The use of $\mathcal{C}^*$-algebras in singular foliations and their representation theory

Iakovos Androulidakis

National and Kapodistrian University of Athens

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Examples

\( M \): compact manifold.

1. Orbits of (some) Lie group actions on \( M \). Vector fields: image of infinitesimal action \( g \mapsto \mathfrak{X}(M) \).

Focus on \( \mathcal{F} = \langle X \rangle \):

2. \( X \) nowhere vanishing vector field of \( M \) \( \rightsquigarrow \) action of \( \mathbb{R} \) on \( M \).

3. Irrational rotation on torus \( T^2 \): "Kronecker" flow of \( X = \frac{d}{dx} + \theta \frac{d}{dy} \). \( \mathbb{R} \) injected as a dense leaf.

4. "Horocyclic" foliation:
   - Let \( \Gamma \) cocompact subgroup of \( \text{SL}(2, \mathbb{R}) \). Put \( M = \text{SL}(2, \mathbb{R})/\Gamma \).
   - \( \mathbb{R} \) is embedded in \( \text{SL}(2, \mathbb{R}) \) by \( \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \ t \in \mathbb{R} \).
   - Therefore \( \mathbb{R} \) acts on \( M \). Action is ergodic, \( \exists \) dense leaves.
Laplacians of Kronecker foliation

Kronecker foliation on $M = T^2$: $\mathcal{F} = \left\langle \frac{d}{dx} + \theta \frac{d}{dy} \right\rangle$. $L = \mathbb{R}$

Two Laplacians:

- $\Delta_L = -\frac{d^2}{dx^2}$ acting on $L^2(\mathbb{R})$
- $\Delta_M = -X^2$ acting on $L^2(M)$

By Fourier:

- $\Delta_L \rightsquigarrow \text{mult. by } \xi^2$ on $L^2(\mathbb{R})$. Spectrum: $[0, +\infty)$.
- $\Delta_M \rightsquigarrow \text{mult. by } (n + \theta k)^2$ on $L^2(\mathbb{Z}^2)$. Spectrum dense in $[0, +\infty)$.

**Qn 1:** Do $\Delta_L$ and $\Delta_M$ have the same spectrum for every (regular) foliation?

**Qn 2:** If so, how to calculate this spectrum?

**Tools:** Holonomy groupoid $\mathcal{H}(\mathcal{F})$, Longitudinal pseudodifferential calculus, Groupoid C*-algebra(s).
The \( C^* \)-algebra of a Lie groupoid (Connes, Renault)

For \( f, g \in C^\infty_c(G) \):

- we put \( f^*(x) = \overline{f(x^{-1})} \)
- we want to form \( f \ast g \) by a formula

\[
f \ast g(x) = \int_{yz=x} f(y)g(z)
\]

In other words, we want to have an integration along the fibers of the composition \( G \times_{s,t} G \rightarrow G \).

Use either Haar systems or half densities.

**Proposition**

The above involution and product make \( C^\infty_c(G) \) a \(^*\)-algebra.

”Reduced” \( C^*_r(G) \): completion with left regular representation

”Full” \( C^*(G) \): completion with all representations

Quotient \( C^*(G) \rightarrow C^*_r(G) \).
Basic tool: Pseudodifferential calculus (Connes)

The Lie algebra of vector fields tangent to the foliation acts by unbounded multipliers on $C_c^\infty(G)$. The algebra generated is the algebra of differential operators.

Using Fourier transform one can write a differential operator $P$ (acting by left multiplication on $f \in C_c^\infty(G)$) as:

$$(Pf)(x, y) = \int \exp(i\langle \phi(x, z), \xi \rangle) \alpha(x, \xi) \chi(x, z)f(z, y) \, d\xi \, dz$$

Proposition (Connes)

- Negative order pseudodifferential operators $\in C^*(M, F)$
- Zero order pseudodifferential operators: multipliers of $C^*(M, F)$.

Together with multiplicativity of the principal symbol this gives an exact sequence of $C^*$-algebras:

$$0 \to C^*(M, F) \to \Psi^*(M, F) \to C(SF^*) \to 0$$
Laplacians revisited

**Theorem (Connes, Kordyukov, Vassout)**

Elliptic operators of positive order are **regular** unbounded multipliers (in the sense of Baaj-Woronowicz: \( \text{graph}(\mathcal{D}) \oplus \text{graph}(\mathcal{D})^\perp \) is dense).

More generally \( M \) compact, \((M, \mathcal{F})\) regular foliation.

- Lie algebra \( \mathcal{F} = C^\infty(M, \mathcal{F}) \) acts on \( C^\infty(G) \) by unbounded multipliers.
- Laplacian \( \Delta = \sum X_i^2 \) is an **unbounded (regular) multiplier** of \( C^*(M, \mathcal{F}) \).

\( L^2(L), L^2(M) \) are representations of \( C^*(M, \mathcal{F}) \).

**Proposition (Baaj, Woronowicz)**

Every representation extends to regular multipliers.

We recover Laplacians \( \Delta_L, \Delta_M \).
Proof of theorems 1 and 2

**Theorem 1**

$\Delta_M$ and $\Delta_L$ are essentially self-adjoint.

- $L^2(M)$ and $L^2(L)$: representations of the foliation C*-algebras.
- Recall *(Baaj, Woronowicz)*: Every representation extends to regular multipliers.  
  \[ \text{image of the adjoint} = \text{adjoint of the image} \]

**Theorem 2 (Kordyukov)**

If all leaves $L$ are dense + amenability assumptions, $\Delta_M$ and $\Delta_L$ have the same spectrum.

- *(Fack and Skandalis)*: If the foliation is minimal *(i.e. all leaves are dense)* then the foliation C*-algebra is simple. Whence all representations are faithful.
- Every injective morphism of C*-algebras is isometric and isospectral.
Elliptic operators - Gaps of their spectrum

**Theorem 3 (Connes)**

In many cases, one can predict the possible gaps in the spectrum.

More precisely:

- Gaps in the spectrum $\implies$ projections in $C^*(M, F)$.
- Projectionless $C^*(M, F)$: spectrum connected.
- Sometimes dimension function on projections (related with K-theory).
  - Values in $\mathbb{N}$: few projections.
  - Values in a dense subset of $\mathbb{R}_+$: many projections.
Examples

Horocyclic foliation: no gaps in the spectrum

Let the "ax + b" group act on a compact manifold M. e.g. \( M = \text{SL}(2, \mathbb{R})/\Gamma \) where \( \Gamma \) discrete co-compact group.

Leaves = orbits of the "x + b" group (assume it is minimal).

The spectrum of the Laplacian is an interval \([m, +\infty)\)

Proof: We show \( C^*(M, F) \) projectionless.

- \exists \text{ measure on } M \text{ invariant by } ax + b \text{ (amenable). } x + b \text{ invariance} \implies \text{ trace on } C^*(M, F) \text{ faithful since } C^*(M, F) \text{ simple (Fack-Skandalis).}
- The "ax" subgroup \( \rightarrow \) action of \( \mathbb{R}^*_+ \) on \( C^*(M, F) \) which scales the trace.
- Image of \( K_0 \) countable subgroup of \( \mathbb{R} \), invariant under \( \mathbb{R}^*_+ \) action.

Similarly, Kronecker flow: Image of the trace \( \mathbb{Z} + \theta \mathbb{Z} \)

Can be (more or less) any closed subset of \( \mathbb{R}_+ \)
Conclusions

Theorems 1 and 2 generalize to any singular foliation!

Definition (Stefan, Sussmann, A-Skandalis)

A (singular) foliation is a finitely generated sub-module $\mathcal{F}$ of $C^\infty(M; TM)$, stable under brackets.

Examples

1. $\mathbb{R}$ foliated by 3 leaves: $(−\infty, 0), \{0\}, (0, +\infty)$. $\mathcal{F}$ generated by $x^n \frac{\partial}{\partial x}$. Different foliation for every $n$.
2. $\mathbb{R}^2$ foliated by 2 leaves: $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$. No obvious best choice. $\mathcal{F}$ given by the action of a Lie group $\text{GL}(2, \mathbb{R}), \text{SL}(2, \mathbb{R}), \mathbb{C}^*$

IA+A-Skandalis (2006-today): Holonomy groupoid, foliation $C^*$-algebras, longitudinal pseudodifferential calculus...

Need to calculate $K_0(\mathcal{C}^*(\mathcal{F}))$!
What does BC say? (I)

Γ discrete group, torsion-free.

\[ \mu : K_\ast(\mathbb{E}\Gamma) \to K_\ast(C_r^\ast(\Gamma)) \text{ isomorphism} \]

- \( \mathbb{E}\Gamma \) = classifying space of proper \( \Gamma \)-actions (CW-complex)
- lhs = \( \Gamma \)-equiv. \( K \)-homology
- rhs = \( K \)-theory of reduced \( C^\ast \)-algebra
- completion with \( \mathbb{C}\Gamma \to B(\ell^2(\Gamma), \ell^2(\Gamma)) \), \( g \mapsto r_g \)

e.g. \( \Gamma = \mathbb{Z}^n \):

- \( \mathbb{E}\mathbb{Z}^n = B\mathbb{Z}^n = T^n \)
- \( C_r^\ast(\mathbb{Z}^n) = C(T^n) \) (Fourier)
- \( \mu \) is Poincaré duality
What does BC say? (II)

G Lie group, $K$ compact subgroup. $G$ acts on $M = K \backslash G$ on the right. Assume $M$ has $\text{Spin}^c$-structure. Put $R(K)$ the free abelian group of (classes of) irreducible representations of $K$.

Define **Dirac induction** $\mu : R(K) \rightarrow K(C^*_r(G))$ as follows:

- Take $\rho \in R(K)$, say $\rho : K \rightarrow \text{GL}(V)$. Define a vector bundle $V_\rho = G \times_K V$ over $M$.
- Levi-Civita connection of spinor bundle $S \rightarrow M$ and Riemannian metric on $M$ give Dirac operator $D_\rho : \Gamma(V_\rho \otimes S) \rightarrow \Gamma(V_\rho \otimes S)$
- Pull back to $G$ and put

$$\mu : R(K) \rightarrow K(C^*_r(G)), \quad \rho \mapsto \mu$$

**Facts:**

- $G$ compact, $K = \{\text{pt}\}$, get $\mu = \text{id}$.
- $K(C^*_r(G)) = K^G(\text{pt}) = R(G)$.
- $K$ maximal compact, $EG = K \backslash G = M$.
- Then, can identify $R(K)$ with $K^G_j(M)$, where $j = \text{dim}(M) \mod 2$. 
What does BC say? (III)

General geometric situations formulated in terms of a Lie groupoid $\mathcal{G} \rightrightarrows M$: There exists an assembly map

$$\mu : K_{\text{top}}(\mathcal{G}) \to K_*(\mathcal{C}_r^*(\mathcal{G}))$$

defined as an analytic index map. (Wrong-way functoriality...)

Baum-Connes conjecture

The assembly map is an isomorphism. (Part of the conjecture is to specify explicitly the lhs!)

• How to read it: "All analytic representations come from geometry!"
• Analogue: Geometric quantization (apply Dirac induction to coadjoint orbits...)
• Counterexample by Higson, Lafforgue, Skandalis.
• Injectivity implies Novikov conjecture.
• Surjectivity implies Kaplansky conjecture.
• Use of BC: Calculate $K(\mathcal{C}^*(\mathcal{G}))$!
Careful look at action $SO(3) \lhd \mathbb{R}^3$ (I)

$\dim(\text{Lie}(SO(3))) = 3$, so $\mathcal{F} = \text{span}_{\mathcal{C}^\infty(M)}\langle X, Y, Z \rangle$.

Take any $(M, \mathcal{F})$. At $x \in M$ put $\mathcal{F}_x = \mathcal{F}/I_x \mathcal{F}$. Get exact sequence

$0 \to g_x \to \mathcal{F}_x \xrightarrow{\text{ev}_x} T_x L_x \to 0$

- $L_x$ regular $\Rightarrow \mathcal{F}_x = T_x L_x$
- $L_x$ singular $\Rightarrow \dim(\mathcal{F}_x) > \dim(L_x)$.
- $\dim(\mathcal{F}_x)$ (upper) semicontinuous

For $(\mathbb{R}^3, \mathcal{F})$ we have:

- $\mathcal{F}_0 = g_x = \text{Lie}(SO(3))$, so $\dim(\mathcal{F}_0) = 3$
- For $x \neq 0$, $\dim(\mathcal{F}_x) = 2$

$H(\mathcal{F}) = (S^2 \times S^2 \times \mathbb{R}^*_+) \cup SO(3) \times \{0\}$
Careful look at action $SO(3) \subset \mathbb{R}^3$ (II)

$H(F) = (S^2 \times S^2 \times \mathbb{R}^+ \cup SO(3) \times \{0\}$ decomposes $\mathbb{R}^3$:

- $\Omega_1 = \{x \in \mathbb{R}^3 : \dim(F_x) \leq 3\} = \mathbb{R}^3$
- $\Omega_0 = \{x \in \mathbb{R}^3 : \dim(F_x) \leq 2\} = \mathbb{R}^3 \setminus \{0\}$

Generalize to arbitrary $(M, F)$:

- $\dim(F_x)$ upper semicontinuous $\Rightarrow \Omega_i = \{x \in M : \dim(F_x) \leq i\}$ open
- Also, $Y_i = \Omega_i \setminus \Omega_{i-1}$ closed and saturated.

Definition

1. **Decomposition sequence** of $(M, F)$:
   \[ \Omega_0 \subseteq \Omega_1 \subseteq \ldots \subseteq \Omega_{k-1} \subseteq \Omega_k = M \]

2. We say that $(M, F)$ has **height** $k$. ($k = +\infty$ allowed and possible!)
Careful look at action $\mathbf{SO}(3) \subset \mathbb{R}^3$ (II)

So foliation $(\mathbb{R}^3, \mathcal{F})$ has height $k = 1$:

$$\Omega_0 = \mathbb{R}_3 \setminus \{0\}, \quad \Omega_1 = \mathbb{R}^3, \quad Y_0 = \Omega_0, \quad Y_1 = \{0\}.$$ 

- $C^*(M, \mathcal{F})|_{\Omega_0} = C_0(\Omega_0) \cdot C^*(M, \mathcal{F}) = C_0(\mathbb{R}_*^+) \otimes \mathcal{K}(L^2(S^2))$
- $C^*(M, \mathcal{F})|_{Y_1} = C^*(M, \mathcal{F})/C^*(M, \mathcal{F})|_{\mathbb{R}^2 \setminus Y_1} = C^*(\mathbf{SO}(3))$

Exact sequence of (full) $C^*$-algebras:

$$0 \to C_0(\mathbb{R}_*^+) \otimes \mathcal{K}(L^2(S^2)) \to C^*(M, \mathcal{F}) \xrightarrow{\pi_{\mathcal{F}}} C^*(\mathbf{SO}(3)) \to 0$$

Action groupoid $\mathcal{G} = \mathbb{R}^2 \rtimes \mathbf{SO}(3) \to \mathbb{R}^3$:

- $\mathcal{G}|_{Y_1} = H(\mathcal{F})|_{Y_1} = \mathbf{SO}(3) \times \{0\}$
- Exact sequence:

$$0 \to C_0(\mathbb{R}_*^+) \otimes (C(S^2) \rtimes \mathbf{SO}(3)) \to C_0(\mathbb{R}^3) \rtimes \mathbf{SO}(3) \to C^*(\mathbf{SO}(3)) \to 0$$
Nicely decomposable foliations

**Definition**

Let \((M, \mathcal{F})\) singular foliation, decomposition sequence

\[ \Omega_0 \subseteq \Omega_1 \subseteq \ldots \subseteq \Omega_j \ldots \subseteq M \]

Put \(Y_0 = \Omega_0 \quad Y_j = \Omega_j \setminus \Omega_{j-1}\).

A nice decomposition is

1. sequence \((W_j)_{0 \leq j \leq k}\) of open sets such that
   \[ Y_j \subset W_j \subset \Omega_j \quad W_j \cap \Omega_{j-1} \subset W_{j-1} \]

2. Lie groupoids \(\mathcal{G}_j \rightrightarrows W_j\) which define \(\mathcal{F}|_{W_j}\) and
   \[ \mathcal{G}_j|_{Y_j} = H(\mathcal{F})|_{Y_j} \]

3. morphisms \(q_j : \mathcal{G}_j|_{\Omega_{j-1} \cap W_j} \to \mathcal{G}_{j-1}\) (for \(j > 0\)) which are submersions
**SO(3) ⊆ \( \mathbb{R}^3 \): calculation (I)**

SO(3) compact, whence amenable. So \( C^*(\mathcal{F}) = C_r^*(\mathcal{F}) \).

\[
\pi : \mathbb{R}^3 \ni SO(3) \to H(\mathcal{F}) = (S^2 \times S^2 \times \mathbb{R}^+_{\ast}) \cup SO(3) \times \{0\}
\]

\[
\begin{array}{ccccccccc}
0 & \to & C_0(\mathbb{R}^+_{\ast}) \otimes (C(S^2) \times SO(3)) & \overset{i}{\to} & C_0(\mathbb{R}^3) \times SO(3) & \to & C^*(SO(3)) & \to & 0 \\
& & & & & \downarrow \pi & & \\
& & & & & C^*(SO(3)) & \to & 0 & & (ES4) \\
0 & \to & C_0(\mathbb{R}^+_{\ast}) \otimes \mathcal{K}(L^2(S^2)) & \overset{\hat{q}}{\to} & C_r^*(\mathbb{R}^3, \mathcal{F}) & \to & C^*(SO(3)) & \to & 0 & (ES5) \\
& & & \downarrow & & & \downarrow & & & \\
& & & 0 & & 0 & & &
\end{array}
\]

where \( q \): integration along fibers of \((s, t) : S^2 \times SO(3) \to S^2 \times S^2\).
Height 1 foliations

**Proposition**

Given a diagram of exact sequences of $C^*$-algebras and morphisms:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & I & \overset{i}{\rightarrow} & B_1 & \rightarrow & Q & \rightarrow & 0 \\
& \downarrow{\pi} & & & \downarrow & & \parallel & & \\
0 & \rightarrow & B_0 & \overset{i'}{\rightarrow} & A & \rightarrow & Q & \rightarrow & 0
\end{array}
\]

the mapping cone $\mathcal{C}_{(\pi,i)}$ of the map $(\pi, i) : I \rightarrow B_0 \oplus B_1$ is canonically $E^1$-equivalent to $A$ (KK-equivalent).

**Conclusion**: Need to formulate the Baum-Connes conjecture for mapping cones!
\( \rho : C^*(SO(3)) \rightarrow \mathcal{K}(L^2(S^2)) \) natural repn of \( SO(3) \) on \( L^2(S^2) \).

\( j : C^*(SO(3)) \rightarrow C(S^2) \rtimes SO(3) \) induced by unital inclusion \( \mathbb{C} \rightarrow C(S^2) \).

\[
\begin{array}{ccc}
C^*(SO(3)) & \xrightarrow{j} & C(S^2) \rtimes SO(3) \\
\downarrow{\rho} & & \downarrow{q} \\
\mathcal{K}(L^2(S^2)) & & \\
\end{array}
\]
Calculation of $K$-theory for action of $SO(3)$ on $\mathbb{R}^3$

**SO(3) $\hookrightarrow \mathbb{R}^3$: calculation (III)**

$C_0(\mathbb{R}^3) = \text{mapping cone of } \mathbb{C} \to C(S^2)$. Taking crossed products by the action of $SO(3)$ and using the first diagram, we find:

- $C_0(\mathbb{R}^3) \rtimes SO(3)$ in (EC5) is mapping cone $\mathcal{C}_j$, where

  $$j : C^*(SO(3)) \to C(S^2) \rtimes SO(3)$$

- Foliation algebra $C^*(\mathcal{F})$ in (EC6) is mapping cone $\mathcal{C}_\rho$. 
Calculation of $K$-theory for action of $SO(3)$ on $\mathbb{R}^3$

$SO(3) \hookrightarrow \mathbb{R}^3$: calculation (IV)

To describe $C^*(\mathcal{F})$ it suffices to describe the representation

$$\rho : C^*(SO(3)) \to \mathcal{K}(L^2(S^2)).$$

- Peter-Weyl: $C^*(SO(3)) = \bigoplus_{m\in\mathbb{N}} M_{2m+1}(\mathbb{C})$ and $K_0(C^*(SO(3))) = \mathbb{Z}^{(\mathbb{N})}$ (and $K_1(C^*(SO(3))) = \{0\}$).

- In order to compute the map $\rho_* : K_0(C^*(SO(3))) \to \mathbb{Z}$, we have to understand how many times the repn $\sigma_m$ ($\dim(\sigma_m) = 2m + 1$) appears in $\rho$, i.e. count dimension of $\text{Hom}_{SO(3)}(\sigma_m, \rho)$.

- Since $S^2 = SO(3)/S^1$, $\rho = \text{Ind}_{S^1}^{SO(3)}(\varepsilon)$ where $\varepsilon$ trivial repn of $S^1$.

- Frobenius reciprocity thm: $\dim(\text{Hom}_{SO(3)}(\sigma_m, \rho)) = \dim(\text{Hom}_{S^1}(\sigma_m, \varepsilon)) = 1$.

- So $\rho_* : K_0(C^*(SO(3))) \to \mathbb{Z}$ maps each generator $[\sigma_m]$ of $K_0(C^*(SO(3)))$ to 1.

$$K_0(C^*(\mathcal{F})) = \ker \rho_* \cong \mathbb{Z}^{(\mathbb{N})} \quad K_1(C^*(\mathcal{F})) = 0$$
Calculation of K-theory for action of $SO(3)$ on $\mathbb{R}^3$

Height 1 foliations

**Proposition**

Given a diagram of exact sequences of $C^*$-algebras and morphisms:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & I & \overset{i}{\longrightarrow} & B_1 & \longrightarrow & Q & \longrightarrow & 0 \\
\downarrow{\pi} & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_0 & \overset{i'}{\longrightarrow} & A & \longrightarrow & Q & \longrightarrow & 0
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\]

the mapping cone $C(\pi, i)$ of the map $(\pi, i): I \to B_0 \oplus B_1$ is canonically $E^1$-equivalent to $A$ (KK-equivalent).

**Conclusion**: Need to formulate the Baum-Connes conjecture for mapping cones!
Height $k > 1$ foliations

Proposition

The previous result extends to foliations $(\mathcal{M}, \mathcal{F})$ of any height: The foliation $C^*$-algebra is “$K$”-equivalent ($\mathcal{E}$-equivalent) to a mapping telescope.

Examples of higher height arise looking at flag manifolds... For instance:

- Let $P$ be the minimal parabolic subgroup of $\text{GL}(n, \mathbb{R})$ ($P =$ uppertriangular matrices).
- Let $P \times P$ act on $\text{GL}(n, \mathbb{R})$ by left and right multiplication.
- Orbits labeled by symmetric group $S_n$ (Bruhat decomposition).
BC for singular foliations

Theorem (I.A. and G. Skandalis)

Let \((M, \mathcal{F})\) be a nicely decomposable foliation such that the classifying spaces of all the groupoids \(G_k \to W_k\) involved in this decomposition are manifolds and if the \textit{full} Baum-Connes conjecture holds for all of them, then the \textit{full} Baum-Connes map is an isomorphism.

Corollary

Let \((M, \mathcal{F})\) be a nicely decomposable foliation. If all the groupoids \(G_k \to W_k\) involved in this decomposition are amenable and their classifying spaces are manifolds, then the Baum-Connes map is an isomorphism.

Thank you Aristides!