The use of C^* -algebras in singular foliations and their representation theory

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Examples

M: compact manifold.

1 Orbits of (some) Lie group actions on M. Vector fields: image of infinitesimal action $\mathfrak{g} \to \mathfrak{X}(M)$.

Focus on $\mathcal{F} = \langle X \rangle$:

- 2 X nowhere vanishing vector field of $M \rightsquigarrow \text{action of } \mathbb{R}$ on M.
- 3 Irrational rotation on torus T²: "Kronecker" flow of $X = \frac{d}{dx} + \theta \frac{d}{dy}$. \mathbb{R} injected as a dense leaf.
- 4 "Horocyclic" foliation:
 - Let Γ cocompact subgroup of $SL(2,\mathbb{R})$. Put $M = SL(2,\mathbb{R})/\Gamma$.
 - \mathbb{R} is embedded in $SL(2,\mathbb{R})$ by $\left(\begin{array}{cc} 1 & 0 \\ t & 1 \end{array}\right)$, $t\in\mathbb{R}.$
 - ▶ Therefore \mathbb{R} acts on M. Action is ergodic, \exists dense leaves.

Laplacians of Kronecker foliation

Kronecker foliation on $M=T^2$: $\mathcal{F}=\langle \frac{d}{dx}+\theta \frac{d}{dy}\rangle$. $L=\mathbb{R}$ Two Laplacians:

- $\Delta_L = rac{d^2}{dx^2}$ acting on $L^2(\mathbb{R})$
- $\Delta_M = -X^2$ acting on $L^2(M)$

By Fourier:

- $\Delta_L \leadsto \text{mult. by } \xi^2 \text{ on } L^2(\mathbb{R}).$ Spectrum: $[0, +\infty)$.
- $\Delta_M \rightsquigarrow \text{mult. by } (n + \theta k)^2 \text{ on } L^2(\mathbb{Z}^2)$. Spectrum dense in $[0, +\infty)$.

Qn 1: Do Δ_L and Δ_M have the same spectrum for every (regular) foliation?

Qn 2: If so, how to calculate this spectrum?

Tools: Holonomy groupoid $H(\mathfrak{F})$, Longitudinal pseudodifferential calculus, Groupoid C^* -algebra(s).

The C^* -algebra of a Lie groupoid (Connes, Renault)

For f, $g \in C_c^{\infty}(G)$:

- we put $f^*(x) = \overline{f(x^{-1})}$
- ▶ we want to form f * g by a formula

$$f * g(x) = \int_{yz=x} f(y)g(z)$$

In other words, we want to have an integration along the fibers of the composition $G \times_{s,t} G \to G$.

Use either Haar systems or half densities.

Proposition

The above involution and product make $C_c^{\infty}(G)$ a *-algebra.

"Reduced" $C_r^*(G)$: completion with left regular representation

"Full" $C^*(G)$: completion with all representations

Quotient $C^*(G) \to C^*_r(G)$.

Basic tool: Pseudodifferential calculus (Connes)

The Lie algebra of vector fields tangent to the foliation acts by unbounded multipliers on $C_c^\infty(G)$. The algebra generated is the algebra of differential operators.

Using Fourier transform one can write a differential operator P (acting by left multiplication on $f \in C_c^\infty(G)$) as:

$$(Pf)(x,y) = \int exp(i\langle \varphi(x,z),\xi\rangle)\alpha(x,\xi)\chi(x,z)f(z,y)d\xi dz$$

Proposition (Connes

- ▶ Negative order pseudodifferential operators $\in C^*(M, F)$
- ▶ Zero order pseudodifferential operators: multipliers of $C^*(M, F)$.

Together with multiplicativity of the principal symbol this gives an exact sequence of C^* -algebras:

$$0 \rightarrow C^*(M, F) \rightarrow \Psi^*(M, F) \rightarrow C(SF^*) \rightarrow 0$$

Laplacians revisited

Theorem (Connes, Kordyukov, Vassout)

Elliptic operators of positive order are regular unbounded multipliers (in the sense of Baaj-Woronowicz: $graph(D) \oplus graph(D)^{\perp}$ is dense).

More generally M compact, (M, F) regular foliation.

- Lie algebra $\mathfrak{F}=C^\infty(M,F)$ acts on $C^\infty(G)$ by unbounded multipliers.
- Laplacian $\Delta = \sum X_i^2$ is an unbounded (regular) multiplier of $C^*(M, \mathcal{F})$.

 $L^2(L), L^2(M)$ are representations of $C^*(M, \mathcal{F})$.

Proposition (Baaj, Woronowicz)

Every representation extends to regular multipliers.

We recover Laplacians $\Delta_{\rm I}$, $\Delta_{\rm M}$.

Proof of theorems 1 and 2

Theorem 1

 Δ_M and Δ_L are essentially self-adjoint.

- $ightharpoonup L^2(M)$ and $L^2(L)$: representations of the foliation C^* -algebras.
- Recall (Baaj, Woronowicz): Every representation extends to regular multipliers.

image of the adjoint = adjoint of the image

Theorem 2 (Kordyukov)

If all leaves L are dense + amenability assumptions, Δ_M and Δ_L have the same spectrum.

- (Fack and Skandalis): If the foliation is minimal (i.e. all leaves are dense) then the foliation C*-algebra is simple. Whence all representations are faithful.
- ▶ Every injective morphism of C*-algebras is isometric and isospectral.

Elliptic operators - Gaps of their spectrum

Theorem 3 (Connes)

In many cases, one can predict the possible gaps in the spectrum.

More precisely:

- Gaps in the spectrum \longrightarrow projections in $C^*(M, F)$.
- ▶ Projectionless $C^*(M, F)$: spectrum connected.
- Sometimes dimension function on projections (related with K-theory).
 - Values in N: few projections.
 - values in a dense subset of \mathbb{R}_+ : many projections.

Examples

Horocyclic foliation: no gaps in the spectrum

Let the " $\alpha x + b$ " group act on a compact manifold M. e.g. $M = SL(2,\mathbb{R})/\Gamma$ where Γ discrete co-compact group. Leaves = orbits of the "x + b" group (assume it is minimal).

The spectrum of the Laplacian is an interval $[m, +\infty)$

Proof: We show $C^*(M, F)$ projectionless.

- ▶ ∃ measure on M invariant by ax + b (amenable). x + b invariance \Longrightarrow trace on $C^*(M, F)$ faithful since $C^*(M, F)$ simple (Fack-Skandalis).
- ► The "ax" subgroup \longrightarrow action of \mathbb{R}_+^* on $C^*(M, F)$ which scales the trace.
- ▶ Image of K_0 countable subgroup of \mathbb{R} , invariant under \mathbb{R}_+^* action.

Similarly, Kronecker flow: Image of the trace $\mathbb{Z} + \theta \mathbb{Z}$

Can be (more or less) any closed subset of \mathbb{R}_+

Conclusions

Theorems 1 and 2 generalize to any singular foliation!

Definition (Stefan, Sussmann, A-Skandalis)

A (singular) foliation is a finitely generated sub-module ${\mathfrak F}$ of $C^\infty(M;TM)$, stable under brackets.

Examples

- I \mathbb{R} foliated by 3 leaves: $(-\infty, 0)$, $\{0\}$, $(0, +\infty)$. \mathcal{F} generated by $x^n \frac{\partial}{\partial x}$. Different foliation for every n.
- 2 \mathbb{R}^2 foliated by 2 leaves: $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$. No obvious best choice. $\mathcal F$ given by the action of a Lie group

$$GL(2,\mathbb{R})$$
, $SL(2,\mathbb{R})$, \mathbb{C}^*

IA+Skandalis (2006-today): Holonomy groupoid, foliation C^* -algebras, longitudinal pseudodifferential calculus...

Need to calculate $K_0(C^*(\mathcal{F}))!$

What does BC say? (I)

 Γ discrete group, torsion-free.

$$\mu: \mathsf{K}^{\Gamma}_*(\underline{\mathsf{E}\Gamma}) \to \mathsf{K}_*(C^*_{\mathfrak{r}}(\Gamma))$$
 isomorphism

- $E\Gamma$ = classifying space of proper Γ-actions (CW-complex)
- Ihs = Γ-equiv. K-homology
- ▶ rhs = K-theory of reduced C*-algebra
- ▶ completion with $\mathbb{C}\Gamma \to \mathrm{B}(\ell^2(\Gamma), \ell^2(\Gamma)), \quad g \mapsto r_g$

e.g.
$$\Gamma = \mathbb{Z}^n$$
:

- $\underline{E}\underline{\mathbb{Z}^n} = B\mathbb{Z}^n = T^n$
- $C_r^*(\mathbb{Z}^n) = C(T^n)$ (Fourier)
- μ is Poicaré duality

What does BC say? (II)

G Lie group, K compact subgroup. G acts on $M = K \setminus G$ on the right. Assume M has $Spin^c$ -structure. Put R(K) the free abelian group of (classes of) irreducible representations of K.

Define Dirac induction $\mu: R(K) \to K(C^*_r(G))$ as follows:

- ▶ Take $\rho \in R(K)$, say $\rho : K \to GL(V)$. Define a vector bundle $V_{\rho} = G \times_{K} V$ over M.
- ▶ Levi-Civita connection of spinor bundle $S \to M$ and Riemannian metric on M give Dirac operator $D_{\rho} : \Gamma(V_{\rho} \otimes S) \to \Gamma(V_{\rho} \otimes S)$
- ▶ Pull back to G and put

$$\mu: R(K) \to K(C_r^*(G)), \quad \rho \stackrel{\mu}{\longrightarrow} Ind(D_{\rho})$$

Facts:

- G compact, $K = \{pt\}$, get $\mu = id$.
- $K(C_r^*(G)) = K^G(pt) = R(G)$.
- K maximal compact, $\underline{EG} = K \backslash G = M$.
- ▶ Then, can identify R(K) with $K_j^G(M)$, where j = dim(M)mod2.

What does BC say? (III)

General geometric situations formulated in terms of a Lie groupoid

$$\mathscr{G} \Longrightarrow M$$
: There exists an assembly map

$$\mu: K_*^{top}(\mathscr{G}) \to K_*(C_r^*(\mathscr{G}))$$

defined as an analytic index map. (Wrong-way functoriality...)

Baum-Connes conjecture

The assemly map is an isomorphism. (Part of the conjecture is to specify explicitly the lhs!)

- ▶ How to read it: "All analytic representations come from geometry!"
- Analogue: Geometric quantization (apply Dirac induction to coadjoint orbits...)
- Counterexample by Higson, Lafforgue, Skandalis.
- Injectivity implies Novikov conjecture.
- Surjectivity implies Kaplansky conjecture.
- ▶ Use of BC: Calculate K(C*(𝒢))!

Careful look at action $SO(3) \subset \mathbb{R}^3$ (I)

$$dim(Lie(SO(3)))=3, \text{ so } \mathfrak{F}=span_{C^{\infty}(M)}\langle X,Y,Z\rangle.$$

Take any (M, \mathcal{F}) . At $x \in M$ put $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$. Get exact sequence

$$0 \to \mathfrak{g}_x \to \mathcal{F}_x \xrightarrow{e\nu_x} \mathsf{T}_x \mathsf{L}_x \to 0$$

- L_x regular $\Rightarrow \mathcal{F}_x = T_x L_x$
- $L_x \text{ singular} \Rightarrow \dim(\mathcal{F}_x) > \dim(L_x)$.
- $dim(\mathcal{F}_x)$ (upper) semicontinuous

For $(\mathbb{R}^3, \mathcal{F})$ we have:

- $\mathcal{F}_0 = \mathfrak{g}_x = \text{Lie}(SO(3))$, so $\dim(\mathcal{F}_0) = 3$
- For $x \neq 0$, $\dim(\mathcal{F}_x) = 2$

$$H(\mathcal{F}) = (S^2 \times S^2 \times \mathbb{R}^+_*) \cup SO(3) \times \{0\}$$

Careful look at action $SO(3) \subset \mathbb{R}^3$ (II)

$$H(\mathfrak{F}) = (S^2 \times S^2 \times \mathbb{R}^+_*) \cup SO(3) \times \{0\} \text{ decomposes } \mathbb{R}^3 \text{:}$$

- $\Omega_1 = \{x \in \mathbb{R}^3 : dim(\mathcal{F}_x) \leqslant 3\} = \mathbb{R}^3$
- $\Omega_0 = \{x \in \mathbb{R}^3 : \dim(\mathcal{F}_x) \leqslant 2\} = \mathbb{R}^3 \setminus \{0\}$

Generalize to arbitrary (M, \mathcal{F}) :

- $\quad \textbf{dim}(\mathfrak{F}_x) \text{ upper semicontinuous} \Rightarrow \Omega_{\mathfrak{i}} = \{x \in M : \text{dim}(\mathfrak{F}_x) \leqslant \mathfrak{i}\} \text{ open}$
- Also, $Y_i = \Omega_i \backslash \Omega_{i-1}$ closed and saturated.

Definition

1 Decomposition sequence of (M, \mathcal{F}) :

$$\Omega_0 \subseteq \Omega_1 \subseteq \ldots \subseteq \Omega_{k-1} \subseteq \Omega_{\textbf{k}} = M$$

2 We say that (M, \mathcal{F}) has height **k**. (**k** = $+\infty$ allowed and possible!)

Careful look at action $SO(3) \subset \mathbb{R}^3$ (II)

So foliation $(\mathbb{R}^3, \mathcal{F})$ has height $\mathbf{k} = 1$:

$$\Omega_0=\mathbb{R}_3\backslash\{0\}, \qquad \Omega_1=\mathbb{R}^3, \qquad Y_0=\Omega_0, \qquad Y_1=\{0\}.$$

- $\label{eq:continuous} \bullet \ C^*(M, \mathfrak{F})|_{\Omega_0} = C_0(\Omega_0) \cdot C^*(M, \mathfrak{F}) = C_0(\mathbb{R}_*^+) \otimes \mathfrak{K}(L^2(S^2))$
- $\quad \quad \mathsf{C}^*(\mathsf{M},\mathfrak{F})|_{Y_1} = \mathsf{C}^*(\mathsf{M},\mathfrak{F})/\mathsf{C}^*(\mathsf{M},\mathfrak{F})|_{\mathbb{R}^2 \backslash Y_1} = \mathsf{C}^*(\mathsf{SO}(3))$

Exact sequence of (full) C^* -algebras:

$$0 \longrightarrow C_0(\mathbb{R}^+_*) \otimes \mathfrak{K}(L^2(S^2)) \longrightarrow C^*(M, \mathfrak{F}) \xrightarrow{\pi_{\mathfrak{F}}} C^*(SO(3)) \longrightarrow 0$$

Action groupoid $\mathscr{G} = \mathbb{R}^2 \rtimes SO(3) \Longrightarrow \mathbb{R}^3$:

- $\mathscr{G}|_{Y_1} = H(\mathcal{F})|_{Y_1} = SO(3) \times \{0\}$
- Exact sequence:

$$0 \longrightarrow C_0(\mathbb{R}_+^*) \otimes (C(S^2) \rtimes SO(3)) \longrightarrow C_0(\mathbb{R}^3) \rtimes SO(3) \longrightarrow C^*(SO(3)) \longrightarrow 0$$

Nicely decomposable foliations

Definition

Let (M, \mathcal{F}) singular foliation, decomposition sequence

$$\Omega_0 \subseteq \Omega_1 \subseteq \ldots \subseteq \Omega_j \ldots \subseteq M$$

$$\text{Put} \qquad Y_0 = \Omega_0 \qquad Y_j = \Omega_j \backslash \Omega_{j-1}.$$

A nice decomposition is

1 sequence $(W_j)_{0 \le j \le k}$ of open sets such that

$$Y_j \subset W_j \subset \Omega_j \qquad W_j \cap \Omega_{j-1} \subset W_{j-1}$$

2 Lie groupoids $\mathscr{G}_j \Longrightarrow W_j$ which define $\mathscr{F}|_{W_j}$ and

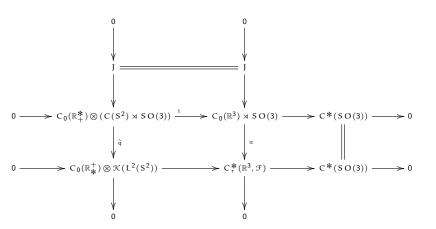
$$\mathscr{G}_{j}|_{Y_{j}}=\mathsf{H}(\mathfrak{F})|_{Y_{j}}$$

3 morphisms $q_j: \mathscr{G}_j|_{\Omega_{j-1} \cap W_j} \to \mathscr{G}_{j-1}$ (for j > 0) which are submersions

$SO(3) \subset \mathbb{R}^3$: calculation (I)

SO(3) compact, whence amenable. So $C^*(\mathfrak{F})=C^*_r(\mathfrak{F}).$

$$\pi: \mathbb{R}^3 > SO(3) \rightarrow H(\mathcal{F}) = (S^2 \times S^2 \times \mathbb{R}^+_*) \cup SO(3) \times \{0\}$$



where q: integration along fibers of $(s, t): S^2 \times SO(3) \rightarrow S^2 \times S^2$.

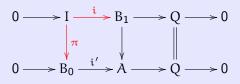
(ES4)

(ES5)

Height 1 foliations

Proposition

Given a diagram of exact sequences of C*-algebras and morphisms:



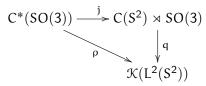
the mapping cone $\mathcal{C}_{(\pi,i)}$ of the map $(\pi,i):I\to B_0\oplus B_1$ is canonically E^1 -equivalent to A (KK-equivalent).

<u>Conclusion</u>: Need to formulate the Baum-Connes conjecture for mapping cones!

$SO(3) \subset \mathbb{R}^3$: calculation (II)

 $\rho: C^*(SO(3)) \to \mathcal{K}(L^2(S^2))$ natural repn of SO(3) on $L^2(S^2).$

 $j: C^*(SO(3)) \to C(S^2) \rtimes SO(3) \text{ induced by unital inclusion } \mathbb{C} \to C(S^2).$



$SO(3) \subset \mathbb{R}^3$: calculation (III)

 $C_0(\mathbb{R}^3)=$ mapping cone of $\mathbb{C}\to C(S^2).$ Taking crossed products by the action of SO(3) and using the first diagram, we find:

• $C_0(\mathbb{R}^3) \rtimes SO(3)$ in (EC5) is mapping cone $\mathfrak{C}_{\mathfrak{j}}$, where

$$j: C^*(SO(3)) \to C(S^2) \rtimes SO(3)$$

▶ Foliation algebra $C^*(\mathfrak{F})$ in (EC6) is mapping cone \mathfrak{C}_{ρ} .

$SO(3) \subset \mathbb{R}^3$: calculation (IV)

To describe $C^*(\mathfrak{F})$ it suffices to describe the representation

$$\rho: C^*(SO(3)) \to \mathcal{K}(L^2(S^2)).$$

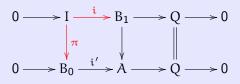
- $\begin{array}{l} \hbox{$\blacktriangleright$ $ Peter-Weyl: $C^*(SO(3))=\oplus_{m\in\mathbb{N}}M_{2m+1}(\mathbb{C})$ and } \\ K_0(C^*(SO(3)))=\mathbb{Z}^{(\mathbb{N})} \mbox{ (and } K_1(C^*(SO(3)))=\{0\}). \end{array}$
- ▶ In order to compute the map $\rho_*: \mathsf{K}_0(C^*(SO(3))) \to \mathbb{Z}$, we have to understand how many times the repn $\sigma_\mathfrak{m}$ (dim $(\sigma_\mathfrak{m}) = 2\mathfrak{m} + 1$) appears in ρ , *i.e.* count dimension of $\mathsf{Hom}_{SO(3)}(\sigma_\mathfrak{m}, \rho)$.
- Since $S^2 = SO(3)/S^1$, $\rho = Ind_{S^1}^{SO(3)}(\epsilon)$ where ϵ trivial repn of S^1 .
- Frobenius reciprocity thm: $\dim(\mathsf{Hom}_{SO(3)}(\sigma_{\mathfrak{m}},\rho)) = \dim(\mathsf{Hom}_{S^1}(\sigma_{\mathfrak{m}},\epsilon)) = 1.$
- ▶ So ρ_* : $K_0(C^*(SO(3))) \to \mathbb{Z}$ maps each generator $[\sigma_m]$ of $K_0(C^*(SO(3)))$ to 1.

$$K_0(C^*(\mathfrak{F})) = \ker \rho_* \simeq \mathbb{Z}^{(\mathbb{N})}$$
 $K_1(C^*(\mathfrak{F})) = 0$

Height 1 foliations

Proposition

Given a diagram of exact sequences of C^* -algebras and morphisms:



the mapping cone $\mathcal{C}_{(\pi,i)}$ of the map $(\pi,i):I\to B_0\oplus B_1$ is canonically E^1 -equivalent to A (KK-equivalent).

<u>Conclusion</u>: Need to formulate the Baum-Connes conjecture for mapping cones!

Height k > 1 foliations

Proposition

The previous result extends to foliations (M, \mathcal{F}) of any height: The foliation C^* -algebra is "K"-equivalent (E-equivalent) to a mapping telescope.

Examples of higher height arise looking at flag manifolds... For instance:

- Let P be the minimal parabolic subgroup of $GL(n, \mathbb{R})$ (P = uppertriangular matrices).
- Let $P \times P$ act on $GL(n, \mathbb{R})$ by left and right multiplication.
- Orbits labeled by symmetric group S_n (Bruhat decomposition)

BC for singular foliations

Theorem (I.A. and G. Skandalis)

Let (M,\mathcal{F}) be a nicely decomposable foliation such that the classifying spaces of all the groupoids $\mathscr{G}_k \Longrightarrow W_k$ involved in this decomposition are manifolds and if the *full* Baum-Connes conjecture holds for all of them, then the *full* Baum-Connes map is an isomorphism.

Corollary

Let (M,\mathcal{F}) be a nicely decomposable foliation. If all the groupoids $\mathscr{G}_k \Longrightarrow W_k$ involved in this decomposition are amenable and their classifying spaces are manifolds, then the Baum-Connes map is an isomorphism.

Thank you Aristides!