The use of C*-algebras in singular foliations and their representation theory

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Examples

M: compact manifold.

1. Orbits of (some) Lie group actions on M. Vector fields: image of infinitesimal action $\mathfrak{g} \to \mathfrak{X}(M)$.

Focus on $\mathcal{F} = \langle X \rangle$:

2. X nowhere vanishing vector field of $M \rightsquigarrow$ action of $\mathbb{R}$ on M.
3. Irrational rotation on torus T^2: "Kronecker" flow of $X = \frac{d}{dx} + \theta \frac{d}{dy}$. $\mathbb{R}$ injected as a dense leaf.
4. "Horocyclic" foliation:
   - Let Γ cocompact subgroup of $SL(2, \mathbb{R})$. Put $M = SL(2, \mathbb{R})/\Gamma$.
   - $\mathbb{R}$ is embedded in $SL(2, \mathbb{R})$ by $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, $t \in \mathbb{R}$.
   - Therefore $\mathbb{R}$ acts on M. Action is ergodic, $\exists$ dense leaves.

Laplacians of Kronecker foliation

Kronecker foliation on $M = T^2$: $\mathcal{F} = \langle \frac{d}{dx} + \theta \frac{d}{dy} \rangle$. $L = \mathbb{R}$

Two Laplacians:

- $\Delta_L = -\frac{d^2}{dx^2}$ acting on $L^2(\mathbb{R})$
- $\Delta_M = -X^2$ acting on $L^2(M)$

By Fourier:

- $\Delta_L \rightsquigarrow$ mult. by ξ^2 on $L^2(\mathbb{R})$. Spectrum: $[0, +\infty)$.
- $\Delta_M \rightsquigarrow$ mult. by $(n + \theta k)^2$ on $L^2(\mathbb{Z}^2)$. Spectrum dense in $[0, +\infty)$.

Qn 1: Do Δ_L and Δ_M have the same spectrum for every (regular) foliation?

Qn 2: If so, how to calculate this spectrum?

Tools: Holonomy groupoid $H(\mathcal{F})$, Longitudinal pseudodifferential calculus, Groupoid C^*-algebra(s).

The C^*-algebra of a Lie groupoid (Connes, Renault)

For $f, g \in C_c^\infty(G)$:

- we put $f^*(x) = \overline{f(x^{-1})}$
- we want to form $f * g$ by a formula

$$f * g(x) = \int_{yz=x} f(y)g(z)$$

In other words, we want to have an integration along the fibers of the composition $G \times_{s,t} G \to G$.
Use either Haar systems or half densities.

Proposition

The above involution and product make $C_c^\infty(G)$ a $*$-algebra.

"Reduced" $C_r^*(G)$: completion with left regular representation

"Full" $C^*(G)$: completion with all representations

Quotient $C^*(G) \to C_r^*(G)$.

Basic tool: Pseudodifferential calculus (Connes)

The Lie algebra of vector fields tangent to the foliation acts by unbounded multipliers on $C_c^\infty(G)$. The algebra generated is the algebra of differential operators.

Using Fourier transform one can write a differential operator P (acting by left multiplication on $f \in C_c^\infty(G)$) as:

$$(Pf)(x, y) = \int \exp(i\langle \phi(x, z), \xi\rangle)\alpha(x, \xi)\chi(x, z)f(z, y)d\xi dz$$

Proposition (Connes)

- Negative order pseudodifferential operators $\in C^*(M, F)$
- Zero order pseudodifferential operators: multipliers of $C^*(M, F)$.

Together with multiplicativity of the principal symbol this gives an exact sequence of C^*-algebras:

$$0 \to C^*(M, F) \to \Psi^*(M, F) \to C(SF^*) \to 0$$

Laplacians revisited

Theorem (Connes, Kordyukov, Vassout)

Elliptic operators of positive order are regular unbounded multipliers (in the sense of Baaj-Woronowicz: $\mathrm{graph}(D) \oplus \mathrm{graph}(D)^{\perp}$ is dense).

More generally M compact, (M, F) regular foliation.

- Lie algebra $\mathcal{F} = C^{\infty}(M, F)$ acts on $C^{\infty}(G)$ by unbounded multipliers.
- Laplacian $\Delta = \sum X_i^2$ is an unbounded (regular) multiplier of $C^*(M, \mathcal{F})$.

$L^2(L), L^2(M)$ are representations of $C^*(M, \mathcal{F})$.

Proposition (Baaj, Woronowicz)

Every representation extends to regular multipliers.

We recover Laplacians Δ_L, Δ_M.

Proof of theorems 1 and 2

Theorem 1

Δ_M and Δ_L are essentially self-adjoint.

- $L^2(M)$ and $L^2(L)$: representations of the foliation C^*-algebras.
- Recall (Baaj, Woronowicz): Every representation extends to regular multipliers.

image of the adjoint = adjoint of the image

Theorem 2 (Kordyukov)

If all leaves L are dense + amenability assumptions, Δ_M and Δ_L have the same spectrum.

- (Fack and Skandalis): If the foliation is minimal (*i.e.* all leaves are dense) then the foliation C^*-algebra is simple. Whence all representations are faithful.
- Every injective morphism of C^*-algebras is isometric and isospectral.

Elliptic operators - Gaps of their spectrum

Theorem 3 (Connes)

In many cases, one can predict the possible gaps in the spectrum.

More precisely:

- Gaps in the spectrum $\longrightarrow$ projections in $C^*(M, F)$.
- Projectionless $C^*(M, F)$: spectrum connected.
- Sometimes dimension function on projections (related with K-theory).
  - Values in $\mathbb{N}$: few projections.
  - values in a dense subset of $\mathbb{R}_+$: many projections.

Examples

Horocyclic foliation: no gaps in the spectrum

Let the "$ax+b$" group act on a compact manifold M.
e.g. $M = SL(2,\mathbb{R})/\Gamma$ where Γ discrete co-compact group.
Leaves = orbits of the "$x+b$" group (assume it is minimal).

$$\boxed{\text{The spectrum of the Laplacian is an interval } [m,+\infty)}$$

Proof: We show $C^*(M,F)$ projectionless.

- $\exists$ measure on M invariant by $ax+b$ (amenable). $x+b$ invariance $\Longrightarrow$ trace on $C^*(M,F)$ faithful since $C^*(M,F)$ simple (Fack-Skandalis).
- The "ax" subgroup $\longrightarrow$ action of $\mathbb{R}_+^*$ on $C^*(M,F)$ which scales the trace.
- Image of K_0 countable subgroup of $\mathbb{R}$, invariant under $\mathbb{R}_+^*$ action.

Similarly, Kronecker flow: Image of the trace $\mathbb{Z}+\theta\mathbb{Z}$

$$\boxed{\text{Can be (more or less) any closed subset of } \mathbb{R}_+}$$

Conclusions

Theorems 1 and 2 generalize to any singular foliation!

Definition (Stefan, Sussmann, A-Skandalis)

A (singular) foliation is a finitely generated sub-module $\mathcal{F}$ of $C^\infty(M; TM)$, stable under brackets.

Examples

1. $\mathbb{R}$ foliated by 3 leaves: $(-\infty, 0), \{0\}, (0, +\infty)$.
   $\mathcal{F}$ generated by $x^n \frac{\partial}{\partial x}$. Different foliation for every n.
2. $\mathbb{R}^2$ foliated by 2 leaves: $\{0\}$ and $\mathbb{R}^2 \backslash \{0\}$.
   No obvious best choice. $\mathcal{F}$ given by the action of a Lie group

$$GL(2, \mathbb{R}), SL(2, \mathbb{R}), \mathbb{C}^*$$

IA+Skandalis (2006-today): Holonomy groupoid, foliation C^*-algebras, longitudinal pseudodifferential calculus...

Need to calculate $K_0(C^*(\mathcal{F}))$!

What does BC say? (I)

Γ discrete group, torsion-free.

$$\mu : K_*^{\Gamma}(\underline{E\Gamma}) \to K_*(C_r^*(\Gamma)) \text{ isomorphism}$$

- $\underline{E\Gamma} =$ classifying space of proper Γ-actions (CW-complex)
- lhs $=$ Γ-equiv. K-homology
- rhs $=$ K-theory of reduced C^*-algebra
- completion with $\mathbb{C}\Gamma \to B(\ell^2(\Gamma), \ell^2(\Gamma)), \quad g \mapsto r_g$

e.g. $\Gamma = \mathbb{Z}^n$:

- $\underline{E\mathbb{Z}^n} = B\mathbb{Z}^n = T^n$
- $C_r^*(\mathbb{Z}^n) = C(T^n)$ (Fourier)
- μ is Poicaré duality

What does BC say? (II)

G Lie group, K compact subgroup. G acts on $M = K\backslash G$ on the right. Assume M has $\mathcal{S}pin^c$-structure. Put $R(K)$ the free abelian group of (classes of) irreducible representations of K.

Define Dirac induction $\mu : R(K) \to K(C^*_r(G))$ as follows:

- Take $\rho \in R(K)$, say $\rho : K \to GL(V)$. Define a vector bundle $V_\rho = G \times_K V$ over M.
- Levi-Civita connection of spinor bundle $S \to M$ and Riemannian metric on M give Dirac operator $D_\rho : \Gamma(V_\rho \otimes S) \to \Gamma(V_\rho \otimes S)$
- Pull back to G and put

$$\mu : R(K) \to K(C^*_r(G)), \quad \rho \xrightarrow{\mu} \mathrm{Ind}(D_\rho)$$

Facts:

- G compact, $K = \{pt\}$, get $\mu = id$.
- $K(C^*_r(G)) = K^G(pt) = R(G)$.
- K maximal compact, $\underline{EG} = K\backslash G = M$.
- Then, can identify $R(K)$ with $K^G_j(M)$, where $j = \dim(M) \mathrm{mod} 2$.

What does BC say? (III)

General geometric situations formulated in terms of a Lie groupoid $\mathscr{G} \rightrightarrows M$: There exists an assembly map

$$\mu : K_*^{top}(\mathscr{G}) \to K_*(C_r^*(\mathscr{G}))$$

defined as an analytic index map. (Wrong-way functoriality...)

Baum-Connes conjecture

The assemly map is an isomorphism. (Part of the conjecture is to specify explicitly the lhs!)

- How to read it: "All analytic representations come from geometry!"
- Analogue: Geometric quantization (apply Dirac induction to coadjoint orbits...)
- Counterexample by Higson, Lafforgue, Skandalis.
- Injectivity implies Novikov conjecture.
- Surjectivity implies Kaplansky conjecture.
- Use of BC: Calculate $K(C^*(\mathscr{G}))$!

Careful look at action $SO(3) \circlearrowright \mathbb{R}^3$ (I)

$\mathrm{dim}(\mathrm{Lie}(SO(3))) = 3$, so $\mathcal{F} = span_{C^\infty(M)}\langle X, Y, Z\rangle$.

Take any $(M, \mathcal{F})$. At $x \in M$ put $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$. Get exact sequence

$$0 \to \mathfrak{g}_x \to \mathcal{F}_x \xrightarrow{ev_x} T_xL_x \to 0$$

- L_x regular $\Rightarrow \mathcal{F}_x = T_xL_x$
- L_x singular $\Rightarrow \mathrm{dim}(\mathcal{F}_x) > \mathrm{dim}(L_x)$.
- $\mathrm{dim}(\mathcal{F}_x)$ (upper) semicontinuous

For $(\mathbb{R}^3, \mathcal{F})$ we have:

- $\mathcal{F}_0 = \mathfrak{g}_x = \mathrm{Lie}(SO(3))$, so $\mathrm{dim}(\mathcal{F}_0) = 3$
- For $x \neq 0$, $\mathrm{dim}(\mathcal{F}_x) = 2$

$$H(\mathcal{F}) = (S^2 \times S^2 \times \mathbb{R}^+_*) \cup SO(3) \times \{0\}$$

Careful look at action $SO(3) \circlearrowright \mathbb{R}^3$ (II)

$H(\mathcal{F}) = (S^2 \times S^2 \times \mathbb{R}_*^+) \cup SO(3) \times \{0\}$ decomposes $\mathbb{R}^3$:

- $\Omega_1 = \{x \in \mathbb{R}^3 : \dim(\mathcal{F}_x) \leqslant 3\} = \mathbb{R}^3$
- $\Omega_0 = \{x \in \mathbb{R}^3 : \dim(\mathcal{F}_x) \leqslant 2\} = \mathbb{R}^3 \backslash \{0\}$

Generalize to arbitrary $(M, \mathcal{F})$:

- $\dim(\mathcal{F}_x)$ upper semicontinuous $\Rightarrow \Omega_i = \{x \in M : \dim(\mathcal{F}_x) \leqslant i\}$ open
- Also, $Y_i = \Omega_i \backslash \Omega_{i-1}$ closed and saturated.

Definition

1. Decomposition sequence of $(M, \mathcal{F})$:

$$\Omega_0 \subseteq \Omega_1 \subseteq \ldots \subseteq \Omega_{k-1} \subseteq \Omega_{\mathbf{k}} = M$$

2. We say that $(M, \mathcal{F})$ has height $\mathbf{k}$. ($\mathbf{k} = +\infty$ allowed and possible!)

Careful look at action $SO(3) \circlearrowright \mathbb{R}^3$ (II)

So foliation $(\mathbb{R}^3, \mathcal{F})$ has height $\mathbf{k} = 1$:

$$\Omega_0 = \mathbb{R}_3 \backslash \{0\}, \qquad \Omega_1 = \mathbb{R}^3, \qquad Y_0 = \Omega_0, \qquad Y_1 = \{0\}.$$

- $C^*(M, \mathcal{F})|_{\Omega_0} = C_0(\Omega_0) \cdot C^*(M, \mathcal{F}) = C_0(\mathbb{R}^+_*) \otimes \mathcal{K}(L^2(S^2))$
- $C^*(M, \mathcal{F})|_{Y_1} = C^*(M, \mathcal{F}) / C^*(M, \mathcal{F})|_{\mathbb{R}^2 \backslash Y_1} = C^*(SO(3))$

Exact sequence of (full) C^*-algebras:

$$0 \longrightarrow C_0(\mathbb{R}^+_*) \otimes \mathcal{K}(L^2(S^2)) \longrightarrow C^*(M, \mathcal{F}) \xrightarrow{\pi_{\mathcal{F}}} C^*(SO(3)) \longrightarrow 0$$

Action groupoid $\mathscr{G} = \mathbb{R}^2 \rtimes SO(3) \rightrightarrows \mathbb{R}^3$:

- $\mathscr{G}|_{Y_1} = H(\mathcal{F})|_{Y_1} = SO(3) \times \{0\}$
- Exact sequence:

$$0 \longrightarrow C_0(\mathbb{R}^*_+) \otimes (C(S^2) \rtimes SO(3)) \longrightarrow C_0(\mathbb{R}^3) \rtimes SO(3) \longrightarrow C^*(SO(3)) \longrightarrow 0$$

Nicely decomposable foliations

Definition

Let $(M, \mathcal{F})$ singular foliation, decomposition sequence

$$\Omega_0 \subseteq \Omega_1 \subseteq \ldots \subseteq \Omega_j \ldots \subseteq M$$

Put $\qquad Y_0 = \Omega_0 \qquad Y_j = \Omega_j \backslash \Omega_{j-1}.$

A nice decomposition is

1. sequence $(W_j)_{0 \leqslant j \leqslant k}$ of open sets such that
$$Y_j \subset W_j \subset \Omega_j \qquad W_j \cap \Omega_{j-1} \subset W_{j-1}$$
2. Lie groupoids $\mathscr{G}_j \rightrightarrows W_j$ which define $\mathcal{F}|_{W_j}$ and
$$\mathscr{G}_j|_{Y_j} = H(\mathcal{F})|_{Y_j}$$
3. morphisms $q_j : \mathscr{G}_j|_{\Omega_{j-1} \cap W_j} \to \mathscr{G}_{j-1}$ (for $j > 0$) which are submersions

$SO(3) \circlearrowright \mathbb{R}^3$: calculation (I)

$SO(3)$ compact, whence amenable. So $C^*(\mathcal{F}) = C^*_r(\mathcal{F})$.

$$\pi : \mathbb{R}^3 \rtimes SO(3) \to H(\mathcal{F}) = (S^2 \times S^2 \times \mathbb{R}^+_*) \cup SO(3) \times \{0\}$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & \\
 & J \arrow[r, equal] \arrow[d] & J \arrow[d] & & & \\
0 \arrow[r] & C_0(\mathbb{R}^*_+) \otimes (C(S^2) \rtimes SO(3)) \arrow[r, "i"] \arrow[d, "\hat{q}"] & C_0(\mathbb{R}^3) \rtimes SO(3) \arrow[r] \arrow[d, "\pi"] & C^*(SO(3)) \arrow[r] \arrow[d, equal] & 0 & (ES4) \\
0 \arrow[r] & C_0(\mathbb{R}^+_*) \otimes \mathcal{K}(L^2(S^2)) \arrow[r] \arrow[d] & C^*_r(\mathbb{R}^3, \mathcal{F}) \arrow[r] \arrow[d] & C^*(SO(3)) \arrow[r] & 0 & (ES5) \\
 & 0 & 0 & & &
\end{tikzcd}$$

where q: integration along fibers of $(s,t) : S^2 \rtimes SO(3) \to S^2 \times S^2$.

Height 1 foliations

Proposition

Given a diagram of exact sequences of C^*-algebras and morphisms:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "\pi"] & B_1 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B_0 \arrow[r, "i'"] & A \arrow[r] & Q \arrow[r] & 0 \end{tikzcd}$$

the mapping cone $\mathcal{C}_{(\pi,i)}$ of the map $(\pi, i) : I \to B_0 \oplus B_1$ is canonically E^1-equivalent to A (KK-equivalent).

Conclusion: Need to formulate the Baum-Connes conjecture for mapping cones!

$SO(3) \circlearrowright \mathbb{R}^3$: calculation (II)

$\rho : C^*(SO(3)) \to \mathcal{K}(L^2(S^2))$ natural repn of $SO(3)$ on $L^2(S^2)$.

$j : C^*(SO(3)) \to C(S^2) \rtimes SO(3)$ induced by unital inclusion $\mathbb{C} \to C(S^2)$.

$$\begin{tikzcd} C^*(SO(3)) \arrow[r, "j"] \arrow[dr, "\rho"'] & C(S^2) \rtimes SO(3) \arrow[d, "q"] \\ & \mathcal{K}(L^2(S^2)) \end{tikzcd}$$

$SO(3) \circlearrowright \mathbb{R}^3$: calculation (III)

$C_0(\mathbb{R}^3) =$ mapping cone of $\mathbb{C} \to C(S^2)$. Taking crossed products by the action of $SO(3)$ and using the first diagram, we find:

- $C_0(\mathbb{R}^3) \rtimes SO(3)$ in $(EC5)$ is mapping cone $\mathcal{C}_j$, where

$$j : C^*(SO(3)) \to C(S^2) \rtimes SO(3)$$

- Foliation algebra $C^*(\mathcal{F})$ in $(EC6)$ is mapping cone $\mathcal{C}_\rho$.

$SO(3) \circlearrowright \mathbb{R}^3$: calculation (IV)

To describe $C^*(\mathcal{F})$ it suffices to describe the representation

$$\rho : C^*(SO(3)) \to \mathcal{K}(L^2(S^2)).$$

- Peter-Weyl: $C^*(SO(3)) = \oplus_{m\in\mathbb{N}} M_{2m+1}(\mathbb{C})$ and $K_0(C^*(SO(3))) = \mathbb{Z}^{(\mathbb{N})}$ (and $K_1(C^*(SO(3))) = \{0\}$).
- In order to compute the map $\rho_* : K_0(C^*(SO(3))) \to \mathbb{Z}$, we have to understand how many times the repn σ_m $(\dim(\sigma_m) = 2m+1)$ appears in ρ, *i.e.* count dimension of $\mathrm{Hom}_{SO(3)}(\sigma_m, \rho)$.
- Since $S^2 = SO(3)/S^1$, $\rho = \mathrm{Ind}_{S^1}^{SO(3)}(\varepsilon)$ where ε trivial repn of S^1.
- Frobenius reciprocity thm: $\dim(\mathrm{Hom}_{SO(3)}(\sigma_m, \rho)) = \dim(\mathrm{Hom}_{S^1}(\sigma_m, \varepsilon)) = 1.$
- So $\rho_* : K_0(C^*(SO(3))) \to \mathbb{Z}$ maps each generator $[\sigma_m]$ of $K_0(C^*(SO(3)))$ to 1.

$$K_0(C^*(\mathcal{F})) = \ker \rho_* \simeq \mathbb{Z}^{(\mathbb{N})} \qquad K_1(C^*(\mathcal{F})) = 0$$

Height 1 foliations

Proposition

Given a diagram of exact sequences of C^*-algebras and morphisms:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "\pi"] & B_1 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B_0 \arrow[r, "i'"] & A \arrow[r] & Q \arrow[r] & 0 \end{tikzcd}$$

the mapping cone $\mathcal{C}_{(\pi,i)}$ of the map $(\pi, i) : I \to B_0 \oplus B_1$ is canonically E^1-equivalent to A (KK-equivalent).

Conclusion: Need to formulate the Baum-Connes conjecture for mapping cones!

Height $k > 1$ foliations

Proposition

The previous result extends to foliations $(M, \mathcal{F})$ of any height: The foliation C^*-algebra is "K"-equivalent (E-equivalent) to a mapping telescope.

Examples of higher height arise looking at flag manifolds... For instance:

- Let P be the minimal parabolic subgroup of $GL(n, \mathbb{R})$ ($P =$ uppertriangular matrices).
- Let $P \times P$ act on $GL(n, \mathbb{R})$ by left and right multiplication.
- Orbits labeled by symmetric group S_n (Bruhat decomposition)

BC for singular foliations

Theorem (I.A. and G. Skandalis)

Let $(M, \mathcal{F})$ be a nicely decomposable foliation such that the classifying spaces of all the groupoids $\mathscr{G}_k \rightrightarrows W_k$ involved in this decomposition are manifolds and if the *full* Baum-Connes conjecture holds for all of them, then the *full* Baum-Connes map is an isomorphism.

Corollary

Let $(M, \mathcal{F})$ be a nicely decomposable foliation. If all the groupoids $\mathscr{G}_k \rightrightarrows W_k$ involved in this decomposition are amenable and their classifying spaces are manifolds, then the Baum-Connes map is an isomorphism.

Thank you Aristides!