Pseudodifferential calculus on a singular foliation [1] [2]

by Iakovos Androulidakis [3] and Georges Skandalis

Mathematsches Institute, Universitaet Goettingen
Bunsenstrasse 3-5
D-37073 Goettingen
e-mail: iakovos@uni-math.gwdg.de

Institut de Mathématiques de Jussieu, UMR 7586
CNRS - Université Diderot - Paris 7
175, rue du Chevaleret, F–75013 Paris
e-mail: skandal@math.jussieu.fr

Abstract

In a previous paper ([1]), we associated a holonomy groupoid and a C^*-algebra to any singular foliation $(M, \mathcal{F})$. Using these, we construct the associated pseudodifferential calculus. This calculus gives meaning to a Laplace operator of any singular foliation $\mathcal{F}$ on a compact manifold M, and we show that it can be naturally understood as a positive, unbounded, self-adjoint operator on $L^2(M)$.

Introduction

This paper is a continuation of our previous paper [1]. There, we defined the holonomy groupoid of any singular foliation $\mathcal{F}$ on a smooth manifold M. Although this groupoid is a rather ill behaved object, we could define

- the convolution algebra $\mathcal{A}(M, \mathcal{F})$ of "smooth compactly supported" functions on this groupoid;
- the full and reduced C^*-algebra of the foliation which are suitable (Hausdorff) completions of this convolution algebra.

A key notion in [1] is that of a bi-submersion which will be also of importance here. This is loosely speaking a cover of an open subset of the holonomy groupoid. It is given by a manifold U with two submersions $s, t : U \to M$, each of which lifts the leaves of $\mathcal{F}$ to the fibers of s and t.

Here, we proceed and construct the longitudinal pseudodifferential calculus for our foliations.

[1] AMS subject classification: Primary 47G30, 57R30. Secondary 46L87.

[2] This research was supported in part by Fundação para a Ciência e a Tecnologia (FCT) through the Centro de Matemática da Universidade do Porto ⟨www.fc.up.pt/cmup⟩

[3] IA devotes this work to Odysseas' 0th birthday.

The longitudinal differential operators are very easily defined: They are generated by vector fields along the foliation.

The longitudinal pseudodifferential operators are obtained as images of distributions on bi-submersions with "pseudodifferential singularities" along a bisection:

Let (U,t,s) be a bi-submersion and $V \subset U$ an identity bisection. Denote by N the normal bundle to V in U and let a be a (classical) symbol on N^*. Let χ be a smooth function on U supported on a tubular neighborhood of V in U and let $\phi : U \to N$ be an inverse of the exponential map (defined on the neighborhood of V). A pseudodifferential kernel on U is a (generalized) function $k_a : u \mapsto \displaystyle\int a(p(u),\xi)\exp(i\phi(u)\xi)\chi(u)\,d\xi$ (here $p : U \to M$ is the composition $U \xrightarrow{\phi} N \xrightarrow{q} V$ where q is the vector bundle projection $(x,\xi) \mapsto x$ - the integral is an oscillatory integral, taken over the vector space $N^*_{p(u)}$).

The principal symbol of such an operator is a homogeneous function on a locally compact space $\mathcal{F}^*$ which is a family of vector spaces (of non constant dimension).

As in the case of foliations and Lie groupoids (*cf.* [5, 12, 13, 18]), we show:

- The kernel k_a defines a multiplier of $\mathcal{A}(M,\mathcal{F})$ (more precisely, of the image of $\mathcal{A}(M,\mathcal{F})$ in the C^*-algebra of the foliation).
- Those multipliers form an algebra.
- The algebra of pseudodifferential operators is filtered by the order of a. The class of k_a only depends up to lower order on the germ of the principal part of a on $\mathcal{F}^*$.
- Negative order pseudodifferential operators are elements of the C^*-algebra (full and therefore reduced) of the foliation.
- Zero order pseudodifferential operators define bounded multipliers of the C^*-algebra of the foliation.
- We therefore have an exact sequence of C^*-algebras
$$0 \to C^*(M,\mathcal{F}) \to \Psi^*(M,\mathcal{F}) \to B \to 0$$
where $\Psi^*(M,\mathcal{F})$ denotes the closure of the algebra of zero order pseudodifferential operators and B is (a quotient of) the algebra $C_0(S^*\mathcal{F})$ of continuous functions on the "cosphere bundle" which vanish at infinity.
- Longitudinally elliptic operators of positive order (*i.e.* operators whose principal symbol is invertible when restricted to $\mathcal{F}^*$) give rise to regular quasi-invertible operators.
- We may form a Laplacian of $\mathcal{F}$, which is an example of such a positive order longitudinally elliptic operator. It defines a regular positive self-adjoint multiplier of the C^*-algebra, and therefore a positive self-adjoint operator in any non-degenerate representation of $C^*(\mathcal{F})$; in particular a self-adjoint element of $B(L^2(M))$.

One can also take coefficients on a smooth vector bundle over M. This allows to build an index theory, which we intend to treat in a subsequent paper.

The paper is organized as follows:

- In section 1 we recall basic facts about pseudodifferential calculus: we define distributions on a manifold U with singularities on a submanifold V and state the main classical results that will be used in the subsequent sections, namely:

  a) Such distributions have a principal symbol which is a smooth function in the co-shpere bundle of N^*, where N is the normal bundle of V in U.

  b) We discuss pull backs and push forwards (partial integrations) of pseudodifferential distributions.

  c) If V_1, V_2 are transversal to each other then the product of $P_1 \in \mathcal{P}(U, V_1)$ and $P_2 \in \mathcal{P}(U, V_2)$ is a well defined distribution; a partial integral gives rise to an element of $\mathcal{P}(W, V_1 \cap V_2)$ whose principal symbol is the product of the principal symbols.

  d) The algebra $C_c^\infty(U)$ is dense in $\mathcal{P}(U, V)$.

- In section 2, for the convenience of the reader, we briefly recall the framework we introduced in [1] and give some slight modifications of results there. We moreover define the "cotangent space" $\mathcal{F}^*$ together with its natural locally compact topology.

- In section 3 we define the longitudinal pseudodifferential operators. Namely, we define an algebra $\Psi^\infty(\mathcal{U}, \mathcal{V})$ of pseudodifferential operators associated with an atlas of bi-submersions $\mathcal{U}$ and a family of identity bisections $\mathcal{V}$ covering M.

  For this algebra to be defined reasonably, one needs to bear in mind the following: In case the foliation is regular (or defined by a Lie groupoid), the longitudinal pseudodifferential operators form a subalgebra of the multipliers of the groupoid convolution algebra. A general singular foliation may not arise from a Lie groupoid, but it always comes from an atlas of bi-submersions. So, in order for $\Psi^\infty(\mathcal{U}, \mathcal{V})$ to generalize properly the pseudodifferential calculus of the regular case, we define its elements a priori as multipliers of the image of the convolution algebra $\mathcal{A}_\mathcal{U}$ in its Hausdorff completion $C^*(\mathcal{U})$.

  This is achieved by showing in §3.3 that every pseudodifferential kernel $P \in \mathcal{P}(U, V)$ defines a multiplier $\tilde{\theta}_{U,V}(P)$ of the image of the natural morphism $\theta : \mathcal{A}_\mathcal{U} \to C^*(\mathcal{U})$ from $\mathcal{A}_\mathcal{U}$ to its Hausdorff completion. To this end, we need to show in §3.2 that a non-degenerate representation Π of $C^*(\mathcal{U})$ on a Hilbert space $\mathcal{H}$ admits an appropriate extension to compactly supported pseudodifferential kernels.

  In §3.5 we show that our pseudodifferential operators have a principal symbol which is a homogeneous function on the subset of non-zero elements in $\mathcal{F}^*$. Last, in §3.6, we show that thus defined, pseudodifferential operators form a $*$-algebra.

- In section 4 we show that our longitudinal pseudodifferential calculus has the classical ellipticity properties. Namely the existence of parametrices for elliptic operators and the existence of square roots for even order, self-adjoint operators with positive principal symbol.
- In section 5 we establish the extension

$$0 \to C^*(M, \mathcal{F}) \to \Psi^*(M, \mathcal{F}) \to B \to 0$$

discussed above.

- In section 6 we show how our pseudodifferential calculus allows for the Laplacian of a singular foliation to be realized as a self-adjoint element of $B(L^2(M))$.

Contents

1 Generalized functions with pseudodifferential singularities

In this section we recall some well known facts on pseudodifferential distributions and operators.

1.1 Symbols

The symbols that we consider are the "classical" or "polyhomogeneous" symbols. Let us briefly recall how they are defined:

- Let $k, n \in \mathbb{N}$, V be an open subset of $\mathbb{R}^n$. For $m \in \mathbb{Z}$, define the space $S^m(V \times \mathbb{R}^k)$ of symbols of order (less than or equal to) m to be the set of smooth functions $a : V \times \mathbb{R}^k \to \mathbb{C}$ such that for any compact set $K \subset V$ and any multi-indices $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^k$ there is a constant $C_{K,\alpha,\beta} \in \mathbb{R}_+$ such that, for all $x \in K$ and $\xi \in \mathbb{R}^n$ we have

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{K,\alpha,\beta}(1 + |\xi|)^{m-|\beta|}.$$

- A symbol $a \in S^m(V \times \mathbb{R}^k)$ is called *classical* or *polyhomogeneous* if $a \sim \sum_{k=-\infty}^{m} a_k$, where a_k are *positively homogeneous* functions of degree k in the second variable, namely they satisfy $a_k(x, t\xi) = t^k a(x, \xi)$ for all $\xi \neq 0$ and $t > 0$. The notation "$\sim$" means that $a(x, \xi) - \chi(\xi) \sum_{k=m-M+1}^{m} a_k(x, \xi) \in S^{m-M}(V \times \mathbb{R}^k)$ for all $M \in \mathbb{N}$. Here χ is a cut-off function with $\chi(\xi) = 0$ if $|\xi| < 1/2$ and $\chi(\xi) = 1$ if $|\xi| \geq 1$. Note that this property does not depend on the cut-off function χ. We will consider only classical symbols in this paper.

- These notions are diffeomorphism invariant, and thus allow to define symbols on vector bundles: given a smooth manifold V and a smooth vector bundle N over V, we may define the space of classical symbols $S^m_{cl}(V, N)$ on the bundle N: these are functions a on the total space N which admit an expansion $a \sim \sum_{k=-\infty}^{m} a_k$ as above in any chart where the bundle N is trivial.

- We will be mostly interested in the subspace $S^m_{cl,c}(V, N) \subset S^m_{cl}(V, N)$ of symbols whose support is compact on the V direction, *i.e.* such that there exists a compact subset K in V with $a(\xi) = 0$ whenever $p(\xi) \notin K$ ($p : N \to V$ is the bundle projection).

- The above definitions extend to give spaces $S^m_{cl}(V, N; E)$ and $S^m_{cl,c}(V, N; E)$ of symbols with values in a smooth vector bundle E over V (considering E as a subbundle of a trivial bundle).

1.2 Pseudodifferential generalized functions and submersions

Remark 1.1 (on densities)**.** In order to make our constructions (which use integration) independent of choices of Lebesgue measures, we use densities everywhere. We just indicate which densities one has to take, with no further explanations most of the time.

1.2.1 Distributions transverse to a submersion

Let M, N be manifolds, $p : N \to M$ a submersion and E a vector bundle on M. Let $P \in C_c^{-\infty}(N; \Omega^1 \ker dp \otimes p^*E)$ be a distribution with compact support on N. It defines a distribution $p_! P \in C_c^{-\infty}(M; E)$ by a formula $\langle p_! P, f \rangle = \langle P, f \circ p \rangle$ ($f \in C^\infty(M; \Omega^1 M \otimes E^*)$.

Let F be a vector bundle on N. A distribution $P \in C^{-\infty}(N; F)$ on N is said to be *transverse* to p if for every $f \in C_c^\infty(N; \Omega^1 \ker dp \otimes F^*)$, the distribution $p_!(f.P)$ is smooth on M. If P is transverse to p, it restricts to a distribution P' on $N' = p^{-1}(M')$ for every submanifold M' of M. The distribution P' is obviously transverse to the restriction $p' : N' \to M'$ of p.

1.2.2 Generalized functions with pseudodifferential singularities

On a vector bundle. Let V be a smooth manifold and N a smooth vector bundle over V. A symbol $a \in S^m_{cl,c}(V, N^*; \Omega^1 N^*)$ defines a generalized function of pseudodifferential type on the total space of N which is given by a formal expression (for $u \in N$):

$$P_a(u) = \int_{N^*_{p(u)}} a(p(u), \xi) e^{i\langle u, \xi \rangle} \tag{1.1}$$

where $p : N \to V$ is the bundle map and the integral is an "oscilatory integral". This "function" P_a makes sense as a distribution on the total space of N *i.e.* elements of the dual space of the space of smooth functions with compact support on the total space of N (actually smooth sections of a suitable bundle of one densities).

We have - here k is the dimension of the bundle N

$$\langle P_a, f\rangle = (2\pi)^{-k} \int_V \Big(\int_{N^*_x \times N_x} a(x, \xi) e^{-i\langle u, \xi\rangle} f(u) \Big) = (2\pi)^{-k} \int_V \Big(\int_{N^*_x} a(x, \xi) e^{-i\langle u, \xi\rangle} \hat{f}(\xi) \Big).$$

he distribution P_a is transverse to the projection $p : N \to M$; we may therefore consider P_a as a $C^\infty(V)$ linear map from $C^\infty_c(N; \Omega^1 N) \to C^\infty(V)$ through a formula (for $x \in V$)

$$\langle P_a, f\rangle(x) = (2\pi)^{-k} \int_{N^*_x} a(x, \xi) \hat{f}(\xi). \tag{1.2}$$

Along a submanifold. Let U be a smooth manifold and V a closed smooth submanifold of U. Denote by N the normal bundle to V.

We will use the tubular neighborhood construction. Let us briefly fix the notation: this is given by a neighborhood U_1 of V in U and a local diffeomorphism $\phi : U_1 \to N$ such that, for $v \in V$, $\phi(v) = (v, 0)$ and $d\phi$ restricted to V is the identity in the normal direction. More explicitly, note that for $v \in V$, $T_{(v,0)}N = T_v V \oplus N_v$; the above condition means that $d\phi_v$ composed with the second projection is the projection $T_v U \to N_v = T_v U / T_v V$.

A *generalized function* on U *with pseudodifferential singularity* on V is a generalized function, which far from V is smooth, and near V coincides with a generalized function of pseudodifferential type through a tubular neighborhood construction.

In other words P is of the form $P = h + \chi \cdot P_a \circ \phi$ where

- (U_1, ϕ) is a tubular neighborhood construction as above,
- $h \in C^\infty(U)$,
- χ is a smooth "bump" function equal to 1 in a a neighborhood of V and to 0 outside U_1;
- $a \in S^m_{cl}(V, N^*; \Omega^1 N^*)$ is a (classical) symbol.

Concretely, such a pseudodifferential function is a distribution on U: if $f \in C_c^\infty(U; \Omega^1(TU))$, we put

$$\langle P, f\rangle = \int_U h(u)f(u) + (2\pi)^{-k} \int_{N^*U_1} a(p\circ\phi(u),\xi)\chi(u)f(u)e^{-i\langle\phi(u),\xi\rangle}$$

The generalized functions on U with pseudodifferential singularities on V form a vector space that will be denoted by $\mathcal{P}(U,V)$. We denote by $\mathcal{P}_c(U,V)$ those which vanish outside a compact subset of U (*i.e.* of the form χP where $\chi \in C_c^\infty(U)$ and $P \in \mathcal{P}(U,V)$).

Example 1.2. Choose a metric on U and thus a trivialization of all densities. Let X be a vector field with compact support on U. The map $q_X : f \mapsto \int_V Xf$ is an example of a (pseudo)differential distribution. Note that if X is tangent to V, then $\int_V Xf = -\int_V \operatorname{div}(X)f$. In other words, q_X only depends up to order zero operators on the image of X in the normal bundle.

Let us state a few facts about these generalized functions that we will use extensively:

- We may extend the construction of pseudodifferential functions and define pseudodifferential *sections* of any smooth (complex) vector bundle E over U. These also give rise to distributions as above, *i.e.* linear mappings on $C_c^\infty(U; \Omega^1 TU \otimes E^*)$. We denote by $\mathcal{P}(U,V;E)$ the space they form - and $\mathcal{P}_c(U,V;E)$ the subspace of those with compact support.
- The space of generalized functions with pseudodifferential singularities doesn't depend on the choice of $\phi : U \to N$ with the above requirements.

We immediately deduce:

Proposition 1.3. *A pseudodifferential distribution $P \in \mathcal{P}(U,V;E)$ is transverse to any submersion $p : U \to M$ which is transverse to V.* □

Notice that the smooth function on $U \setminus V$ associated with a generalized function P doesn't determine P: if the symbol is a polynomial - then P is differential and is supported by V - *i.e.* vanishes outside V.

1.2.3 Density of smooth functions

Let $P \in \mathcal{P}(U,V)$ be given by a formal formula

$$P(u) = h(u) + (2\pi)^{-k}\chi(u)\int_{N^*_{p(u)}} a(p\circ\phi(u),\xi)e^{-i\langle\phi(u),\xi\rangle}$$

Let χ_1 be a smooth nonnegative function with compact support on $\mathbb{R}_+$ which is equal to 1 in a neighborhood of 0. Put then

$$P_n(u) = h(u) + (2\pi)^{-k}\chi(u)\int_{N^*_{p(u)}} a(p\circ\phi(u),\xi)\chi_1(\|\xi\|/n)e^{-i\langle\phi(u),\xi\rangle}$$

Then $P_n \in C^\infty_c(U)$ and converges to P in the topology of $C^{-\infty}$. Furthermore, for every submersion $q : U \to M'$ which is transverse to V, and every $f \in C^\infty(U; \Omega^1 \ker dp)$, the sequence of $p_!(fP_n)$ of smooth functions on M' converges to $p_!(fP)$ in the topology of $C^\infty_c(M')$.

1.2.4 Principal symbol

A generalized function $P \in \mathcal{P}(U, V)$ of order m with pseudodifferential singularities has a principal symbol. If P is associated with a symbol a of order m, then the principal symbol $\sigma_m(P)$ of P is the homogeneous part of a of order m. It is defined outside the zero section on the total space of N and $\sigma_m(P)(x,\xi)$ is a 1-density on N^*_x (for $x \in V$ and $\xi \in N^*_x$ non zero). By choosing smoothly a euclidean metric of the bundle N, it can be defined as an element $\sigma_m(P) \in C^\infty(S^*N)$, where S^*N is the co-sphere bundle of N^*.

Proposition 1.4. *We have an exact sequence*

$$0 \to \Psi^{m-1}(V, U) \to \Psi^m(V, U) \xrightarrow{\sigma_m} C^\infty(S^*N) \to 0$$

Proof. The only thing that has to be proved is that $\sigma_m(P)$ only depends on P. This is a classical fact (see *e.g.* [13]). Let us recall this briefly:

We may assume $U = N$. Then P defines a $C^\infty(V)$-linear map $C^\infty_c(N) \to C^\infty_c(V)$ (using appropriate densities). Let $x \in V$ and $\xi \in N_x$ a non zero covector. Then $\sigma_m(x,\xi) = \lim\limits_{\tau\to+\infty}(i\tau)^{-m}P(e^{i\tau\varphi}\chi)(x)$ where $\varphi \in C^\infty_c(N)$ with derivative ξ at $x \in V$ (the zero point of N_x) along N_x - and $\chi \in C^\infty_c(N)$ is equal to 1 in a neighborhood of x. □

We may of course add bundles into the picture: if $P \in \mathcal{P}(U, V; E)$ then $\sigma_m(P)(x,\xi) \in \Omega^1(N^*_x) \otimes E_x$ (for $x \in V$ and $\xi \in N^*_x$ non zero).

It is easy to see that generalized functions satisfy all the properties of classical pseudodifferential operators. For future reference in this sequel we recall the following one; it is the key ingredient that provides the existence of parametrices for elliptic pseudodifferential operators.

Theorem 1.5. *Let $(Q_n)_{n\in\mathbb{N}}$ be a sequence of pseudodifferential functions such that $Q_n - Q_{n+1}$ is of order $m - n$. Then there exists a pseudodifferential function Q such that $Q - Q_n$ is of order $m - n$ for all n.*

Example 1.6. The principal symbol of q_X in example 1.2 is $\xi \mapsto i\langle X|\xi\rangle$.

1.3 Pull-back, push-forward, product

1.3.1 Pull-back (restriction)

Let U and U' be smooth manifolds and $V \subset U$ a closed submanifold. Let $p : U' \to U$ be a smooth map, transverse to V and put $V' = p^{-1}(V)$. It is a submanifold of U'. Let $P \in \mathcal{P}(U, V)$. Locally (near a point of V') we may assume $U = V \times \mathbb{R}^k$, $U' = V' \times \mathbb{R}^k$ and $p(x', u) = (p(x'), u)$ for $x' \in V'$ and $u \in \mathbb{R}^k$.

We may then assume that P is given (formally - see equation 1.1)) by a formula

$$P(x, u) = (2\pi)^{-k} \int_{\mathbb{R}^k} e^{-i\langle u,\xi\rangle} a(x, \xi) + h(x, u)$$

where h is smooth and a is a symbol.

We then define p^*P setting $(p^*P)(x', u) = P(p(x'), u)$

Under the identification of the normal bundle N' of V' in U' with p^*N, the principal symbol of p^*P is given by $\sigma_m(p^*P) = \sigma_m(P) \circ p$.

1.3.2 Push-forward (partial integration)

Proposition 1.7 (Push-forward). *Let U and U' be smooth manifolds and $V \subset U$ a closed submanifold. Let $p : U \to U'$ be a submersion which restricts to a diffeomorphism $p : V \to V'$ where V' is a submanifold of U'. Let E' be a vector bundle on U'.*

a) *Integration along the fibers of p gives rise to a map $p_! : \mathcal{P}_c(U, V; \Omega^1 \ker dp \otimes p^*E') \to \mathcal{P}_c(U', V'; E')$ defined by $\langle p_!(P), f\rangle = \langle P, f \circ p\rangle$ for $f \in C^\infty(U; \Omega^1 TU \otimes E^*)$. The principal symbol of $p_!P$ is $\sigma'_m(x, \xi) = \sigma_m(x, p^*\xi)$ where $x \in V' \simeq V$, and p^* is the (injective) map $(T_xU'/T_xV)^* \to (T_xU/T_xV)^*$ induced by p.*

b) *If p is onto, then $p_!$ is onto too.*

Proof. a) We may assume that $p : V \times \mathbb{R}^k \times \mathbb{R}^\ell \to V \times \mathbb{R}^k$ is the projection and that P is given by a formula

$$\langle P, f\rangle = (2\pi)^{-(k+\ell)} \int e^{-i(\langle u,\xi\rangle + \langle v,\eta\rangle)} a(x, \xi, \eta)\chi_1(u)\chi_2(v)f(x, u, v).$$

We thus get

$$\langle p_!P, f\rangle = \langle P, f \circ p\rangle = (2\pi)^{-(k+\ell)} \int_{V\times\mathbb{R}^\ell\times(\mathbb{R}^{k+\ell})^*} a(x, \xi, \eta)e^{-i\langle u,\xi\rangle}\chi_1(u)\widehat{\chi_2}(\eta)f(x, u)$$

Using a Taylor expansion of the form

$$a(x, \xi, \eta) \sim a(x, \xi, 0) + \sum_{1\le|\alpha|} \frac{\eta^\alpha}{\alpha!} \frac{\partial^{|\alpha|}}{(\partial\eta)^\alpha} a(x, \xi, 0)$$

we find the principal term

$$(2\pi)^{-(k+\ell)} \int_{V\times\mathbb{R}^\ell\times(\mathbb{R}^{k+\ell})^*} a(x,\xi,0)e^{-i\langle u,\xi\rangle}\chi_1(u)\widehat{\chi_2}(\eta)f(x,u).$$

Since $(2\pi)^{-\ell}\displaystyle\int\widehat{\chi_2}(\eta) = \chi_2(0) = 1$ we find

$$(2\pi)^{-k} \int_{V\times\mathbb{R}^\ell\times(\mathbb{R}^{k})^*} a(x,\xi,0)e^{-i\langle u,\xi\rangle}\chi_1(u)f(x,u)$$

b) Let $P'\in\mathcal{P}_c(U',V';E')$. One obviously may extend the principal symbol of P' to get a homogeneous section on the normal bundle of V in U. It follows that there exists an operator $P_1\in\mathcal{P}_c(U,V;p^*E')$ such that $p_!P_1-P'$ is of order $m-1$. Using induction, one constructs a sequence $P_n\in\mathcal{P}_c(U,V;p^*E')$ such that $p_!P_n-P'$ is of order $m-n$ and $P_{n+1}-P_n$ is of order $m-n$. Using theorem 1.5 , one then gets Q such that $Q-P_n$ is of order $m-n$, whence $p_!Q-P'$ is smoothing. Finally, using partitions of the identity, it is obvious that $p_!:C_c^\infty(U;p^*E')\to C_c(U',E')$ is onto. □

Remarks 1.8. a) We will also need a slightly more general statement: In the above proposition, we may just assume that p induces a submersion $p:V\to V'$ where V' is a submanifold of U'. In that case, for $P\in\mathcal{P}(U,V;\Omega^1(\ker dp)\otimes p^*(E'))$, the principal symbol $\sigma'(x',\xi')$ of $p_!P$ is the integral of $\sigma(x,p_x^*(\xi'))$ for x running in the fiber $V\cap p^{-1}(x')$ (and $p_x^*:(T_{x'}U'/T_{x'}V')^*\to(T_xU/T_xV)^*$ is the (injective) map induced by $(dp)_x$) ([4]).

To establish this, one may assume $U'=V'\times\mathbb{R}^k$, $U=V'\times\mathbb{R}^j\times\mathbb{R}^k\times\mathbb{R}^\ell$, $V=V'\times\mathbb{R}^j\times\{(0,0)\}$ and p is the obvious projection $V'\times\mathbb{R}^j\times\mathbb{R}^k\times\mathbb{R}^\ell\to V'\times\mathbb{R}^k$.

b) Obviously, we have $(q\circ p)_!=q_!\circ p_!$ if $p:U\to U'$ and $q:U'\to U''$ are submersions satisfying requirements of (a).

1.3.3 Products

In order to understand the product of pseudodifferential operators in our context, we give the following Lemma. We say that a submersion $p:U\to U'$ is *strictly transverse* to a submanifold $V\subset U$ if at any point $x\in V$ we have $T_xV\oplus\ker dp_x=T_xU$.

Lemma 1.9. *Let U be a manifold, V_1,V_2 two closed submanifolds of U that are transverse to each other and $p:U\to U'$ a submersion strictly transverse to both V_1 and V_2. Then there are charts of U of the form $W\times\mathbb{R}^k\times\mathbb{R}^k$ covering $V=V_1\cap V_2$, such that $V_1=W\times\mathbb{R}^k\times\{0\}$ and $V_2=W\times\{0\}\times\mathbb{R}^k$ and p can be written as $(v,\xi,\eta)\mapsto(v,\xi+\eta)$.*

Proof. Notice that V is a manifold due to the transversality of V_1 to V_2. Then V is covered by charts such that U can be written as $U'\times T$, where $T=\mathbb{R}^k$ is the fiber of p, and p is the

[4] An easy check shows that the densities match correctly.

projection. Since T is transverse to V_1, for this chart we can write $V_1 = U' \times \{0\}$. Since V_2 is trasversal to V_1, it is the graph of a submersion $q : V_1 \to T$; we have $V = q^{-1}(\{0\})$, hence we can write $V_1 = V \times \mathbb{R}^k$ and q the projection. Under these identifications, we find a chart $V \times \mathbb{R}^k \times \mathbb{R}^k$, for which $V_1 = V \times \mathbb{R}^k \times \{0\}$, $V_2 = \{(x, \xi, \eta);\ \xi = \eta\}$ and $p(x, \xi, \eta) = \xi$. The result follows by applying the diffoemorphism $(v, \xi, \eta) \mapsto (v, \xi + \eta, \eta)$ on $V \times \mathbb{R}^k \times \mathbb{R}^k$. □

Proposition 1.10. *Let U be a manifold and V_1, V_2 two closed submanifolds of U that are transverse to each other. Let $P_i \in \mathcal{P}(U, V_i)$.*

a) *The product $P_1 \cdot P_2$ makes sense as a distribution on U.*

b) *Assume that $P_1 \cdot P_2$ has compact support. Let $p : U \to U'$ be a submersion which is both strictly transverse to V_1 and V_2 and whose restriction to $V_1 \cap V_2$ is injective and proper. Then $p_!(P_1.P_2)$ is a pseudodifferential function in $\mathcal{P}(U', p(V_1 \cap V_2))$. If σ_i is the principal symbol of P_i $(i = 1, 2)$, then the principal symbol of $p_!(P_1.P_2)$ is $p_!(\sigma_1) \cdot p_!(\sigma_2)$.*

Proof. a) The statement is local. We may therefore assume $U = V \times \mathbb{R}^k \times \mathbb{R}^\ell$, $V_1 = V \times \mathbb{R}^k \times \{0\}$ and $V_2 = V \times \{0\} \times \mathbb{R}^\ell$. Also, by an obvious choice of the tubular neighborhood construction, we may write

$$P_1(x, u, v) = \int e^{i\langle v|\eta\rangle} a_1(x, u, \eta) d\eta \quad \text{and} \quad P_2(x, u, v) = \int e^{i\langle u|\xi\rangle} a_2(x, \xi, v) d\xi.$$

Here, a_1 and a_2 are classical (polyhomogeneous) symbols.

The product is then given by a formula:

$$\langle P_1 \cdot P_2, f\rangle = \int \left(\int e^{i\langle u|\xi\rangle} e^{i\langle v|\eta\rangle} a_1(x, u, \eta) a_2(x, \xi, v) f(x, u, v)\, dudv \right) dxd\xi d\eta$$

which makes perfect sense when $f \in C_c^\infty(U)$.

b) Due to 1.9 we can assume $U = V \times \mathbb{R}^k \times \mathbb{R}^k$, $U' = V \times \mathbb{R}^k$ and $p(x, u, v) = (x, u + v)$.

We thus have to compute

$$\left(\int \left(\int e^{i\langle u|\xi\rangle} e^{i\langle v|\eta\rangle} \chi(x, u, v) a_1(x, u, \eta) a_2(x, \xi, v) f(x, u + v)\, dudv \right) d\xi d\eta \right) dx.$$

This is a "classical" oscillatory integral on $\mathbb{R}^{4k}$ which is treated by the usual integration by parts methods. It actually amounts to composing (families indexed by V of) pseudodifferential operators in $\mathbb{R}^k$. □

2 Singular foliations; cotangent space

2.1 Foliations, bi-submersions, atlas, *-algebra

Let us first recall some definitions, notation and conventions taken in [1].

2.1.1 Foliations

Definition 2.1. a) Let M be a smooth manifold. A *foliation* on M is a locally finitely generated submodule of $C_c^\infty(M; TM)$ stable under Lie brackets.

b) There is an obvious notion of a pull-back foliation: if $(M, \mathcal{F})$ is a foliation and $f : L \times M \to M$ is the second projection, the pull back foliation $f^{-1}(\mathcal{F})$ is the space of vector fields whose M component as a map from L to $C_c^\infty(M, TM)$ takes its values in $\mathcal{F}$. In the same way, one defines pull back foliations by submersions. See [1], subsection 1.2.3.

c) For $x \in M$, put $I_x = \{f \in C^\infty(M) : f(x) = 0\}$. The *fiber* of $\mathcal{F}$ is the quotient $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$. The *tangent space of the leaf* is the image F_x of the evaluation map $ev_x : \mathcal{F} \to T_xM$.

The spaces F_x and $\mathcal{F}_x$ differ on singular leaves: the dimension of F_x is lower semi-continuous and the dimension of $\mathcal{F}_x$ is upper semi-continuous. They coincide in a dense open subset of M, namely on points x that lie in a regular leaf.

We get a surjective linear map $e_x : \mathcal{F}_x \to F_x$ whose kernel is a Lie algebra $\mathfrak{g}_x$.

2.1.2 Bi-submersions, bisections

The main ingredient in the constructions of [1] is the notion of bi-submersion that we now recall.

Definition 2.2. A *bi-submersion* of $(M, \mathcal{F})$ is a smooth manifold U endowed with two smooth maps $s, t : U \to M$ which are submersions and satisfy:

a) $s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F})$.

b) $s^{-1}(\mathcal{F}) = C_c^\infty(U; \ker ds) + C_c^\infty(U; \ker dt)$.

If (U, t_U, s_U) and (V, t_V, s_V) are bi-submersions then (U, s_U, t_U) is a bi-submersion, as well as (W, s_W, t_W) where $W = U \times_{s_U, t_V} V$, $s_W(u, v) = s_V(v)$ and $t_W(u, v) = t_U(u)$ ([1, Prop. 2.4]). The bi-submersion (U, s_U, t_U) is called the inverse of (U, t_U, s_U) and is noted $(U, t_U, s_U)^{-1}$, or just U^{-1}; the bi-submersion (W, s_W, t_W) is called the composition of (U, t_U, s_U) and (V, t_V, s_V) and is noted $(U, t_U, s_U) \circ (V, t_V, s_V)$ - or just $U \circ V$.

In [1, Prop. 2.10] it is shown that there are enough bi-submersions $\{(U_i, t_i, s_i)\}_{i \in I}$ such that $\bigcup_{i \in I} s_i(U_i) = M$: For an $x \in M$, if $X_1, \ldots, X_n \in \mathcal{F}$ form a base of $\mathcal{F}_x$ then we can find a neighborhood U of $(x, 0)$ in $M \times \mathbb{R}^n$ where the exponential $t_U(y, \xi) = \exp(\sum_{i=1}^n \xi_i X_i)(y)$ is defined and such that (U, t_U, s_U) is a bi-submersion, where s_U denotes the first projection. Such a bi-submersion is sometimes called an *identity bi-submersion*.

Definition 2.3 (morphisms of bi-submersions)**.** Let (U_i, t_i, s_i) $(i = 1, 2)$ be bi-submersions. A smooth map $f : U_1 \to U_2$ is a *morphism of bi-submersions* if $s_1 = s_2 \circ f$ and $t_1 = t_2 \circ f$.

In order to compare bi-submersions, we used the notion of bisections:

Definition 2.4. A *bisection* of (U, t, s) is a locally closed submanifold V of U such that s and t restricted to V are diffeomorphisms to open subsets of M. We say that V is an *identity bisection* if moreover the restrictions of s and t to V coincide.

We say that $u \in U$ *carries* the foliation-preserving local diffeomorphism ϕ if there exists a bisection V such that $u \in V$ and $\phi = t\mid_V \circ (s\mid_V)^{-1}$.

In [1, §2.3] it is shown that if (U_j, t_j, s_j) are bi-submersions, $j = 1, 2$ then a $u_1 \in U_1$ carries the same diffeomorphism with $u_2 \in U_2$ iff there exists a morphism of bi-submersions g defined in an open neighborhood of $u_1 \in U_1$ such that $g(u_1) = u_2$.

Actually the proof given in [1, §2.3] proves a stronger statement which will be useful here:

Proposition 2.5. *Let (U_j, t_j, s_j) be bi-submersions, $j = 1, 2$, $V_i \subset U_i$ identity bisections and $u_j \in V_j$ such that $s_1(u_1) = s_2(u_2)$. Then there exists a morphism of bi-submersions g defined in an open neighborhood U_1' of $u_1 \in U_1$ such that $g(u_1) = u_2$ and $g(V_1 \cap U_1') \subset V_2$.* □

2.1.3 Minimal bi-submersions

Definition 2.6. If (U, t, s) is a bi-submersion and $u \in U$, then the dimension of the manifold U is at least $\dim M + \dim \mathcal{F}_{s(u)}$. We say that (U, t, s) is *minimal* at u if $\dim U = \dim M + \dim \mathcal{F}_{s(u)}$.

If $f : (U', t', s') \to (U, t, s)$ is a morphism of bi-submersions and U is minimal at $f(u')$, then $df_{u'}$ is onto. Therefore, there is a neighborhood W' of u' in U' such that the restriction of f to W' is a submersion.

For every bi-submersion (U, t, s) and every $u \in U$, there exists a bi-submersion (U', t', s'), and $u' \in U'$ such that U' is minimal at u' and carries at u' the same diffeomorphisms as U at u. It follows that there is a neighborhood W of u in U and a submersion which is a morphism $f : (W, t_{|W}, s_{|W}) \to (U', t', s')$.

The following result will be used in the sequel:

Proposition 2.7. *Let (U, t, s) be a bi-submersion and V an identity bisection. Let $u \in V$ and assume U is minimal at u. Then there is a neighborhood U' of u in U and a submersion $p : U' \circ U' \to U$ which is a morphism of bi-submersions strictly transverse to $U \circ V$ and to $V \circ U$.*

Proof. The composition $U \circ U$ carries at (u, u) the identity bisection $V \circ V$. It follows that there exists a neighborhood of (u, u) in $U \circ U$ thet we may assume of the form $U' \circ U'$ and a morphism $p : U' \circ U' \to U$. By minimality of U at u, we may assume that p is a submersion. Moreover $U \circ V$ and $V \circ U$ are bi-submersions. Again, by minimality of U at u, it follows that, up to restricting U', the restrictions of p to $(U \circ V) \cap (U' \circ U')$ and $(V \circ U) \cap (U' \circ U')$ are submersions, hence p is transverse to $U \circ V$ and to $V \circ U$ - strictly by equality of dimensions. □

2.1.4 Atlas of bi-submersions

Definition 2.8. Let $\mathcal{U} = \big((U_i, t_i, s_i)\big)_{i\in I}$ be a family of bi-submersions. A bi-submersion (U, t, s) is *adapted* to $\mathcal{U}$ if for all $u \in U$ there exists an open subset $U' \subset U$ containing u, an $i \in I$, and a morphism of bi-submersions $U' \to U_i$.

We say that $\mathcal{U}$ is an *atlas* if

a) $\displaystyle\bigcup_{i\in I} s_i(U_i) = M$.

b) The inverse of every element in $\mathcal{U}$ is adapted to $\mathcal{U}$.

c) The composition $U \circ V$ of any two elements in $\mathcal{U}$ is adapted to $\mathcal{U}$.

An atlas $\mathcal{V} = \{(V_j, t_j, s_j)\}_{j\in J}$ is adapted to $\mathcal{U}$ if every element of $\mathcal{V}$ is adapted to $\mathcal{U}$. We say $\mathcal{U}$ and $\mathcal{V}$ are *equivalent* if they are adapted to each other. There is a *minimal atlas* which is adapted to any other atlas: this is the atlas generated by "identity bi-submersions".

2.1.5 The groupoid of an atlas

The groupoid of an atlas $\mathcal{U} = \big((U_i, t_i, s_i)\big)_{i\in I}$ is the quotient of $U = \displaystyle\coprod_{i\in I} U_i$ by the equivalence relation for which $u \in U_i$ is equivalent to $u' \in U_j$ if U_i carries at u the same local diffeomorphisms as U_j at u'.

For every bi-submersion U adapted to $\mathcal{U}$ we have a well defined (quotient) map $\zeta_U : U \to \mathcal{G}_{\mathcal{U}}$.

2.1.6 The C*-algebra of a foliation

In [1, §4.3] we associated to an atlas $\mathcal{U}$ a $*$-algebra $\mathcal{A}(\mathcal{U}) = \displaystyle\bigoplus_{i\in I} C_c^\infty(U_i; \Omega^{1/2}U_i)/\mathcal{I}$. Here $\Omega^{1/2}$ denotes the bundle of half densities on $\ker ds \oplus \ker dt$ and $\mathcal{I}$ is the space spanned by the $p_!(f)$, where $p : W \to U$ is a submersion which is a morphism of bi-submersions and $f \in C_c(W; \Omega^{1/2}W)$ is such that there exists a morphism $q : W \to V$ of bi-submersions which is a submersion and such that $q_!(f) = 0$.

To any bi-submersion V adapted to $\mathcal{U}$ we can associate a linear map $Q_V : C_c^\infty(V; \Omega^{1/2}V) \to \mathcal{A}_{\mathcal{U}}$. Involution and convolution in $\mathcal{A}_{\mathcal{U}}$ are then defined by

$$\Big(Q_{V_1}(f_1)\Big)^* = (Q_{V_1^{-1}})(f_1^*) \quad\text{and}\quad Q_{V_1}(f_1)Q_{V_2}(f_2) = Q_{V_1\circ V_2}(f_1 \otimes f_2)$$

for (V_i, t_i, s_i) bi-submersions adapted to $\mathcal{U}$ and $f_i \in C_c^\infty(V_i; \Omega^{1/2}V_i)$, $i = 1, 2$.

The C^*-algebra $C^*(\mathcal{U})$ of the atlas $\mathcal{U}$ is the (Hausdorff)-completion of $\mathcal{A}_{\mathcal{U}}$ with a natural C^*-norm [1, §4.4, §4.5]. Actually, two natural completions were considered: the full and the reduced one.

If $\mathcal{U}$ is adapted to $\mathcal{V}$, we have natural $*$-morphisms $\mathcal{A}(\mathcal{U}) \to \mathcal{A}(\mathcal{V})$ and $C^*(\mathcal{U}) \to C^*(\mathcal{V})$.

When $\mathcal{U}$ is the mininal atlas we write $C^*(M, \mathcal{F})$ for the full and $C^*_r(M, \mathcal{F})$ for the reduced completion.

The representations of the full C^*-algebras were described in [1, §5] in terms of representations of the associated groupoid. We will come back to this description below (§3.2).

2.2 The cotangent space

The symbols should be functions on the total space of a "cotangent bundle". Let us discuss this space.

Definition 2.9. The *cotangent bundle* of the foliation $\mathcal{F}$ (although it is in general not a bundle) is the union $\mathcal{F} = \coprod_{x \in M} \mathcal{F}_x^*$. It is endowed with a natural projection $p : \mathcal{F}^* \to M$ ($(x, \xi) \mapsto x$). Also, for each $X \in \mathcal{F}$, we have a natural map $q_X : (x, \xi) \mapsto \xi \circ e_x(X)$. We endow $\mathcal{F}^*$ with the weakest topology for which the maps p and q_X are continuous.

Proposition 2.10. *The space $\mathcal{F}^*$ is locally compact.*

Proof. It is enough to show that $p^{-1}(U)$ is locally compact for every small enough open set U of M. We may therefore assume that $\mathcal{F}$ is finitely generated *i.e.* it is a quotient of $C_c^\infty(M; \mathbb{R}^n)$. Then $\mathcal{F}^*$ consists of elements of $(x, y) \in M \times (\mathbb{R}^n)^*$ such that the map $C_c^\infty(M; \mathbb{R}^n) \ni \varphi \mapsto \langle y, \varphi(x) \rangle$ factors through the quotient $\mathcal{F}$ of $C_c^\infty(M; \mathbb{R}^n)$ (*i.e.* vanishes on the kernel of the map $C_c^\infty(M; \mathbb{R}^n) \to \mathcal{F}$). It is a closed subset of $M \times (\mathbb{R}^n)^*$ and is therefore locally compact. ☐

Example 2.11. Let $\mathcal{F}$ be the foliation on $\mathbb{R}^3$ defined by the (infinitesimal) action of $SO(3)$. It is easy to see that $\mathcal{F}^* = \cup_{\xi \in \mathbb{R}^3} \{x \in \mathbb{R}^3 : \langle x | \xi \rangle = 0\}$.

Let (U, t, s) be a bi-submersion and V an identity bisection. Identifying the normal bundle NV with $\ker ds$ (or $\ker dt$), there are epimorphisms $d_x t : N_x V \to \mathcal{F}_x$, $x \in V$ (or $d_x s$). Dualizing these maps we find that locally the cotangent bundle $\mathcal{F}^*$ is a closed subspace of N^*V. Thus we can restrict symbols to $\mathcal{F}^*$.

Let (U', t', s') and (U, t, s) be bi-submersions with identity bisections V' and V. Let $p : U' \to U$ be a smooth map that is a morphism of bi-submersions such that $p(V') \subseteq V$. Then for every $x \in s'(V')$ the inclusion $\mathcal{F}_x^* \to (N'_x)^*$ factors as the composition of $p_x^* : N_x^* \to (N'_x)^*$ with the inclusion $\mathcal{F}_x^* \to N_x^*$.

3 The space of pseudodifferential kernels on a foliation

3.1 Pseudodifferential kernels on a bi-submersion

Let (U, t, s) be a bi-submersion. The bundle (over U) of half densities on $\ker ds \oplus \ker dt$ will be simply denoted by $\Omega^{1/2}$.

We define $\mathcal{P}^m(U,V;\Omega^{1/2})$ (*resp.* $\mathcal{P}_c^m(U,V;\Omega^{1/2})$) to be the space of generalized sections of the bundle $\Omega^{1/2}$ with pseudodifferential singularities along V (*resp.* those with compact support) of order $\leq m$. We drop the superscript m when we make no order requirements.

Let N be the normal bundle of the inclusion $V \to U$. Note that N canonically identifies with both the restrictions to V of the bundles $\ker ds$ and $\ker dt$. Under these identifications the bundle $\Omega^1 N^* \otimes \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dt)$ is trivial.

The principal symbol of $P \in \mathcal{P}^m(U,V;\Omega^{1/2})$ is therefore an element of $S^m(V,N^*)$, *i.e.* a function on N^* which is homogeneous of degree m.

Definition 3.1. Let (U,t,s) be a bi-submersion and V an identity bisection.

a) Denote by U^{-1} the inverse bi-submersion and $\kappa : U \to U^{-1}$ the (identity) isomorphism. For an operator $P \in \mathcal{P}_c(U,V;\Omega^{1/2})$ define $P^* = \overline{\kappa_! P}$.

b) Finally, let U, U' be bi-submersions, $V \subset U$ an identity bisection, $P \in \mathcal{P}_c(U,V;\Omega^{1/2})$ and $f \in C_c^\infty(U';\Omega^{1/2})$. Using the first projection which is a submersion $U \circ U' \to U$ we may pull back P to a generalized function with pseudodifferential singularities on $V \circ U'$; multiplying it by f, we get $P \star f \in \mathcal{P}_c(U \circ U', V \circ U';\Omega^{1/2})$. In the same way, we construct $f \star P \in \mathcal{P}_c(U' \circ U, U' \circ V;\Omega^{1/2})$.

3.2 Extending representations to pseudodifferential kernels

We fix an atlas $\mathcal{U}$. Denote by $\mathcal{G}_\mathcal{U}$ the associated groupoid.

3.2.1 Quasi-invariant measures

Recall that a measure μ on M is said to be quasi-invariant by the atlas $\mathcal{U}$ if for every bi-submersion (U,t,s) adapted to the atlas $\mathcal{U}$ the measures $\mu \circ \lambda^s$ and $\mu \circ \lambda^t$ are equivalent (here λ^s and λ^t are Lebesgue measures along the fibers of s and t respectively).

In that case, there is a measurable almost everywhere invertible section ρ^U of $\Omega^{-1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dt)$ such that for every $f \in C_c(U;\Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dt))$ we have

$$\int_M \Big(\int_{s^{-1}\{x\}} (\rho_u^U)^{-1} \cdot f(u) \Big) d\mu(x) = \int_M \Big(\int_{t^{-1}(x)} \rho_u^U \cdot f(u) \Big) d\mu(x).$$

If $p : (U,t,s) \to (U',t',s')$ is a morphism of bi-submersions, there is a canonical isomorphism of the bundles $\Omega^{-1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dt)$ and $p^*\Big(\Omega^{-1/2}(\ker ds') \otimes \Omega^{1/2}(\ker dt')\Big)$. Under this isomorphism, the maps ρ^U and $p^*(\rho^{U'})$ coincide (almost everywhere). We interpret this by saying that ρ is defined on the groupoid $\mathcal{G}_\mathcal{U}$. Furthermore, one may see that it is naturally a homomorphism.

Remark 3.2. The morphism ρ defined here takes also into account the Radon Nykodym derivative $(D^U)^{1/2}$ that we used in [1, §5].

Remark 3.3 (on measurable functions on $\mathcal{G}_{\mathcal{U}}$). A function f on $\mathcal{G}_{\mathcal{U}}$ is just a function $f \circ \zeta$ on $\coprod_{i \in I} U_i$ which is constant on the equivalence classes. We will say that the function f is measurable with respect to the quasi-invariant measure μ if $f \circ \zeta_U$ is measurable (with respect to the measures $\mu \circ \lambda$).

Lemma 3.4. *Let (W, t, s) and (W_i, t_i, s_i) be bi-submersions, $Y \subset W$ a submanifold and $p^i : W \to W_i$ be morphisms of bi-submersions which are submersions transverse to Y (here $i \in \{1, 2\}$). Let μ be a quasi-invariant measure on M and β a measurable bounded function on $\mathcal{G}_{\mathcal{U}}$.*

For $Q \in \mathcal{P}_c(W, Y; \Omega^{1/2})$, we have

$$\int_M \left(\int_{t_1^{-1}\{x\}} \rho_w^{W_1} \cdot p_!^1(Q)(w) \beta \circ \zeta_{W_1}(w) \right) d\mu(x) = \int_M \left(\int_{t_2^{-1}\{x\}} \rho_w^{W_2} \cdot p_!^2(Q)(w) \beta \circ \zeta_{W_2}(w) \right) d\mu(x)$$

Note that, by the transversality assumption, the $p_!^i(Q)$ are smooth functions. This explains the meaning of this formula.

Proof. For $Q \in C_c^\infty(W; \Omega^{1/2})$ these two expressions coincide (by Fubini) with

$$\int_M \left(\int_{t^{-1}\{x\}} \rho_u^W \cdot Q(u) \beta \circ \zeta_U(u) \right) d\mu(x).$$

The Lemma follows from §1.2.3. $\square$

We will use an immediate generalization of this lemma:

Lemma 3.5. *Let (W, t, s) and $(W_i, t_i, s_i)_{i \in I}$ be bi-submersions and $Y \subset W$ a closed submanifold. Assume that there is an open cover $(Z_i)_{i \in I}$ of W and $p^i : Z_i \to W_i$ morphisms of bi-submersions which are submersions transverse to $Y \cap Z_i$. Let (χ_i) be a smooth partition of the identity adapted to the cover Z_i. Let μ be a quasi-invariant measure on M and β a measurable bounded function on $\mathcal{G}_{\mathcal{U}}$.*

For $Q \in \mathcal{P}_c(W, Y; \Omega^{1/2})$, the quantity

$$\int_M \sum_{i \in I} \left(\int_{t_i^{-1}\{x\}} \rho_w^{W_1} \cdot p_!^i(\chi_i Q)(w) \beta \circ \zeta_{W_i}(w) \right) d\mu(x))$$

does not depend on the choices of I and the family $(Z_i, \chi_i, W_i, p_i)_{i \in I}$ with the above requirements. $\square$

Let (W, t, s) be a bi-submersion and $Y \subset W$ a submanifold. We say that Y is *transverse to* ζ_W if there is an open cover $(Z_i)_{i \in I}$ of W and $p^i : Z_i \to W_i$ morphisms of bi-submersions which are submersions transverse to $Y \cap Z_i$. The above Lemma makes sense of

$$\int_M \left(\int_{t^{-1}\{x\}} \rho_u^W \cdot Q(u) \beta \circ \zeta_U(u) \right) d\mu(x)$$

for $Q \in \mathcal{P}_c(W, Y; \Omega^{1/2})$.

3.2.2 Extension of representations

We now fix an atlas $\mathcal{U}$ and a non degenerate $*$-representation Π of $C^*(\mathcal{U})$ on a Hilbert space $\mathcal{H}$.

Let us fix some notation:

Denote by $\theta : \mathcal{A}_{\mathcal{U}} \to C^*(\mathcal{U})$ the natural morphism from $\mathcal{A}_{\mathcal{U}}$ to its Hausdorff-completion. If U is a bi-submersion adapted to $\mathcal{U}$, denote by $\theta_U : C_c^\infty(U;\Omega^{1/2}) \to C^*(\mathcal{U})$ the composition $C_c^\infty(U;\Omega^{1/2}) \xrightarrow{Q_U} \mathcal{A}_{\mathcal{U}} \xrightarrow{\theta} C^*(\mathcal{U})$. Finally, put $\Pi_U = \Pi \circ \theta_U$.

According to [1, §5], there is a triple (μ, H, π) where:

a) μ is a quasi-invariant measure on M;

b) $H = (H_x)_{x\in M}$ is a measurable (with respect to μ) field of Hilbert spaces over M.

c) For every bi-submersion (U,t,s) adapted to $\mathcal{U}$, π^U is a measurable (with respect to $\mu \circ \lambda$) section of the field of unitaries $\pi_u^U : H_{s(u)} \to H_{s(u)}$.

Moreover:

a) π is 'defined on $\mathcal{G}_{\mathcal{U}}$':
if $f : U \to V$ is a morphism of bi-submersions, for almost all $u \in U$ we have $\pi_u^U = \pi_{f(u)}^V$.

b) π is a homomorphism:
If U and V are bi-submersions adapted to $\mathcal{U}$, we have $\pi_{(u,v)}^{U\circ V} = \pi_u^U \pi_v^V$ for almost all $(u,v) \in U \circ V$.

Then $\mathcal{H} = \displaystyle\int_M^\oplus H_x\, d\mu(x)$ is the space of square integrable sections of H. For every bi-submersion (U,t,s) adapted to $\mathcal{U}$, we have:

$$\Pi_U(f)(\xi)(x) = \int_{t^{-1}\{x\}} (\rho_u^U \cdot f(u))\pi_u^U(\xi_{s(u)})$$

μ-a.e. for every $\xi \in \mathcal{H}$ and $x \in M$.

It follows that $\langle \eta, \Pi_U(f)\xi\rangle = \displaystyle\int_M \Big(\int_{t^{-1}\{x\}} (\rho_u^U \cdot f(u))\langle \eta_x, \pi_u^U(\xi)\rangle\Big) d\mu(x)$.

Proposition 3.6. *Let (W,t,s) be a bi-submersion and $Y \subset W$ a submanifold transverse to ζ_W. There is a linear map $\Pi_{W,Y} : \mathcal{P}_c(W,Y;\Omega^{1/2}) \to \mathcal{L}(H)$ such that for every open subset $Z \subset W$, every morphism $p : Z \to U$ of bi-submersions which is a submersion transverse to $Y \cap Z$ and every $Q \in \mathcal{P}_c(Z, Y\cap Z;\Omega^{1/2}) \subset \mathcal{P}_c(W,Y;\Omega^{1/2})$ we have $\Pi_{W,Y}(Q) = \Pi_U(p_!(Q))$.*

Proof. Let $\xi, \eta \in \displaystyle\int^\oplus H_x\, d\mu(x) = \mathcal{H}$ be bounded square integrable sections, and define a bounded measurable function β on $\mathcal{G}_{\mathcal{U}}$ by putting $\beta \circ \zeta_U(u) = \langle \eta_x, \pi_u^U(\xi_{s_U(u)})\rangle$. It follows from Lemma 3.5 that, for $Q \in \mathcal{P}_c(W, Y; \Omega^{1/2})$, using any partition of unit adapted to a nice cover of W to construct $\Pi_{W,Y}(Q)$ we have

$$\langle \eta, \Pi_{W,Y}(Q)\xi\rangle = \int_M \left(\int_{t^{-1}\{x\}} \rho_u^W \cdot Q(u)\beta \circ \zeta_U(u)\right) d\mu(x)$$

which is well defined by Lemma 3.5. The conclusion follows by density of bounded sections in $\mathcal{H}$. □

Taking Π to be faithful, we find:

Corollary 3.7. *Let (W, t, s) be a bi-submersion and $Y \subset W$ a submanifold transverse to ζ_W. There is a linear map $\theta_{W,Y} : \mathcal{P}_c(W, Y; \Omega^{1/2}) \to \theta(\mathcal{A}(\mathcal{U}))$ such that for every open subset $Z \subset W$, every morphism $p : Z \to U$ of bi-submersions which is a submersion transverse to $Y \cap Z$ and every $Q \in \mathcal{P}_c(Z, Y \cap Z; \Omega^{1/2}) \subset \mathcal{P}_c(W, Y; \Omega^{1/2})$ we have $\theta_{W,Y}(Q) = \theta_U(p_!(Q))$.* □

3.3 Pseudodifferential kernels and multipliers

Let (U, t, s) be a bi-submersion and $V \subset U$ an identity bisection. Let U' and U'' be bi-submersions. For $P \in \mathcal{P}_c(U, V; \Omega^{1/2})$, $f \in C_c^\infty(U'; \Omega^{1/2})$ and $g \in C_c^\infty(U''; \Omega^{1/2})$, we defined $P \star f \in \mathcal{P}_c(U \circ U', V \circ U'; \Omega^{1/2})$ and $g \star P \in \mathcal{P}_c(U'' \circ U, U'' \circ V; \Omega^{1/2})$ (see Definition 3.1).

Note that $V \circ U'$ and $U'' \circ V$ are bi-submersions and therefore transverse to $\zeta_{U\circ U'}$ and $\zeta_{U''\circ U}$ respectively.

Theorem 3.8. *Let (U, t, s) be a bi-submersion and $V \subset U$ an identity bisection. Let U' and U'' be bi-submersions. For $P \in \mathcal{P}_c(U, V; \Omega^{1/2})$, $f \in C_c^\infty(U'; \Omega^{1/2})$ and $g \in C_c^\infty(U''; \Omega^{1/2})$ we have $\theta_{U''}(g)\theta_{U\circ U', V\circ U'}(P \star f) = \theta_{U''\circ U, U''\circ V}(g \star P)\theta_{U'}(f)$. In other words, there is a multiplier $\tilde{\theta}_{U,V}(P)$ of $\theta(\mathcal{A}(\mathcal{U}))$ such that*

$$\tilde{\theta}_{U,V}(P)\theta_{U'}(f) = \theta_{U\circ U', V\circ U'}(P \star f) \quad \textit{and} \quad \theta_{U''}(g)\tilde{\theta}_{U,V}(P) = \theta_{U''\circ U, U''\circ V}(g \star P).$$

Proof. It is enough to prove this theorem for P, f, g with small enough support so that we may assume that there exist morphisms of bi-submersions $U \circ U' \to W'$ and $U'' \circ U \to W''$ which are submersions respectively transverse to $V \circ U'$ and $U'' \circ V$. Then the morphisms $\mathrm{id} \times p : (u'', u', u') \mapsto (u'', p(u, u'))$ and $q \times \mathrm{id}$ are morphisms of bi-submersions and are submersions respectively transverse to $U'' \circ V \circ U'$.

We have $\theta_{U''}(g)\theta_{U\circ U', V\circ U'}(P \star f) = \theta_{U''\circ W'}((\mathrm{id} \times p)(g \star P \star f)) = \theta_{U''\circ U\circ U', U''\circ V\circ U'}(g \star P \star f)$ by Corollary 3.7. In the same way $\theta_{U''\circ U, U''\circ V}(g \star P)\theta_{U'}(f) = \theta_{U''\circ U\circ U', U''\circ V\circ U'}(g \star P \star f)$. □

Note that $\tilde{\theta}_{U,V}(P)$ is a closable multiplier, since its adjoint contains $\tilde{\theta}_{U^{-1},V}(P^*)$ and is therefore densely defined.

Let $p : U' \to U$ be a submersion and a morphism of bi-submersions such that $p(V') \subset V$. Recall from prop. 1.7 that there is a natural map $p_! : \mathcal{P}(U', V'; \Omega^{1/2}) \to \mathcal{P}(U, V; \Omega^{1/2})$ obtained by integration along fibers.

Proposition 3.9. *With the above notation, we have $\tilde{\theta}_{U,V} \circ p_! = \tilde{\theta}_{U',V'}$.* □

3.4 The space of pseudodifferential multipliers

We have fixed an atlas $\mathcal{U} = (U_i, t_i, s_i)_{i\in I}$ together with identity bisections $V_i \subset U_i$ such that $\bigcup_{i\in I} s_i(V_i) = M$ ([5]).

Denote by $\widetilde{U}$ the disjoint union $\coprod_{i\in I} U_i$ and by $\widetilde{V} \subset \widetilde{U}$ the disjoint union $\coprod_{i\in I} V_i$.

Definition 3.10. Let $m \in \mathbb{Z}$. We form a space $\Psi_c^m(\mathcal{U}, \mathcal{V})$. This is the image in the multiplier algebra of $\theta(\mathcal{A}(\mathcal{U}))$ of $\bigoplus_{i\in I} \mathcal{P}_c^m(U_i, V_i; \Omega^{1/2}) = \mathcal{P}_c^m(\widetilde{U}, \widetilde{V}; \Omega^{1/2})$.

We define the *space of pseudodifferential multipliers* to be the union $\Psi_c^\infty(\mathcal{U}, \mathcal{V})$ of $\Psi_c^m(\mathcal{U}, \mathcal{V})$.

An element in $\bigcap_{m\in\mathbb{Z}} \Psi_c^m(\mathcal{U}, \mathcal{V})$ is called *regularizing.*

Let (U, t, s) be a bi-submersion adapted to $\mathcal{U}$ and V an identity bisection in U. We just constructed a linear map $\tilde{\theta}_{U,V} : \mathcal{P}_c(U, V; \Omega^{1/2}) \to \Psi_c^\infty(\mathcal{U}, \mathcal{V})$.

Elements of $\mathcal{A}(\mathcal{U})$ give obviously rise to regularizing operators. On the other hand, a regularizing operator is for every k the image of a function $f_k \in C_c^k(\widetilde{U}; \Omega^{1/2})$; as the map θ is not injective, it is not clear whether f_k can be taken constant (*i.e.* in $\mathcal{A}(\mathcal{U})$).

3.5 The longitudinal principal symbol

Definition 3.11. The *longitudinal principal symbol* of an element of $\mathcal{P}_c^m(\widetilde{U}, \widetilde{V}; \Omega^{1/2})$ is a homogeneous function on the space $\mathcal{F}^* \setminus M$ of non zero elements in $\mathcal{F}^*$.

To construct it, we may assume that $s_i : V_i \to M$ are injective. Identify V_i with its image in M. Let $P_i \in \mathcal{P}(U_i, V_i; \Omega^{1/2})$ of order m. The principal symbol $\tilde{\sigma}_m(P_i)$ is defined to be the restriction to the subspace $\mathcal{F}^* \setminus M$ of $N^* \setminus V$ of the principal symbol of P_i. Extending it by linearity, we define the longitudinal principal symbol of any element of $\bigoplus_{i\in I} \mathcal{P}_c^m(U_i, V_i; \Omega^{1/2})$.

Remark 3.12. It is not obvious whether the longitudinal principal symbol is defined in the image $\Psi_c^m(\mathcal{U}, \mathcal{V})$. This happens if the groupoid $\mathcal{G}_\mathcal{U}$ is longitudinally smooth. In that case, one may define $\tilde{\sigma}_m(P)(x, \xi)$ using the regular representation on $L^2((\mathcal{G}_\mathcal{U})_x)$ and a formula as in Proposition 1.4: $\tilde{\sigma}_m(x, \xi) = \lim_{n\to+\infty} \langle e^{in\varphi}\chi_n, (in)^{-m} P(e^{in\varphi}\chi)(x)$ where $\varphi \in C_c^\infty((\mathcal{G}_\mathcal{U})_x)$ with derivative ξ at x and χ_n is a suitable function which has L^2 norm 1 and has support around x - it is of the form $\chi_n(y) = n^{k/2}\chi(n\|x - y\|^2)$ where $k = \dim(\mathcal{G}_\mathcal{U})_x = \dim \mathcal{F}_x$.

Even when this longitudinal principal symbol is well defined though, it is not clear whether there is an exact sequence as in Proposition 1.4, since an element in $P \in \mathcal{P}_c^m(U, V; \Omega^{1/2})$ whose longitudinal principal symbol vanishes on $\mathcal{F}^* \subset N^*$ may not be in $\mathcal{P}_c^{m-1}(U, V; \Omega^{1/2})$. Namely, it is not clear to us whether there exists $Q \in \mathcal{P}_c^{m-1}(U, V; \Omega^{1/2})$ which has the same

[5] For a given i, V_i may be empty.

image in $\Psi_c^\infty(\mathcal{U},\mathcal{V})$ as P. Here is an example of this situation: Consider the foliation defined by the the action of $SO(3)$ in $\mathbb{R}^3$ (example 2.11) and take the order 0 symbol $a(x,\xi) = e^{-\frac{1}{\langle x|\xi\rangle^2}}$ outside $\mathcal{F}^*$ and zero in $\mathcal{F}^*$. This is a symbol of order 0 which vanishes on $\mathcal{F}^*$, but there is no pseudodifferential operator of order -1 whose symbol is a in a neighborhood of $\mathcal{F}^*$.

Remark 3.13. If we change atlases, the pseudodifferential operators don't really change. Indeed, let (U,t,s) be a bi-submersion and V an identity bisection. Let $\mathcal{U}$ be any atlas (*e.g.* the minimal one). Then, since $\mathcal{U}$ carries the identity bisection, there is a neighborhood U' of V in U such that (U',t,s) is adapted to $\mathcal{U}$. Thus, there is an open cover (U'_i) of U' and morphisms $f_i : U'_i \to \widetilde{U}$ such that $f_i(V \cap U'_i) \subset \widetilde{V}$ (*cf.* prop. 2.5). Every element of $\mathcal{P}_c(U,V;\Omega^{1/2})$ can be written as $P+h$ with $P \in \mathcal{P}_c(U',V;\Omega^{1/2})$ and $h \in C_c^\infty(U,\Omega^{1/2})$.

We deduce:

a) If we change the identity bisections we don't change at all the space $\Psi_c^\infty(\mathcal{U},\mathcal{V})$.

b) Let $\mathcal{U}$ and $\mathcal{U}'$ be atlases such that $\mathcal{U}$ is adapted to $\mathcal{U}'$. We have a natural morphism $j : C^*(\mathcal{U}) \to C^*(\mathcal{U}')$. If j is injective, then we have an equality $\mathcal{P}(\mathcal{U}',\mathcal{V}') = \mathcal{P}(\mathcal{U},\mathcal{V}) + \theta(\mathcal{A}(\mathcal{U}'))$.

3.6 Convolution: the algebra of pseudodifferential kernels

Lemma 3.14. *Let U,W be bi-submersions adapted to $\mathcal{U}$, $V \subset U$ an identity bisection and $p : U \circ U \to W$ a morphism of bi-submersions which is a submersion strictly transverse to $V \circ U$ and to $U \circ V$. Then, for $Q_1, Q_2 \in \mathcal{P}_c(U,V;\Omega^{1/2})$ we have $\tilde\theta(Q_1)\tilde\theta(Q_2) = \tilde\theta(p_!(Q_1 \star Q_2))$.*

Note that by proposition 1.10, $Q_1 \star Q_2$ makes sense as a distribution and $p_!(Q_1 \star Q_2)$ is pseudodifferntial.

Proof. Let U' be another bi-submersion and $f \in C_c^\infty(U';\Omega^{1/2})$. We have to show that $\tilde\theta(Q_1)\tilde\theta(Q_2)\theta(f) = \tilde\theta(p_!(Q_1 \star Q_2))\theta(f)$. To that end, we may take a faithful representation Π and compute $\langle \eta, \Pi\Big(\tilde\theta(Q_1)\tilde\theta(Q_2)\theta(f)\Big)\xi\rangle$ and $\langle \eta, \Pi\Big(\tilde\theta(p_!(Q_1 \star Q_2))\theta(f)\Big)\xi\rangle$. These two expressions are equal when Q_1 and Q_2 are smooth functions. The general case follows using §1.2.3. □

Theorem 3.15. *The space $\Psi_c^\infty(\mathcal{U},\mathcal{V})$ is a subalgebra of the multiplier algebra of $\theta(\mathcal{A}(\mathcal{U}))$. More precisely, given $P_i \in \mathcal{P}_c^{m_i}(\widetilde{U},\widetilde{V};\Omega^{1/2})$ $(i = 1,2,$ there is $P \in \mathcal{P}_c^m(\widetilde{U},\widetilde{V};\Omega^{1/2})$ such that $\tilde\theta(P_1)\tilde\theta(P_2) = \tilde\theta(P)$ and $\tilde\sigma_{m_1}(P_1)\tilde\sigma_{m_2}(P_2) = \tilde\sigma_m(P)$.*

Proof. By prop. 2.7, there is a cover of M by (open) sets $s(V'_i)$ where V'_i is an identity bisection of a bi-submersion U'_i adapted to $\mathcal{U}$ for which there is a morphism of bi-submersions $p'_i : U'_i \circ U'_i \to W_i$ which is a submersion strictly transverse to $V'_i \circ U'_i$ and to $U'_i \circ V'_i$.

There is a finite open cover (U''_j) of the $Supp(P_1) \cup Supp(P_2)$ such that, putting $V''_j = U''_j \cap \widetilde{V}$, if $s(V''_j) \cap s(V''_k) \neq \emptyset$, then there are morphisms of bi-submersions from U''_j and from U''_k to the same U'_i which are submersions.

Using a partition of the identity adapted to U_j'', we are reduced to the case where $P_1 \in \mathcal{P}_c^{m_1}(U_j'', V_j''; \Omega^{1/2})$ and $P_2 \in \mathcal{P}_c^{m_2}(U_k'', V_k''; \Omega^{1/2})$.

If $V_j'' \circ V_k'' = \emptyset$ then $P_1 \star P_2 \in \mathcal{P}_c(U_j'' \circ U_k'', W; \Omega^{1/2})$ with $W = V_j'' \circ U_k'' \cup U_j'' \circ V_k''$, and therefore $\tilde{\theta}(P_1)\tilde{\theta}(P_2) \in \theta(\mathcal{A}_{\mathcal{U}})$.

If $s(V_j'') \cap s(V_k'') \neq \emptyset$, we may replace P_1 and P_2 by their images Q_1, Q_2 in $\mathcal{P}_c(U_i', V_i'; \Omega^{1/2})$ (prop. 3.9). Now, by proposition 1.10, $Q_1 \star Q_2$ makes sense as a distribution and $(p_i')_!(Q_1 \star Q_2)$ is pseudodifferntial and has the right longitudinal principal symbol. The result follows by Lemma 3.14. □

4 Longitudinal ellipticity

In this section we assume that the manifold M is compact.

Definition 4.1. A pseudodifferential operator $P \in \mathcal{P}_c^m(\mathcal{U}, \mathcal{V})$ is said to be *longitudinally elliptic* if its longitudinal principal symbol $\tilde{\sigma}_m(P)$ is invertible.

4.1 Parametrix

We now state the analogues in our setting of some most important classical results in the pseudodifferental calculus.

Theorem 4.2 (Existence of quasi-inverses). *Let $P \in \mathcal{P}_c^m(\widetilde{U}, \widetilde{V}; \Omega^{1/2})$ be a longitudinally elliptic operator of order m. There is a pseudodifferential operator $Q \in \mathcal{P}_c^{-m}(\widetilde{U}, \widetilde{V}; \Omega^{1/2})$ of order $-m$ such that $1 - \tilde{\theta}(P)\tilde{\theta}(Q)$ and $1 - \tilde{\theta}(Q)\tilde{\theta}(P)$ are regularizing.*

The main ingredient of the proof is:

Lemma 4.3. *Let $P \in \mathcal{P}_c^m(\widetilde{U}, \widetilde{V}; \Omega^{1/2})$ be longitudinally elliptic and $S \in \Psi_c^k(\widetilde{U}, \widetilde{V})$. Then there exists $Q \in \mathcal{P}_c^{k-m}(\widetilde{U}, \widetilde{V}; \Omega^{1/2})$ such that $\tilde{\theta}(P)\tilde{\theta}(Q) - S \in \Psi_c^{k-1}(\widetilde{U}, \widetilde{V})$.*

which, in turn, relies on the following result:

Lemma 4.4. *Let $P \in \mathcal{P}_c^m(\widetilde{U}, \widetilde{V}; \Omega^{1/2})$ be longitudinally elliptic. Then there exists the following data:*

(i) *a finite set I, bi-submersions $(U_i', t_i', s_i')_{i \in I}$, identity bisections $V_i' \subset U_i'$ and morphisms of bi-submersions $p^i : U_i' \circ U_i' \to W_i$ that are submersions strictly transverse to $U_i' \circ V_i'$ and to $V_i' \circ U_i'$;*

(ii) *open relatively compact subsets $U_i \subset U_i'$ such that $V_i = V_i' \cap U_i$ is relatively compact in V_i';*

(iii) $\bigcup_{i \in I} s_i(V_i) = M$*;*

(iv) *operators* $P'_i \in \mathcal{P}^m_c(U'_i, V'_i; \Omega^{1/2})$ *whose (plain) principal symbol* $\sigma_m(P'_i)$ *is invertible on* $\overline{V}_i$*;*

(v) *smooth functions* $\phi_i \in C^\infty_c(U'_i)$ *such that* $\phi_i|_{V_i} = 1$

so that $\phi_i P - P'_i$ *is regularizing.*

Proof of 4.4. Let $x \in M$; consider a pair (U', V') such that (U', t', s') is a bi-submersion, V' is an identity bisection, which is minimal at a point $v \in V'$ with $s(v) = x$. Take $\phi \in C^\infty_c(M)$ to be 1 in a neighborhood of x with support in $s(V')$. There exists an operator $P' \in \mathcal{P}(U', V')$ of order m, such that $\phi P - P'$ is regularizing. Whence $\sigma_m(P)\phi = \sigma_m(P')|_{\mathcal{F}^*}$ and $\sigma_m(P')(v, \xi)$ is invertible for every $\xi \in N^*_v$ since (U', V') is minimal at v. The set of $w \in V'$ for which $\sigma_m(P')(w, \xi)$ is invertible for all $\xi \in N^*_w$ is open (by compactness of the spheres). It follows that $\sigma_m(P')(w, \xi)$ is invertible for every w in a small enough neighborhood of u in V' and $\xi \in N^*_w$.

The result follows by compactness of M (using prop. 2.7). □

Here are the proofs of the previous two results:

Proof of 4.3. Consider the data $(U_i, V_i), (U'_i, V'_i)$ and P'_i of 4.4 associated to P. Let $(\chi_i)_{i\in I}$ be a partition of unity associated to the cover $(V_i)_{i\in I}$.

Since the (plain) symbol of P'_i is invertible over $\overline{V}_i$, there exist $T_i \in \mathcal{P}(U'_i, V'_i; \Omega^{1/2})$ of order $-m$ whose (plain) principal symbol is $\sigma_m(P'_i)^{-1}$ in a neighborhood of $\overline{V}_i$. Then $P'_i T_i \chi_i - \chi \in \mathcal{P}^{-1}_c(U'_i, V'_i; \Omega^{1/2})$ (we use $p^i_!$ to make this composition).

There is an operator $R_i \in \mathcal{P}^k_c(U_i, V_i; \Omega^{1/2})$ whose image in $\Psi^\infty_c(\mathcal{U}, \mathcal{V})$ is $\chi_i S$ up to regularizing operators. Put then $Q = \sum_{i\in I} T_i R_i$. □

Proof of 4.2. Lemma 4.3 allows us to follow the classical proof:
First construct $Q_0 \in \mathcal{P}^{-m}_c(\widetilde{U}, \widetilde{V}; \Omega^{1/2})$ such that $I - Q_0 P$ is of negative order. By putting $Q_k = Q_0(I - PQ_0)^k = (I - Q_0 P)^k Q_0$ we obtain a sequence of operators of order $-m-k$, $i \in \mathbb{N}$. From 1.5 it follows that there exists Q of order $-m$ (asymptotically the sum of the Q_k) such that $I - PQ$ and $I - QP$ are regularizing. □

4.2 Square roots

Theorem 4.5 (square roots)**.** *If* $P \in \mathcal{P}^{2m}_c(\widetilde{U}, \widetilde{V}; \Omega^{1/2})$ *is self-adjoint of even order and* $\tilde{\sigma}_{2m}(P) > 0$*, there is a self-adjoint* $Q \in \mathcal{P}^m_c(\widetilde{U}, \widetilde{V}; \Omega^{1/2})$ *such that* $P - Q^2$ *is smoothing.*

Proof. We use lemma 4.4 and the notation there. Coming back to its proof, we may assume that the (plain) symbol of P'_i restricted to V_i is > 0. Let then $Q'_i \in \mathcal{P}^{-1}_c(U'_m, V'_i; \Omega^{1/2})$ that we may assume self-adjoint, such that the restriction to V_i of its plain principal symbol is

$\sqrt{\sigma_{2m}(P_i')}$. Let χ_i^2 be a partition of the identity adapted to V_i, and put $Q_0 = \sum_{i\in I} \chi_i Q_i' \chi_i$. Then $P - Q_0^2$ is of order $2m-1$. Note that Q_0 is elliptic

Suppose we constructed $Q_0, \dots, Q_{n-1}$ self-adjoint, such that Q_j has order $m-j$ and $R_n = P - (\sum_{j=0}^{n-1} Q_j)^2$ is self adjoint of order $2m-n$. Thanks to lemma 4.3, we find Q_n of order $m-n$ such that $2Q_0Q_n - R_n$ has order $m-n-1$. It is a consequence of prop. 1.10 that Q_0 and Q_n commute up to lower order, therefore $2Q_nQ_0 - R_n$ has order $m-n-1$. Since Q_0 and R_n are selfajoint, we may replace Q_n by $1/2(Q_n + Q_n^*)$. Hence Q_n is a sequence that satisfies 1.5. Put Q' the asymptotic sum of the Q_js and $Q = \frac{Q' + Q'^*}{2}$. This is self-adjoint and also an asymptotic sum for Q_n. By construction, $P - Q^2$ is smoothing. □

5 The extension of zero order pseudodifferential operators

Let us begin by a remark that will allow us to assume that the manifold M is compact.

Remark 5.1. a) Let M' be an open subset of M. Then $C_c^\infty(M')\mathcal{F}$ is a foliation $\mathcal{F}'$ on M'. A bi-submersion (U, t, s) for $(M, \mathcal{F})$ restricts to a bi-submersion of $(M', \mathcal{F}')$ by putting $U' = \{u \in U;\ s(u) \in M',\ t(u) \in M'\}$. In this way, an atlas $\mathcal{U}$ of $(M, \mathcal{F})$ restricts to an atlas of $\mathcal{U}'$ of $(M', \mathcal{F}')$. By extending compactly supported functions on $\mathcal{U}'$, we embed $C^*(\mathcal{U}')$ into $C^*(\mathcal{U})$.

b) Assume M' is relatively compact in M. Then there exists $f \in C_c^\infty(M)$ which is everywhere nonzero on M'. Note that $\{fX;\ X \in \mathcal{F}\}$ is a foliation on M which has the same restriction to M' as M. There is a compact manifold M'' which contains an open subset diffeomorphic to a neighborhood of the support of f. Then M'' carries a foliation which has the same restriction to M' as $\mathcal{F}$.

Lemma 5.2. *Every sufficiently negative order pseudodifferential operator defines an element in $C^*(\mathcal{U})$. More precisely, given a bi-submersion (U, t, s) and an identity bisection $V \subset U$, let $P \in \mathcal{P}_c^{-m}(U, V; \Omega^{1/2})$ with m strictly bigger than the dimension of the fibers of s and t, then $\tilde{\theta}(P) \in C^*(\mathcal{U})$.*

Proof. A continuous function with compact support in U defines an element in $C^*(\mathcal{U})$ (thanks to the L^1 estimate and by density of $C_c^\infty(U)$). Now, if a is of sufficiently negative order, the integral

$$\iint_{N^*v} a(v, \xi)\chi(u)e^{i\langle u, \xi\rangle}$$

makes sense and thus the distribution P is actually a continuous function with compact support in U. □

Theorem 5.3. a) *Negative order pseudodifferetial operators are in $C^*(\mathcal{U})$, as well as those zero order operators whose principal symbol vanishes on $\mathcal{F}^*$.*

b) *Zero order pseudodifferential operators define bounded multipliers of the C^*-algebra of the foliation.*

Proof. Using remark 5.1, we may assume M is compact. By 4.5, if $\|\tilde{\sigma}_P^0\| < t$, there is $Q \in \Psi_c^\infty(\mathcal{U}, \mathcal{V})$ such that $P^*P + Q^*Q = t^2 + R$ where R is of negative enough order, so that it belongs to the C^*-algebra of the foliation (in fact it can even be taken smoothing). We have $(Pf)^*(Pf) + (Qf)^*(Qf) = t^2 f^*f + f^*Rf$ for all $f \in \mathcal{A}(\mathcal{U})$.

It follows that:

- $\|Pf\| \leq k\|f\|$, where $k = \sqrt{t^2 + \|R\|}$, hence P extends to a bounded multiplier and (b) follows.
- if $\sigma_P^0 = 0$, then in the quotient C^*-algebra $\overline{\Psi_c^\infty(\mathcal{U}, \mathcal{V})}/C^*(\mathcal{U})$, the norm of P is $\leq t$ for all $t > 0$, whence $P \in C^*(\mathcal{U})$. □

We thus have an exact sequence of C^*-algebras

$$0 \to C^*(M, \mathcal{F}) \to \Psi^*(M, \mathcal{F}) \to B \to 0 \tag{5.1}$$

where $\Psi^*(M, \mathcal{F})$ denotes the closure of the algebra of zero order pseudodifferential operators with respect to multiplier norm and order 0 symbol. The algebra B is a quotient of the algebra $C_0(S^*\mathcal{F})$ of continuous functions on the cosphere "bundle". As discussed in remark 3.12, if the groupoid $\mathcal{G}_\mathcal{U}$ is longitudinally smooth, then $B = C_0(S^*\mathcal{F})$.

6 Longitudinally elliptic operators of positive order

In this section we assume that M is compact.

6.1 Longitudinally elliptic operators and regular multipliers

Recall [2, 3, 19] that an unbounded multiplier T of a C^*-algebra is said to be *regular* if it is densely defined, its adjoint is densely defined and its graph is *orthocomplemented*, which means that $A \oplus A = G \oplus G^\perp$, where $G = \{(x, Tx);\ x \in \operatorname{dom} T\}$ is the graph of T and $G^\perp = \{(T^*y, -y);\ y \in \operatorname{dom} T^*\}$ its orthogonal complement for the obvious A valued scalar product in $A \oplus A$.

Let Π be a non degenerates representation of A. It extends to a representation $\tilde{\Pi}$ of the multiplier algebra $\mathcal{M}(A)$. Every regular unbounded multiplier T of A gives rise to a closed operator $\widehat{\Pi}(T)$ whose graph is the closure of $\{(\Pi(a)\xi, \Pi(Ta)\xi);\ a \in \operatorname{dom} T;\ \xi \in H_\Pi\}$. The adjoint of $\widehat{\Pi}(T)$ is $\widehat{\Pi}(T^*)$. In particular, if T is self adjoint, so is $\widehat{\Pi}(T)$.

In [18, §3,4] Vassout proved that elliptic pseudodifferential operators (of positive order) on a Lie groupoid G give rise to regular operators. The proof in [18] can be adapted to our setting to show:

Theorem 6.1. *If $P \in \Psi_c^m(\mathcal{U},\mathcal{V})$ is the image of a longitudinally elliptic operator of order m, then $\overline{P}$ is a regular multiplier on $C^*(\mathcal{U})$ ($m > 0$).*

Proof. For every $S \in \Psi_c^0(\mathcal{U},\mathcal{V})$, denote by $\overline{S}$ its closure which is a multiplier of $C^*(\mathcal{U})$.
Let Q be a parametrix of P and write $I - QP = R$. Let $T \in \mathcal{L}(C^*(\mathcal{U}) \oplus C^*(\mathcal{U}))$ be the (adjointable) operator of the C^*-module $C^*(\mathcal{U}) \oplus C^*(\mathcal{U})$ with matrix $\begin{pmatrix} \overline{R} & \overline{Q} \\ \overline{PR} & \overline{PQ} \end{pmatrix}$.

The restriction to $\theta(\mathcal{A}) \oplus \theta(\mathcal{A})$ of T^2 has matrix $\begin{pmatrix} (R+QP)R & (R+QP)Q \\ P(R+QP)R & P(R+QP)Q \end{pmatrix}$, whence $T^2 = T$ (as $R + QP = I$) .

- Since T is an (adjointable) idempotent element in $\mathcal{L}(C^*(\mathcal{U}) \oplus C^*(\mathcal{U}))$, its range is orthocomplemented.
- Since T is continuous, and $\theta(\mathcal{A}(\mathcal{U})) \oplus \theta(\mathcal{A}(\mathcal{U}))$ is dense in $C^*(\mathcal{U}) \oplus C^*(\mathcal{U})$, we deduce that the range of T is the closure of $T(\theta(\mathcal{A}(\mathcal{U})) \oplus \theta(\mathcal{A}(\mathcal{U})))$
- If $(x, y) \in \theta(\mathcal{A}(\mathcal{U}))$, we find $T(x, y) = (Rx + Qy, P(Rx + Qy))$. If furthermore $y = Px$, $T(x, y) = (x, y)$. It follows that $T(\theta(\mathcal{A}(\mathcal{U})) \oplus \theta(\mathcal{A}(\mathcal{U})))$ is the graph of P.

We just proved that the closure of the graph of P is orthocomplemented, *i.e.* $\overline{P}$ is regular. □

Remarks 6.2. In the same way we may adapt the proofs of [18] to our setting to prove:

a) Any two longitudinally elliptic operators $P, P' \in \Psi_c^\infty(\mathcal{U},\mathcal{V})$ of the same order have the same domain.

b) Longitudinally elliptic operators define a filtration of $C^*(\mathcal{U})$ by Sobolev modules. If P is of order $k > 0$ then

   - $H^k = \operatorname{dom} P$ with scalar product $\langle \alpha, \beta\rangle_k = \langle P\alpha, P\beta\rangle + \langle \alpha, \beta\rangle$
   - H^{-k} is the completion of $C^*(\mathcal{U})$ with the norm $\|\xi\|_{-k} = \|(1 + P^*P)^{-1/2}\xi\|$.

   This filtration satisfies the following properties:

   - If $k > k'$ then the identity on $\mathcal{A}_\mathcal{U}$ extends to a compact morphism of Hilbert modules $i_{k,k'} : H^k \hookrightarrow H^{k'}$.
   - Any $P \in \Psi_c^\infty(\mathcal{U},\mathcal{V})$ of order m defines an element of $\mathcal{L}(H^k; H^{k-m})$ for any k.

c) Using the Sobolev spaces above one can define an algebra $\Psi^{-\infty}(\mathcal{U})$ of smoothing pseudodifferential operators without compact support by calling an operator R smoothing iff $R \in \bigcap_{s,t\in\mathbb{R}} \mathcal{L}(H^s; H^t)$. It follows that $\mathcal{A}_\mathcal{U} \subset \Psi^{-\infty}(\mathcal{U})$ is a dense subalgebra.

6.2 Application: Laplacian of a singular foliation

As a particular case we may construct a Laplacian operator for every foliation and prove that it is a positive self-adjoint operator of $L^2(M)$.

Let $(M, \mathcal{F})$ be a foliation. Every vector field $X \in \mathcal{F}$ defines a differential hence pseudodifferential operator $X \in \Psi_c^\infty(\mathcal{U}, \mathcal{V})$ (*cf.* example 1.2).

Since M is compact, $\mathcal{F}$ is generated by finitely many vector fields $X_1, \ldots, X_N$.

Definition 6.3. The element $\Delta = \sum_{k=1}^{N} X_k^* X_k$ is called a *Laplacian* of the foliation $\mathcal{F}$.

From the definition of Δ we have:

Theorem 6.4. *A Laplacian Δ is a formally self adjoint elliptic operator of order 2; it therefore defines a regular (unbounded) self adjoint multiplier of $C^*(\mathcal{U})$.* □

We may apply that to the natural representation of $C^*(\mathcal{U})$ on $L^2(M)$ (which can be seen as the integration of the trivial representation $(\lambda, \mathbb{C})$ of the groupoid in the sense of [1, section 5.1] where λ is the Lebesgue measure on M).

Corollary 6.5. *The Laplacian Δ defines an unbounded, self-adjoint operator Δ of $L^2(M)$. In other words, take vector fields $X_1, \ldots, X_N$ on a compact manifold. Assume that the module they generate is a foliation, i.e. for every (i, j) there exist $f_{i,j,k} \in C^\infty(M)$ such that $[X_i, X_j] = \sum_k f_{i,j,k} X_k$. Then the closure in $L^2(M)$ of $\sum_{k=1}^{N} X_j^* X_j$ is self adjoint.*

Remarks 6.6. a) One can apply theorem 6.4 to other natural representations of $C^*(\mathcal{U})$. One may for instance take the representation on L^2 of a leaf. There, the Laplacian is elliptic and the difficulty comes from the fact that the leaf may not be compact.

b) The spectrum of the image of every regular operator - and in particular of Δ - is the same if we take two weakly equivalent representations of $C^*(\mathcal{U})$. This applies if we compare the representation in $L^2(M)$ and a representation in L^2 of a dense leaf (with some amenability asumptions).

References

[1] I. Androulidakis and G. Skandalis, The holonomy groupoid of a singular foliation. *J. Reine Angew. Math.* **626**(2009), 1-37.

[2] S. Baaj, *Multiplicateurs non bornés.* Thèse, Univ. P. et M.Curie, Paris. (1980)

[3] S. Baaj and P. Julg, Théorie bivariante de Kasparov et opérateurs non bornés dans les C^*-modules hilbertiens, *Note C.R.A.S.* Paris, **296**, Sér. **I** (1983), 875–878.

[4] B. Bigonnet, J. Pradines, Graphe d'un feuilletage singulier, *C. R. Acad. Sci. Paris* **300** (13) (1985) 439–442.

[5] A. Connes, Sur la théorie non commutative de l'intégration. *in* Algèbres d'opérateurs *Springer Lect. Notes in Math.* **725** (1979) 19–143.

[6] A. Connes, A survey of foliations and operator algebras. *in* Operator Algebras and Applications *Proc. Symp. in Pure Math. A.M.S.* **38** part I, (1982) 521–628.

[7] C. Debord, Local integration of Lie algebroids. *Banach Center Publications* **54** (2001) 21–33.

[8] C. Debord, Holonomy groupoids of singular foliations. *J. Diff. Geom.* **58** (2001) 467–500.

[9] Th. Fack et G. Skandalis, Sur les représentations et idéaux de la C^*-algèbre d'un feuilletage. *J. Op. Th.* **8** (1982) 95-129.

[10] M. Khoshkam, G. Skandalis, Regular representation of groupoid C^*-algebras and applications to inverse semigroups. *J. Reine Angew. Math.* **546** (2002), 47–72.

[11] E. Christopher Lance, *Hilbert C^*-modules.* London Mathematical Society Lecture Note Series, **210** Cambridge University Press, Cambridge, 1995.

[12] B. Monthubert and F. Pierrot, Indice analytique et groupoïdes de Lie. *C. R. Acad. Sci. Paris* Ser. I, **325** (1997) no. 2, 193–198.

[13] V. Nistor, A. Weinstein and P. Xu, Pseudodifferential operators on differential groupoids. *Pacific J. Math.* **189** (1999) no. 1, 117–152.

[14] J. Pradines, How to define the differentiable graph of a singular foliation, *C. de Top. et Geom. Diff. Cat.* XXVI(4) (1985) 339–381.

[15] J. Pradines, Feuilletages: holonomie et graphes locaux. *C. R. A. S.* série I, **298** no 13 (1984) 297–300.

[16] J.N. Renault, *A groupoid approach to C^*-algebras. L.N. in Math.* **793** (1979).

[17] M. E. Taylor *Pseudodifferential Operators.* Princeton University Press.

[18] S. Vassout, Unbounded pseudodifferential Calculus on Lie groupoids, *Journal of Functional Analysis* **236** (2006) 161–200.

[19] S.L. Woronowicz and K. Napiórkowski, Operator theory in the C^*-algebra framework, *Rep. Math. Phys.*, **31** (1992) n° 3, 353–371.

[20] H. E. Winkelnkemper, The graph of a foliation. *Ann. Glob. Analysis and Geometry* **1** (3) (1983) 51–75.