, s) be a bi-submersion for $(M, \mathcal{F})$ and V an identity bisection. Put $\widetilde{U} = D(U, V)$. We define a map $\varpi_{(U,V)} : C_c^\infty(\widetilde{U}; \Omega^{1/2}\widetilde{U}) \to C_0(\mathcal{F}^*)$ as follows: Given $f \in C_c^\infty(\widetilde{U}; \Omega^{1/2}\widetilde{U})$,

- first restrict it to $f_0 \in C_c^\infty(NV \times \{0\}; \Omega^1 NV)$;
- then apply the Fourier transform to obtain $\widehat{f_0} \in C_0(N^*V)$;
- since $\mathcal{F}_V^* = \{(x, \xi);\ x \in s(V),\ \xi \in \mathcal{F}_x^*\}$ identifies with a closed subspace of N^*V, consider the restriction to this set and extend it by 0 outside $\mathcal{F}_V^*$ to get an element $\varpi_{(U,V)}(f) = \widehat{f_0}\,|_{\mathcal{F}^*} \in C_0(\mathcal{F}^*)$.

We next show that π is a well defined and surjective homomorphism.

To show that π is a well defined homomorphism, we just need to show that for every $x \in M$ and $\xi \in \mathcal{F}_x^*$ there is a well defined character $\hat{\chi}_{(x,\xi)}$ of $C^*(M \times \mathbb{R}; \mathcal{TF})$ such that the image of the class of $f \in C_c^\infty(\widetilde{U}; \Omega^{1/2}\widetilde{U})$ is $\varpi_{(U,V)}(f)(x, \xi)$. Now, ξ defines a one dimensional representation of the groupoid $G(\widetilde{\mathcal{U}})$ in the sense of [1, §5]: the corresponding measure is the Dirac measure $\delta_{(x,0)}$ on $M \times \mathbb{R}$, the Hilbert space is just $\mathbb{C}$, and $\chi_\xi(x, X) = e^{-i\langle X|\xi\rangle}$ for $X \in \mathcal{F}_x$ (the rest of the groupoid being of measure 0, the value of χ_ξ on an element which is not of the form (x, X) doesn't matter). It is now an elementary calculation to see that the image of f under the character $\hat{\chi}_{(x,\xi)}$ of $C^*(M \times \mathbb{R}; \mathcal{TF})$ corresponding to $(\delta_{(x,0)}, \mathbb{C}, \chi_\xi)$ is $\varpi_{(U,V)}(f)(x, \xi)$.

To show that π is onto, first note that the map $f \mapsto f_0$ is surjective from $C_c^\infty(\widetilde{U}; \Omega^{1/2}\widetilde{U})$ to $C_c^\infty(NV \times \{0\}; \Omega^1 NV)$, and that the Fourier transform has then dense range in $C_0(N^*(V))$. Whence the closure of the image by $\varpi_{(U,V)}$ of $C_c^\infty(\widetilde{U}; \Omega^{1/2}\widetilde{U})$ is the set of functions on $\mathcal{F}^*$ which vanish outside the open set $\mathcal{F}_V$. Since the $s(V_i)$ form an open cover of M, these sets form an open cover of $\mathcal{F}^*$. It follows that π is surjective.

3.3.3 Exactness

We come to the main result of this section:

Theorem 3.7. *The sequence* (3.1) *namely*

$$0 \to C_0(\mathbb{R}^*) \otimes C^*(M, \mathcal{F}) \xrightarrow{j} C^*(M \times \mathbb{R}, \mathcal{TF}) \xrightarrow{\pi} C_0(\mathcal{F}^*) \to 0$$

is exact

Proof. We already showed that j is injective and π is surjective. One sees also easily that $\pi \circ j = 0$.

Put $A = C^*(M \times \mathbb{R}, \mathcal{TF})$ and $J = j(C_0(\mathbb{R}^*) \otimes C^*(M, \mathcal{F}))$.

Let $\tilde{U} = D(U, V)$ be a bi-submersion and $f \in C_c^\infty(\widetilde{U}; \Omega^{1/2}\widetilde{U})$; if f vanishes in a neighborhood of $NV \times \{0\}$, its image in A lies in J; the same holds if f just vanishes on $NV \times \{0\}$ thanks to the L^1 estimate ([1, §4.4]). Indeed, f can then be approximated uniformly with fixed support by a sequence of elements which vanish near NV.

It suffices to show that every irreducible representation of A which vanishes on J also vanishes on $\ker \pi$. So we'll just show that every such irreducible representation θ is actually a point of $\mathcal{F}^*$. Extending θ to the multipliers, we find a representation $\overline{\theta}$ of $C_0(M \times \mathbb{R})$.

Take $f \in C_c^\infty(M \times \mathbb{R})$ and $g \in C_c^\infty(\widetilde{U})$, and put $h = (f \circ \widetilde{t})g - g(f \circ \widetilde{s}) \in C_c^\infty(\widetilde{U})$ which vanishes on $NV \times \{0\}$. Therefore $\overline{\theta}(f)\theta(g) - \theta(g)\overline{\theta}(f) = \theta(h) = 0$. It follows that $\overline{\theta}(C_0(M \times \mathbb{R}))$ lies in the commutant $\mathbb{C}1$ of θ and thus $\overline{\theta}$ is a character, *i.e.* a point $(x, \beta) \in M \times \mathbb{R}$. Now if f vanishes in a neighborhood of $M \times \{0\}$, then $fA \subset J$. It follows that $\beta = 0$.

By [1, §5] θ is an integrated form of a representation $(\mu, \mathcal{H}, \chi)$ of the groupoid $G(\tilde{\mathcal{U}})$. We just showed that the measure μ is a Dirac measure $\delta_{(x,0)}$, and it follows that χ is just a representation of $\mathcal{F}_x$ on the Hilbert space $\mathcal{H}$, whence a direct integral of characters χ_ξ. Therefore, θ is itself a direct integral of characters $\hat{\chi}_{(x,\xi)}$. Since it is irreducible it coincides with a character $\hat{\chi}_{(x,\xi)}$. □

Remark 3.8. Since $C^*(M \times \mathbb{R}; \mathcal{TF})$ is a $C_0(\mathbb{R})$ algebra it restricts to any locally closed subset of $\mathbb{R}$. If $Y = T_1 \setminus T_2$ where $T_2 \subset T_1$ are open sets of $\mathbb{R}$, one puts $C^*(M \times \mathbb{R}; \mathcal{TF})_Y = C_0(T_1)C^*(M \times \mathbb{R}; \mathcal{TF})/C_0(T_2)C^*(M \times \mathbb{R}; \mathcal{TF})$.

Restricting extension (3.1) to $[0, 1]$, we get an exact sequence:

$$0 \to C_0((0, 1]) \otimes C^*(M, \mathcal{F}) \to C^*(M \times \mathbb{R}, \mathcal{TF})_{[0,1]} \xrightarrow{\mathrm{ev}_0} C_0(\mathcal{F}^*) \to 0 \tag{3.2}$$

4 The analytic index via the tangent groupoid

The tangent groupoid exact sequence (3.2) gives rise to an element in $KK(C_0(\mathcal{F}^*), C^*(M, \mathcal{F}))$ and we will show that this element coincides with the analytic index element.

Indeed, since $C_0(\mathcal{F}^*)$ is abelian, the exact sequence (3.2) is semi-split. Moreover the kernel of the homomorphism ev_0 is the contractible C^*-algebra $C_0((0, 1]) \otimes C^*(M, \mathcal{F})$, so that the element $[\mathrm{ev}_0] \in KK(C^*(M \times \mathbb{R}, \mathcal{TF})_{[0,1]}, C_0(\mathcal{F}^*))$ is invertible.

Evaluation at 1 is a morphism $\mathrm{ev}_1 : C^*(M \times \mathbb{R}, \mathcal{TF})_{[0,1]} \to C^*(M, \mathcal{F})$.

The main result in this section is:

Theorem 4.1. *We have the equality* $\mathrm{ind}_a = [\mathrm{ev}_0]^{-1} \otimes [\mathrm{ev}_0] \in KK(C_0(\mathcal{F}^*), C^*(M, \mathcal{F}))$.

Proof. The restriction to $[0,1]$ of the exact sequence of pseudodifferential operators on $\mathcal{TF}$ is written as follows:

$$0 \to C^*(M \times \mathbb{R}; \mathcal{TF})_{[0,1]} \to \Psi(M \times \mathbb{R}; \mathcal{TF})_{[0,1]} \to B_{[0,1]} \to 0.$$

Here $B_{[0,1]}$ is a quotient of $C_0(S^*\mathcal{F} \times [0,1])$.

Extending functions on $S\mathcal{F}^*$ to $S\mathcal{F}^* \times [0,1]$ (by taking them independent on the variable in $[0,1]$) we get a morphism $C_0(S\mathcal{F}^*) \to B_{[0,1]}$. Also, considering multiplication by functions on M as pseudo differential elements we get a diagram

$$\begin{tikzcd}
 & & C_0(M) \arrow[r, "p"] \arrow[d, "\tilde{m}"'] & C_0(S\mathcal{F}^*) \arrow[d, "\tilde{q}"] & \\
0 \arrow[r] & C^*(M \times \mathbb{R}, \mathcal{TF})_{[0,1]} \arrow[r] & \Psi(M \times \mathbb{R}; \mathcal{TF})_{[0,1]} \arrow[r, "\tilde{\sigma}"] & B_{[0,1]} \arrow[r] & 0
\end{tikzcd} \tag{4.1}$$

from which we get a morphism $\tilde{\varphi} : C_0(\mathcal{F}^*) \simeq \mathcal{C}_p \to \mathcal{C}_{\tilde{\sigma}}$ and a KK-element

$$\widetilde{\mathrm{ind}_a} = [\tilde{\varphi}] \otimes_{\mathcal{C}_{\tilde{\sigma}}} [\tilde{e}]^{-1} \in KK(C_0(\mathcal{F}^*), C^*(M \times \mathbb{R}, \mathcal{TF})_{[0,1]})$$

where $\tilde{e} : C^*(M \times \mathbb{R}, \mathcal{TF})_{[0,1]} \to \mathcal{C}_{\tilde{\sigma}}$ is the excision morphism.

The theorem is an immediate consequence of the two following facts:

Claim 1. $(\mathrm{ev}_1)_*(\widetilde{\mathrm{ind}_a}) = \mathrm{ind}_a$

Claim 2. $\widetilde{\mathrm{ind}_a} = [\mathrm{ev}_0]^{-1}$.

Proof of Claim 1. Evaluation at 1 gives the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & C^*(M \times \mathbb{R}, \mathcal{TF})_{[0,1]} \arrow[r] \arrow[d, "\mathrm{ev}_1"] & \Psi(M \times \mathbb{R}; \mathcal{TF})_{[0,1]} \arrow[r, "\tilde{\sigma}"] \arrow[d, "\mathrm{ev}_1^\Psi"] & B_{[0,1]} \arrow[r] \arrow[d, "\mathrm{ev}_1^B"] & 0 \\
0 \arrow[r] & C^*(M, \mathcal{F}) \arrow[r] & \Psi(M, \mathcal{F}) \arrow[r, "\sigma"] & B_1 \arrow[r] & 0
\end{tikzcd}$$

from which we get a commutative diagram:

$$\begin{tikzcd}
C^*(M \times \mathbb{R}, \mathcal{TF})_{[0,1]} \arrow[r, "\tilde{e}"] \arrow[d, "\mathrm{ev}_1"'] & \mathcal{C}_{\tilde{\sigma}} \arrow[d, "\mathrm{ev}_1^{\mathcal{C}}"] \\
C^*(M, \mathcal{F}) \arrow[r, "e"] & \mathcal{C}_\sigma
\end{tikzcd} \tag{4.2}$$

Moreover, $\mathrm{ev}_1^B \circ \tilde{q} = q$ and $\mathrm{ev}_1^\Psi \circ \tilde{m} = m$, whence $\mathrm{ev}_1^{\mathcal{C}} \circ \tilde{\varphi} = \varphi$.

We thus have

$$\begin{aligned}(\mathrm{ev}_1)_*(\widetilde{\mathrm{ind}_a}) &= [\tilde\varphi]\otimes_{\mathcal{C}_{\tilde\sigma}}[\tilde e]^{-1}\otimes[\mathrm{ev}_1]=[\tilde\varphi]\otimes_{\mathcal{C}_{\tilde\sigma}}[\mathrm{ev}_1^{\mathcal{C}}]\otimes_{\mathcal{C}_\sigma}[e]^{-1}\\ &= [\mathrm{ev}_1^{\mathcal{C}}\circ\tilde\varphi]\otimes_{\mathcal{C}_\sigma}[e]^{-1}=\mathrm{ind}_a\end{aligned}$$

Proof of Claim 2. Evaluation at 0 gives the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & C^*(M\times\mathbb{R},\mathcal{TF})_{[0,1]} \arrow[r] \arrow[d, "\mathrm{ev}_0"] & \Psi(M\times\mathbb{R};\mathcal{TF})_{[0,1]} \arrow[r, "\widetilde\sigma"] \arrow[d, "\mathrm{ev}_0^\Psi"] & B_{[0,1]} \arrow[r] \arrow[d, "\mathrm{ev}_0^B"] & 0 \\ 0 \arrow[r] & C_0(\mathcal{F}^*) \arrow[r] & \Psi(M\times\mathbb{R};\mathcal{TF})_0 \arrow[r, "\sigma_0"] & B_0 \arrow[r] & 0 \end{tikzcd} \tag{4.3}$$

Let $x\in M$. Let (U,t,s) be a bi-submersion for $(M,\mathcal{F})$ and V an identity bisection such that $x\in s(V)$. Assume that U is minimal at x. Then $V\times\mathbb{R}\subset D(U,V)$ is an identity bi-section (for the foliation $(M\times\mathbb{R},\mathcal{TF})$). Let $P\in\mathcal{P}^0_c(D(U,V),V\times\mathbb{R};\Omega^{1/2})$ be a pseudodifferential distribution with compact support (*cf.* [2, §1.2.2]). From the definition of the pseudodifferential family, it follows that for $\xi\in\mathcal{F}_x$, we have $\hat\chi_{(x,\xi)}(P)=a(x,\xi,0)$ where a is a symbol of P. It follows that the algebra $\Psi(M\times\mathbb{R};\mathcal{TF})_0$ is the closure of order zero symbols, *i.e.* the algebra $C_0(\overline{\mathcal{F}^*})$ where $\overline{\mathcal{F}^*}$ denotes the closure of $\mathcal{F}^*$ by spheres at infinity (which is homeomorphic to the "bundle" of closed unit balls).

The bottom line in diagram (4.3) is

$$0\to C_0(\mathcal{F}^*)\to C_0(\overline{\mathcal{F}^*})\xrightarrow{p_0}C_0(S^*\mathcal{F})\to 0.$$

Moreover $\mathrm{ev}_0^B\circ\tilde q$ is the identity of $C_0(S\mathcal{F}^*)$. Therefore $(\mathrm{ev}_0)_*(\widetilde{\mathrm{ind}_a})=[e_0]^{-1}\otimes[\psi]$ where $e_0: C_0(\mathcal{F}^*)\to\mathcal{C}_{p_0}$ is the excision map and $\psi:\mathcal{C}_p\to\mathcal{C}_{p_0}$ is the morphism corresponding to the commutative diagram

$$\begin{tikzcd} C_0(M) \arrow[r, "p"] \arrow[d] & C_0(S^*\mathcal{F}) \arrow[d, equal] \\ C_0(\overline{\mathcal{F}^*}) \arrow[r, "p_0"] & C_0(S^*\mathcal{F}) \end{tikzcd}$$

But one easily identifies $\mathcal{C}_{p_0}$ with $C_0(\mathcal{F}^*)$ in such a way both ψ and e_0 are homotopic to the identity. It follows that $(\mathrm{ev}_0)_*(\widetilde{\mathrm{ind}_a})=1_{C_0(\mathcal{F}^*)}$. □

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