Holonomy transformations for singular foliations[1]

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Abstract

In order to understand the linearization problem around a leaf of a singular foliation, we extend the familiar holonomy map from the case of regular foliations to the case of singular foliations. To this aim we introduce the notion of holonomy transformation. Unlike the regular case, holonomy transformations can not be attached to classes of paths in the foliation, but rather to elements of the holonomy groupoid of the singular foliation.

Holonomy transformations allow us to link the linearization problem with the compactness of the isotropy group of the holonomy groupoid, as well as with the linearization problem for proper Lie groupoids. We also study the deformations of a singular foliation, recovering the deformation cocycles of Crainic–Moerdijk as well as those of Heitsch.

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Introduction

Historical overview and motivations

A great deal of foliation theory is based on the understanding of the action of the holonomy pseudogroup on the transversal structure of the foliation. Geometrically, the holonomy of a (regular) foliation (M, F) at a point $x \in M$ is realised by a map $h_x \colon \pi_1(L) \to GermDiff(S)$, where L is the leaf at x, S is a transversal at x, and $GermDiff(S)$ is the space of germs of local diffeomorphisms of S. Its linearisation $Lin(h_x) \colon \pi_1(L) \to GL(N_x L)$ is a representation on the normal space to L at x. When one considers all pairs of points in leaves of M, the linearization gives rise to a representation $Lin(h)$ of the holonomy groupoid on TM/F, the normal bundle to the leaves. Notice that TM/F plays the role of the tangent bundle of the quotient space M/F (*cf.* [Co79, §10.2]), and is the starting point for various invariants carrying geometric, topological, and analytic information of the given foliation (an account of which was given in [AnZa11]).

As in [AnZa11], in the current article we are concerned with the much larger class of *singular* foliations. Recall that we understand a (singular) foliation on a smooth manifold M as a $C^\infty(M)$-submodule $\mathcal{F}$ of vector fields on M which is locally finitely generated and closed by the Lie bracket. We extend the notion of holonomy to the singular case, in order to understand the linearization and stability properties of singular foliations. The results we wish to ultimately generalize to the singular case are the following:

- The *local Reeb stability theorem*, which gives a normal form of a (regular) foliated manifold (M, F) around a compact leaf L. Namely, when the leaf L has finite holonomy, the theorem states that there is a neighborhood of L which is diffeomorphic to $\dfrac{\widetilde{L} \times N_x L}{\pi_1(L)}$ endowed with

the "horizontal" foliation (see [MoMr03, Thm 2.9]). Here the representation of $\pi_1(L)$ on $N_x L$ is exactly $Lin(h_x)$. This quotient is diffeomorphic to NL, the normal bundle of the leaf, endowed with the linearization of the foliation F. Hence the local Reeb stability theorem can be viewed as a linearization result.

- A certain *cocycle* defined by Heitsch [He73], which controls deformations of foliations. Note that Crainic and Moerdijk, studying deformations of Lie algebroids in [CrMo08], introduce a cohomology theory (deformation cohomology) and a certain cocycle which controls such more general deformations; they also show that it recovers Heitsch's cocycle when the Lie algebroid is a regular foliation.

- The notion of *Riemannian foliation*. A Riemannian foliation consists of a regular foliation F on a Riemannian manifold M such that such that the action $Lin(h)$ of the holonomy groupoid on the normal bundle $F^\perp \to M$ is by isometries [Mo88][Hur, §1]. A lot can be said about the structure and topology of Riemannian foliations, see for instance [Mo88][MoMr03].

We elaborate on the singular version of the first item above, that is, the question of whether a singular foliation is isomorphic to its linearization. This question is interesting already in the case of singular foliations generated by one vector field. In this case, it reads as follows and is an interesting problem in the theory of differential equations:

- Consider a vector field X on a smooth manifold M vanishing at a point x. Its linearization is the vector field X_{lin} on $T_x M$ defined by the first-order (linear) term of the Taylor expansion of X at x. Under what assumptions is there a diffeomorphism ϕ from a neighborhood of x to neighborhood V of the origin in $T_x M$ and a nowhere-vanishing function $f \in C^\infty(V)$ such that ϕ identifies X with $f \cdot X_{lin}$? When this occurs, X and X_{lin} are said to be *orbitally equivalent*, as their orbits are identified by ϕ. The literature[2] seems to provide an answer to this question only when M has dimension 2 and in the formal setting, see [IY08, Prop. 4.29].

Recent work by Crainic and Struchiner [CrStr11] on the linearization of proper Lie groupoids does provide a linearization result for those singular foliations $\mathcal{F}$ which arise from such groupoids. One question that arises naturally is what role the holonomy groups (namely the isotropy groups of the holonomy groupoid) play in the linearizability of the foliation. After all, much like regular foliations and the Reeb stability theorem there, also in the singular case it is reasonable to require linearization conditions using the least possible information, and the holonomy groupoid naturally provides the correct framework for this among all other groupoids the foliation is possibly defined from. However, techniques which work in the smooth category, like the ones developed in [CrStr11], cannot be applied to the holonomy groupoid of a singular foliation, given the pathology of its topology (see [AnSk06]). In fact, the range of this applicability in the framework of singular foliations is an endeavour of different order, well worth investigating in a separate article.

Main tools and overview of results

Our current study relies once again on the construction of the holonomy groupoid H for *any* singular foliation $(M, \mathcal{F})$ given in [AnSk06], and the notion of bi-submersion introduced there. Recall that H

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[2]The question of whether there is a diffeomorphism mapping X straight to X_{lin} appears to be treated more in the literature, see Sternberg's [Ste57, Thm. 4] and, in the setting of homeomorphisms, the Hartman-Grobman Theorem [Pe01, §2.8].

is a topological groupoid associated to $(M, \mathcal{F})$ (in particular, it encodes more information than just the partition of M into leaves of $\mathcal{F}$), and that a bi-submersion is a smooth cover of an open subset of the (often) topologically pathological groupoid H. In [AnZa11] we showed that if a leaf satisfies certain regularity conditions, then the restriction of H to the leaf is smooth; we build strongly on this result.

If one tries to define the holonomy of a singular foliation starting from (classes of) paths, as in the regular case, one obtains a very coarse notion, which does not allow for linearization. To remedy this, we introduce the notion of *holonomy transformation*, an equivalence class of germs of diffeomorphisms. It is the correct replacement of $GermDiff(S)$ for a singular foliation, as it encodes the geometric idea of holonomy and specializes correctly in the regular case. We explain this in the first item of the following list, which presents our main results in §2–§3:

- For x, y in the same leaf of a singular foliation $(M, \mathcal{F})$, consider transversal slices S_x, S_y to the leaf at x and y respectively. There is a well-defined map

$$\Phi_x^y \colon H_x^y \to \frac{GermAut_{\mathcal{F}}(S_x; S_y)}{exp(I_x\mathcal{F})|_{S_x}},$$

  suitably constructed restricting the flows of vector fields in $\mathcal{F}$. The above target space is the space of germs of foliation-preserving local diffeomorphisms between S_x and S_y, quotiented by the exponentials of elements in the maximal ideal $I_x\mathcal{F}$ (Thm. 2.7). Elements of the target are, by definition, holonomy transformations.

  The maps Φ_x^y assemble to a morphism of groupoids $\Phi \colon H \to \{\text{holonomy transformations}\}$, which in the case of regular foliations recovers the usual notion of holonomy given assembling the maps h_x introduced earlier. The map Φ is conjecturally injective.

- The map Φ linearizes to a morphism of groupoids $Lin(\Phi) \colon H \to Iso(N)$, whose target is the groupoid of isomorphisms between the fibres of the (singular) normal bundle to the leaves N (linear holonomy). See Prop. 3.1.

- Although the normal "bundle" N is a singular space, its sections form a nice $C^\infty(M)$-module $\mathcal{N} = \mathfrak{X}(M)/\mathcal{F}$. The "bundle" N carries transversal information only up to first order, while all the higher order transversal data is carried by the module $\mathcal{N}$. We show that $Lin(\Phi)$ differentiates to the "Bott connection" $\mathcal{F} \times \mathcal{N} \to \mathcal{N}$ (Prop. 3.12).

The map Φ above encodes the geometric notion of holonomy for singular foliations – quite a non-trivial notion –, and as such it is geometrically relevant and interesting. The well-definess of Φ is the technically hardest result in this paper, and the existence of Φ gives a full geometric justification for the terminology "holonomy groupoid". Our results toward a generalization of the Reeb stability theorem are based on its linearlization $Lin(\Phi)$. More precisely, they are based on the fact that, given a leaf satisfying certain regularity conditions, the restriction of $Lin(\Phi)$ to the leaf is a *Lie groupoid representation*. The main results of §4 are:

- We give two local models for the foliation around a leaf L:
  a) The normal bundle NL is endowed with the foliation $\mathcal{F}_{lin}$ generated by the linearizations of vector fields in $\mathcal{F}$. Under regularity conditions on L, $\mathcal{F}_{lin}$ is the foliation induced by the above-mentioned Lie groupoid representation.

b) Under regularity conditions on the leaf L, the quotient $Q = \dfrac{H_x \times N_x L}{H_x^x}$ is smooth, where H_x^x acts on $N_x L$ by the restriction of the linear holonomy $Lin(\Phi)$. Q is endowed with a canonical singular foliation.

We show that the two models are isomorphic (see Prop. 4.7). Notice that the second model Q is the natural generalization of the model appearing in the Reeb stability theorem for regular foliations.

- Under regularity conditions on the leaf L we show the following equivalence, where $x \in L$: $\mathcal{F}$ is linearizable about L and H_x^x is compact iff $\mathcal{F}$ (in a neighborhood of L) is the foliation induced by a Lie groupoid which is proper at x (see Prop. 4.12).

In this case, $\mathcal{F}$ admits the structure of a singular Riemannian foliation around L (Prop. 4.17).

A key ingredient to prove the last item above is the following observation [AnSk06, Ex. 3.4(4)]: when a foliation $\mathcal{F}$ is induced by a Lie groupoid G, then the holonomy groupoid H of $\mathcal{F}$ is a quotient of G (in particular, the properness of G at x implies the compactness of H_x^x). This observation implies that there is a huge class of linearizable foliations which *cannot* be defined by any proper Lie groupoid, namely those foliations which are defined from a linear action of a non-compact group (see remark 4.13). Further, the problem of when a given Lie algebroid A integrates to a proper Lie groupoid is an open one, and the above observation shows: if the foliation defined by A has non-compact isotropy group H_x^x at some point x, then there is no proper Lie groupoid integrating A. Finally in §5 we consider deformations:

- We define the cohomology groups $H_{def}^*(\mathcal{F})$ (deformation cohomology) and, using the "Bott connection", we define $H^*(\mathcal{F}, \mathcal{N})$ (foliated cohomology). Using the techniques of [CrMo08] we find that deformations of $(M, \mathcal{F})$ with isomorphic underlying $C^\infty(M)$-module structure are controlled by a certain element of $H_{def}^2(\mathcal{F})$. There exists a canonical map $H_{def}^2(\mathcal{F}) \to H^1(\mathcal{F}, \mathcal{N})$, and in the regular case the image of the above cocycle is exactly Heitsch's class.

Notation: Given a manifold M, we use $\mathfrak{X}(M)$ to denote its vector fields, and $\mathfrak{X}_c(M)$ its vector fields with compact support. For a vector field X and $x \in M$, we use $exp_x(X) \in M$ to denote the time-one flow of X applied to x. By $\mathcal{F}$ we will always denote a singular foliation on M and L a leaf. If $X \in \mathcal{F}$, we use $[X]$ to denote the class $X \mod I_x \mathcal{F}$ (here $x \in M$). The notation $\langle X \rangle$ is used to denote either classes under several other equivalence relations or the foliation generated by X. The holonomy groupoid of a singular foliation is denoted by H. Further, $H_x = \mathbf{s}^{-1}(x)$ is the source fiber and $H_x^x = \mathbf{s}^{-1}(x) \cap \mathbf{t}^{-1}(x)$ the isotropy group at x; the same notation applies to bi-submersions $U, W, ...$

1 Background material

For the convenience of the reader we give here an outline of the constructions and results of [AnSk06] and [AnZa11].

1.1 Foliations

Let M be a smooth manifold. Given a vector bundle $E \to M$ we denote $C_c^\infty(M;E)$ the $C^\infty(M)$-module of compactly supported sections of E. Stefan [St74] and Sussmann [Su73] showed that the following definition gives rise to a partition of M into injectively immersed submanifolds (leaves):

Definition 1.1. A *(singular) foliation* on M is a locally finitely generated submodule of the $C^\infty(M)$-module $\mathcal{X}_c(M) = C_c^\infty(M;TM)$, stable by the Lie bracket.

In general, a singular foliation contains more information than the underlying partition of M into leaves. Singular foliations arise in many natural geometric contexts: from actions of Lie groups and, more generally, from Lie groupoids and Lie algebroids. The following apparatus is naturally associated with a foliation, and will be of use in the current article.

  a) A leaf L is *regular* if there exists an open neighborhood W of L in M such that the dimension of L is equal to the dimension of any other leaf intersecting W. Otherwise L will be called *singular*.

  b) For $x \in M$ let $I_x = \{f \in C^\infty(M) : f(x) = 0\}$ and consider the maximal ideal $I_x\mathcal{F}$ of $\mathcal{F}$. Since $\mathcal{F}$ is locally finitely generated, the quotient $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$ is a finite dimensional vector space. Let F_x be the tangent space at x of the leaf L_x passing through x. We have a short exact sequence of vector spaces
  $$0 \to \mathfrak{g}_x \to \mathcal{F}_x \overset{ev_x}{\to} F_x \to 0$$
  Its kernel $\mathfrak{g}_x$ is a Lie algebra, which vanishes iff L_x is a regular leaf. Explicitly, $\mathfrak{g}_x = \mathcal{F}(x)/I_x\mathcal{F}$ where $\mathcal{F}(x) := \{X \in \mathcal{F} : X_x = 0\}$.

  c) Let L be a leaf and put I_L the space of smooth functions on M which vanish on L. Then $A_L = \cup_{x \in L}\mathcal{F}_x$ is a transitive Lie algebroid over L. If L is embedded then $C_c^\infty(L;A_L) = \mathcal{F}/I_L\mathcal{F}$. If L is immersed then the previous equality holds locally (see [AnZa11, Rem. 1.8]).

  d) We can pull back a foliation along a smooth map: If $p : N \longrightarrow M$ is smooth then $p^{-1}(\mathcal{F})$ is the submodule of $\mathcal{X}_c(N)$ consisting of $C_c^\infty(N)$-linear combinations of vector fields on N which are projectable and project to elements of $\mathcal{F}$.

The local picture of a foliation is the following:

Proposition 1.2. (Splitting theorem) *Let $(M,\mathcal{F})$ be a manifold with a foliation and $x \in M$, and set $k := dim(F_x)$. Let $\hat{S}$ be a slice at x, that is, an embedded submanifold such that $T_x\hat{S} \oplus F_x = T_xM$.*

Then there exists an open neighborhood W of x in M and a diffeomorphism of foliated manifolds

$$(W, \mathcal{F}_W) \cong (I^k, TI^k) \times (S, \mathcal{F}_S). \tag{1.1}$$

Here $\mathcal{F}_W$ is the restriction of $\mathcal{F}$ to W, $I := (-1, 1)$, $S := \hat{S} \cap W$ and $\mathcal{F}_S$ consists of the restriction to S of vector fields in W tangent to S.

In particular, if we denote by $s_1, \ldots, s_k$ the canonical coordinates on I^k and $X_1, \ldots, X_l$ are generators of $\mathcal{F}_S$, then $\mathcal{F}_W$ is generated by $\partial_{s_1}, \ldots, \partial_{s_k}$ and the (trivial extensions of) $X_1, \ldots, X_l$.

1.2   Holonomy groupoids

Let $(M, \mathcal{F})$ be a (singular) foliation. We recall the notion of bi-submersion from [AnSk06] and the construction of the holonomy groupoid.

a) A *bi-submersion* of $(M, \mathcal{F})$ is a smooth manifold U endowed with two submersions $\mathbf{t}, \mathbf{s} : U \longrightarrow M$ satisfying:

   (i) $\mathbf{s}^{-1}(\mathcal{F}) = \mathbf{t}^{-1}(\mathcal{F})$,

   (ii) $\mathbf{s}^{-1}(\mathcal{F}) = C_c^\infty(U; \ker d\mathbf{s}) + C_c^\infty(U; \ker d\mathbf{t})$.

   We say $(U, \mathbf{t}, \mathbf{s})$ is *minimal* at u if $dim(U) = dim(M) + dim(\mathcal{F}_{\mathbf{s}(u)})$.

b) Let $x \in M$, and $X_1, \ldots, X_n \in \mathcal{F}$ inducing a basis of $\mathcal{F}_x$. In [AnSk06, Prop. 2.10 a)] it was shown that there is an open neighborhood U of $(x, 0)$ in $M \times \mathbb{R}^n$ such that $(U, \mathbf{t}_U, \mathbf{s}_U)$ is a bi-submersion minimal at $(x, 0)$, where $\mathbf{s}_U(y, \xi) = y$ and $\mathbf{t}_U(y, \xi) = exp_y(\sum_{i=1}^n \xi_i X_i)$. (Recall that the latter is the image of y under the time-1 flow of $\sum_{i=1}^n \xi_i X_i$.) Bi-submersions arising this way are called *path holonomy bi-submersions.*

c) Let $(U_i, \mathbf{t}_i, \mathbf{s}_i)$ be bi-submersions, $i = 1, 2$. Then $(U_i, \mathbf{s}_i, \mathbf{t}_i)$ are bi-submersions, as well as $(U_1 \circ U_2, \mathbf{t}, \mathbf{s})$ where $U_1 \circ U_2 = U_1 \times_{\mathbf{s}_1, \mathbf{t}_2} U_2$, $\mathbf{t}(u_1, u_2) = \mathbf{t}(u_1)$ and $\mathbf{s}(u_1, u_2) = \mathbf{s}(u_2)$. They are called the *inverse* and *composite* bi-submersions respectively.

d) Let $(U, \mathbf{t}_U, \mathbf{s}_U)$ and $(V, \mathbf{t}_V, \mathbf{s}_V)$ be two bi-submersions. A *morphism of bi-submersions* is a smooth map $f : U \longrightarrow V$ such that $\mathbf{s}_V \circ f = \mathbf{s}_U$ and $\mathbf{t}_V \circ f = \mathbf{t}_U$.

e) A *bisection* of $(U, \mathbf{t}, \mathbf{s})$ is a locally closed submanifold V of U on which the restrictions of $\mathbf{s}$ and $\mathbf{t}$ are diffeomorphisms to open subsets of M.

f) We say that $u \in U$ *carries* the foliation-preserving local diffeomorphism ψ if there is a bisection V such that $u \in V$ and $\psi = \mathbf{t}\mid_V \circ(\mathbf{s}\mid_V)^{-1}$.

g) It was shown in [AnSk06, Cor. 2.11(b)] that if $\{(U_i, \mathbf{t}_i, \mathbf{s}_i)\}_{i \in I}$ are bi-submersions, $i = 1, 2$ then $u_1 \in U_1$ and $u_2 \in U_2$ carry the same local diffeomorphism iff there exists a morphism of bi-submersions g defined in an open neighborhood of $u_1 \in U_1$ such that $g(u_1) = u_2$. Such a morphism maps every bisection V of U_1 at u_1 to a bisection $g(V)$ of U_2 at u_2.

Bi-submersions are the key for the construction of the holonomy groupoid. Let us recall this construction:

Given a foliation $(M, \mathcal{F})$, take a family of path holonomy bi-submersions $\{U_i\}_{i \in I}$ covering M, i.e. $\cup_{i \in I} \mathbf{s}(U_i) = M$. Let $\mathcal{U}$ be the family of all finite products of elements of $\{U_i\}_{i \in I}$ and of their inverses

($\mathcal{U}$ is a *path-holonomy atlas*, see [AnSk06, Ex. 3.4(3)]). The *holonomy groupoid* of the foliation $\mathcal{F}$ is the quotient

$$H(\mathcal{F}) := \coprod_{U \in \mathcal{U}} U / \sim$$

by the equivalence relation for which $u \in U$ is equivalent to $u' \in U'$ if there is a morphism of bi-submersions $f : W \to U'$ defined in a neighborhood $W \subset U$ of u such that $f(u) = u'$.

We denote the holonomy groupoid by H when the choice of $\mathcal{F}$ is clear. Its restriction to a leaf L is $H_L = \mathbf{s}^{-1}(L) = \mathbf{t}^{-1}(L)$. Its $\mathbf{s}$-fiber is $H_x = \mathbf{s}^{-1}(x)$ and its isotropy group $H_x^x = \mathbf{s}^{-1}(x) \cap \mathbf{t}^{-1}(x)$, where $x \in L$.

1.3 Essential isotropy groups

The holonomy groupoid H of a singular foliation is a topological groupoid, whose topology is usually very pathological. Given a leaf L of the foliation, in [AnZa11] we gave conditions ensuring the smoothness of H_L and the integrability of A_L. First, let us recall what smoothness means: We say that H_L is *smooth* if there exists a differentiable structure on it such that $\sharp \colon U_L \to H_L$ a submersion[3] for all bi-submersions $(U, \mathbf{t}, \mathbf{s})$ in the path holonomy atlas, where $U_L := \mathbf{s}^{-1}(L)$ and $\sharp$ denotes the quotient map by the (restriction of the) equivalence relation $\sim$. If H_L is smooth, it turns out that it is a Lie groupoid.

What controls the smoothness of H_L is an "essential isotropy" group attached to the leaf L. Here is how it arises: Pick a point $x \in L$ and consider the connected and simply connected Lie group G_x integrating the Lie algebra $\mathfrak{g}_x$. Let $\{X_i\}_{i \leq n} \in \mathcal{F}$ be vector fields whose images in $\mathcal{F}_x$ form a basis of $\mathcal{F}_x$ and such that the images of $\{X_i\}_{i \leq \ell}$ form a basis of $\mathfrak{g}_x$. Let U be the corresponding path holonomy bi-submersion. Choose $\mathbf{t}$-lifts $Y_i \in C^\infty(U; \ker d\mathbf{s})$ of the X_i. We can find a small neighborhood $\widetilde{G}_x$ of the identity in G_x where the map

$$\Delta \colon \tilde{G}_x \to U_x^x \ , \ exp_{\mathfrak{g}_x}(\sum_{i=1}^l k_i[X_i]) \mapsto exp_{(x,0)}(\sum_{i=1}^l k_i Y_i)$$

is a diffeomorphism onto its image. It turns out that the composition $\widetilde{\varepsilon}_x = \sharp \circ \Delta : \widetilde{G}_x \to H_x^x$ is independent of the choice of path holonomy bi-submersion and extends to a morphism of topological groups

$$\varepsilon : G_x \to H_x^x$$

The *essential isotropy group of L* is the kernel of ε. Here is how it controls the smoothness of H_L and the integrability of A_L:

Theorem 1.3. *The transitive groupoid H_L is smooth and integrates the Lie algebroid $A_L = \cup_{x \in L} \mathcal{F}_x$ if and only if the essential isotropy group of L is discrete.*

2 Holonomy transformations

In this section we extend the familiar notion of holonomy of a regular foliation to the singular case. We introduce and motivate the notion of holonomy transformation in §2.2. In §2.3

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[3]When L is not embedded, U_L has to be endowed with the leaf-wise smooth structure (see [AnSk06, Prop. 1.14]). The latter is well-defined as U_L is a leaf of the singular foliation $\mathbf{s}^{-1}(\mathcal{F})$ on U.

we show how to associate holonomy transformations to a foliation: we obtain a map $\Phi\colon H \to$ {holonomy transformations}, generalizing what happens in the regular case. In §2.4 we conjecture that this map is injective, and provide some evidence for this claim.

2.1 Overview of holonomy in the regular case

A regular foliation of M is given by an involutive subbundle of TM. Taking $\mathcal{F}$ to be its module of compactly supported sections we obtain a foliation in the sense of Def. 1.1. One way to see the classical notion of holonomy of a regular foliation is as follows: Consider a path γ from x to y lying in a leaf L and fix transversals S_x at x and S_y at y. The map $\gamma\colon [0,1] \to M$ can be extended to a continuous map

$$\Gamma\colon S_x \times [0,1] \to M$$

with $\Gamma|_{S_x \times \{0\}} = Id_{S_x}$, $\Gamma(S_x \times \{1\}) \subset S_y$ and such that $t \mapsto \Gamma(\tilde{x}, t)$ is a curve in a leaf of $\mathcal{F}$ for all $\tilde{x} \in S_x$. In particular, $\Gamma(x,t) = \gamma(t)$. The *holonomy of γ* is then defined as the germ at x of the map

$$hol_\gamma : S_x \to S_y, \tilde{x} \mapsto \Gamma(\tilde{x}, 1).$$

We have:

- The holonomy of γ is independent of the choice of extension Γ, and depends on the homotopy class of γ in L rather than on γ itself. This gives rise to a map
  $hol\colon$ {homotopy classes of paths in L from x to y} $\to GermAut_\mathcal{F}(S_x; S_y)$.

- The holonomy of γ can be linearised taking the derivative $d_x hol_\gamma \colon T_x S_x \to T_y S_y$. This gives rise to a map
  {homotopy classes of paths in L from x to y} $\to Iso(T_x S_x, T_y S_y)$ (linear holonomy).

Remark 2.1. The choice of transversal is immaterial: if S'_x is another transversal at x, one obtains a canonical identification $S_x \cong S'_x$ near x modifying slightly the above holonomy construction. If further we take a transversal S'_y at y, then the diffeomorphism $S'_x \to S'_y$ obtained as the holonomy of γ and the diffeomorphism $S_x \to S_y$ obtained above coincide upon applying the identifications.

2.2 The singular case

Let us consider a singular foliation $(M, \mathcal{F})$. We make a first attempt to define a notion of holonomy, by the following recipe which makes use of the module $\mathcal{F}$ rather than just of the underlying partition of M into leaves, and which clearly reduces to the notion of holonomy in the regular case. Our attempt will not be completely successful, but it is useful in that it motivates the definition of holonomy transformation.

Let $\gamma\colon [0,1] \to M$ be a curve from x to y lying in a leaf of $\mathcal{F}$, and fix slices S_x and S_y. For every t extend $\dot\gamma(t)$, the velocity of the curve at time t, to a vector field Z^t lying in $\mathcal{F}$, with the property that $\Gamma\colon S_x \times [0,1] \to M$ – defined following the flow of the time-dependent vector field $\{Z^t\}_{t\in[0,1]}$ starting at points of S_x – takes S_x to S_y. Unlike the regular case, the resulting map $S_x \to S_y, \tilde{x} \mapsto \Gamma(\tilde{x}, 1)$ depends on the choice of extension Γ. This can be seen looking at simple examples:

Examples 2.2.   a) Let $M = \mathbb{R}$, and $\mathcal{F} = \langle z\partial_z \rangle$: taking $x = y = 0$ the transversal S_0 is a neighborhood of the origin in M. The constant path at the origin admits many extensions: $\Gamma\colon S_0 \times [0,1] \to M, (\tilde{x}, t) \mapsto \tilde{x}$ (the flow of the zero vector field), $\Gamma'\colon S_0 \times [0,1] \to M, (\tilde{x}, t) \mapsto e^t \tilde{x}$ (the flow of $z\partial_z$). They clearly give quite different germs of diffeomorphisms at the origin.

b) Let $M = \mathbb{R}^2$ with coordinates z, w, and let $\mathcal{F} = \langle z\partial_w - w\partial_z \rangle$. Taking $x = y = 0$ the transversal S_0 is a neighborhood of the origin in M. The constant path at the origin admits many extensions: $\Gamma \colon S_0 \times [0,1] \to M, (\tilde{x}, t) \mapsto \tilde{x}$ (the flow of the zero vector field), or $\Gamma' \colon S_0 \times [0,1] \to M, (\tilde{x}, t) \mapsto R^t \tilde{x}$ where R^t is rotation about the origin by the angle t (the flow of $z\partial_w - w\partial_z$). They clearly give quite different germs of diffeomorphisms at the origin, namely, the identity and the rotation by one gradient.

Now we show that different choices of Γ induce diffeomorphisms $S_x \to S_y$ which differ by the flow of a vector field on $\mathcal{F}$ vanishing at x. In order to do this we introduce the following notation (see §1.1 for the definitions of I_x and $\mathcal{F}(x)$):

- $Aut_\mathcal{F}(M)$ is the subgroup of local diffeomorphisms of M preserving $\mathcal{F}$.

- $GermAut_\mathcal{F}(S_x, S_y)$ is the space of germs at x of local diffeomorphisms in $Aut_\mathcal{F}(M)$ mapping S_x to S_y, restricted to S_x.

- $exp(I_x\mathcal{F})$ is the space of time-one flows of time-dependent vector fields in $I_x\mathcal{F}$. Analogously we define $exp(\mathcal{F}(x))$. Both are subgroups of $Aut_\mathcal{F}(M)$.

- $exp(I_x\mathcal{F})|_{S_x}$ is the space of germs at x of local diffeomorphisms in $exp(I_x\mathcal{F})$ mapping S_x to itself, restricted to S_x. Analogously we define $exp(\mathcal{F}(x))|_{S_x}$.

Proposition 2.3. *Let γ be a path as above. The class of $\Gamma(\cdot, 1) \colon S_x \to S_y$ in the quotient $GermAut_\mathcal{F}(S_x, S_y)/exp(\mathcal{F}(x))|_{S_x}$ is independent of the chosen extension Γ.*

The above quotient is given by the equivalence relation

$$\psi \sim \hat{\psi} \Leftrightarrow \hat{\psi}^{-1} \circ \psi \in exp(\mathcal{F}(x))|_{S_x}$$

Proof. Denote by Z_t and Z_t' the time-dependent vector fields in $\mathcal{F}$ used to define the extensions Γ and Γ', and by ϕ_t, ϕ_t' their flows.

For any two time-dependent vector fields $V = \{V_t\}_{t\in\mathbb{R}}, W = \{W_t\}_{t\in\mathbb{R}}$, the following relation between flows Φ_t at time t holds [Po88, eq. (2)]:

$$\Phi_t(V + W) = \Phi_t(V) \circ \Phi_t[\{(\Phi_s(V))_*^{-1}W_s\}_{s\in\mathbb{R}}] \tag{2.1}$$

where the argument in the square bracket is the time-dependent vector field which, at time s, is obtained pushing forward W_s via $(\Phi_s(V))^{-1}$. With $V_t := Z_t, W_t := Z_t' - Z_t$ the above formula reads

$$\phi_t' = \phi_t \circ (\text{time-}t \text{ flow of } \{(\phi_s)_*^{-1}(Z_s' - Z_s)\}_{s\in\mathbb{R}}). \tag{2.2}$$

Since γ is an integral curve of both Z_s and Z_s', we have $Z_s(\phi_s(x)) = Z_s'(\phi_s(x))$ for all s. Hence $(\phi_s)_*^{-1}(Z_s' - Z_s)$ vanishes at x, for all s. Therefore the time-t flow of $\{(\phi_s)_*^{-1}(Z_s' - Z_s)\}_{s\in\mathbb{R}}$ lies in $exp(\mathcal{F}(x))$. $\square$

The notion of holonomy obtained in our attempt – namely, the class associated to the path γ by Proposition 2.3 – is unsatisfactory. Indeed, as it can be seen in examples 2.2, the ambiguity given by $exp(\mathcal{F}(x))$ is much too large to allow for linearization (i.e., we do not obtain a well-defined map $T_x S_x \to T_y S_y$ associated to the path γ). Hence we replace $\mathcal{F}(x)$ by $I_x\mathcal{F}$ and propose the following notion:

Definition 2.4. Let $(M, \mathcal{F})$ be a singular foliation, and $x, y \in M$ lying in the same leaf. Fix a transversal S_x at x, as well as a transversal S_y at y. A *holonomy transformation from x to y* is an element of
$$\frac{GermAut_{\mathcal{F}}(S_x, S_y)}{exp(I_x\mathcal{F})|_{S_x}}.$$

Remark 2.5. The equivalence relation on $GermAut_{\mathcal{F}}(S_x, S_y)$ in the above definition is finer than the one in Proposition 2.3, as $exp(I_x\mathcal{F})$ is quite smaller than $exp(\mathcal{F}(x))$. As a consequence, holonomy transformations have a well-defined linearization (see §3.1).

Lemma 2.6. *If x belongs to a regular leaf then $exp(I_x\mathcal{F})|_{S_x}$ is trivial.*

Proof. Any element of $exp(I_x\mathcal{F})|_{S_x}$ fixes x and restricts to each leaf of $\mathcal{F}$, so it can be computed in terms of the usual holonomy of the constant path at x, which of course is Id_{S_x}. $\qquad\square$

Consider now the case of regular foliations. Lemma 2.6 says that the holonomy of a path is a holonomy transformation, since $exp(I_x\mathcal{F})|_{S_x}$ is trivial. Further, since an element of the holonomy groupoid H is exactly a class of paths in $(M, \mathcal{F})$ having the same holonomy (see [AnSk06, Cor. 3.10]), there is a canonical injective map

$$H \to \{\text{holonomy transformations}\}.$$

This is the point of view that we carry over to the singular case: while holonomy transformations are certainly not associated to (classes of) paths, in §2.3 we will see that they are associated to elements of the holonomy groupoid H.

2.3 The "action" by Holonomy Transformations

The following theorem assigns a holonomy transformation to each element of the holonomy groupoid. It is the main result of the whole of §2. Its proof is rather involved and will be given in Appendix A.1.

Theorem 2.7. *Let $x, y \in (M, \mathcal{F})$ be points in the same leaf L, and fix transversals S_x at x and S_y at y. Then there is a well defined map*

$$\Phi_x^y \colon H_x^y \to \frac{GermAut_{\mathcal{F}}(S_x, S_y)}{exp(I_x\mathcal{F})|_{S_x}}, \quad h \mapsto \langle \tau \rangle. \tag{2.3}$$

Here τ is defined as follows, given $h \in H_x^y$:

- *take any bi-submersion $(U, \mathbf{t}, \mathbf{s})$ in the path-holonomy atlas with a point $u \in U$ satisfying $[u] = h$,*

- *take any section $\bar{b} \colon S_x \to U$ through u of $\mathbf{s}$ such that $(\mathbf{t} \circ \bar{b})(S_x) \subset S_y$,*

and define $\tau := \mathbf{t} \circ \bar{b} \colon S_x \to S_y$.

Corollary 2.8. *For every point $x \in (M, \mathcal{F})$ fix a slice S_x transverse to the foliation. The maps Φ_x^y of Thm. 2.7 assemble to a groupoid morphism*

$$\boxed{\Phi \colon H \to \cup_{x,y} \frac{GermAut_{\mathcal{F}}(S_x, S_y)}{exp(I_x\mathcal{F})|_{S_x}}}$$

where the union is taken over all pairs of points lying in the same leaf.

Remark 2.9. Heuristically, one can think of Φ as an action of the groupoid H on the union of all slices.

Proof. Clearly $\cup_{x,y} GermAut_{\mathcal{F}}(S_x, S_y)$ is a set-theoretic groupoid over M. For every $\psi \in exp(I_x\mathcal{F})|_{S_x}$ and $\phi \in GermAut_{\mathcal{F}}(S_x, S_y)$ we have $\phi\psi\phi^{-1} \in exp(I_y\mathcal{F})|_{S_y}$. This follows from the fact that for any time-dependent vector field Y on M one has $\phi \circ exp(Y) \circ \phi^{-1} = exp((\phi_* Y))$, and that if $Y \in I_x\mathcal{F}$ then $\phi_* Y \in I_y\mathcal{F}$. Hence the target of Φ is a set-theoretic groupoid over M.

We prove that Φ is a groupoid morphism, as follows. If $u \in U$ and $v \in V$ are points of bi-submersions with $\mathbf{s}_U(u) = \mathbf{t}_V(v)$, and $\bar{b}_U, \bar{b}_V$ are as in Thm. 2.7, then $\bar{b}_U \circ \bar{b}_V \colon S_{\mathbf{s}_V(v)} \to U \circ V$ is a section through (u, v) and it is given by $z \mapsto (\bar{b}_U(\mathbf{t}_V(\bar{b}_V(z))), \bar{b}_V(z))$. So applying $\mathbf{t}_{U\circ V}$ we obtain $(\mathbf{t}_U \circ \bar{b}_U) \circ (\mathbf{t}_V \circ \bar{b}_V)$. $\qquad\square$

Remark 2.10.   a) If x belongs to a regular leaf then Thm. 2.7 recovers the usual notion of holonomy for regular foliations, by the commments at the very end of §2.2. For singular foliations, a point of the holonomy groupoid H does not determine a path in M, nor a homotopy class of paths.

 b) The above result was inspired by [Fe02, Prop. 3.1], which we can not apply directly since in general there is no Lie algebroid (defined over M) integrating to the holonomy groupoid H. It differs from [Fe02, Prop. 3.1] in that we do not make any choices (the price to pay for this is that we have to quotient by $exp(I_x\mathcal{F})|_{S_x}$) and that the domain of the above map Φ is not a set of Lie algebroid paths but rather H.

 c) The use of slices in Theorem 2.7 is unavoidable, as in general there exists no continuous groupoid morphism
$$H \to \cup_{x,y} \frac{GermAut_{\mathcal{F}}(W_x; W_y)}{exp(I_x\mathcal{F})|_{W_x}},$$
where W_x, W_y are suitable open neighborhoods of x, y respectively, and where the union is taken over all pairs of points lying in the same leaf. This is already apparent in the case of regular foliations. Take for example the Möbius strip M with the one-leaf foliation, so $H = M \times M$, and assume that such a continuous groupoid morphism existed. On one hand, the morphism property implies that the point $(x, x) \in H$ is mapped to the class of Id_{W_x}. On the other hand, considering (x, y) as y varies along a non-contractible loop in M starting and ending at x, by continuity the point (x, x) would be mapped to an orientation-reversing diffeomorphism of W_x, which can not lie in the class of Id_{W_x} since elements of $exp(I_x\mathcal{F})|_{W_x}$ are orientation-preserving. Hence we obtain a contradiction.

The following examples for Theorem 2.7 show the dependence from the choice of module $\mathcal{F}$.

Examples 2.11.   (i) Let $M = \mathbb{R}$ and $\mathcal{F}$ be generated by $X := z\partial_z$. We consider $0 \in M$; a transversal S_0 is just a neighborhood of 0 in M. Consider the path-holonomy bi-submersions $U \subset M \times \mathbb{R} \rightrightarrows M$ defined by the generator X and fix $u := (\lambda, 0) \in U$ where λ is a real number. In order to compute the image of $[u]$ under Φ_0^0 we can take any bisection passing through it. The bisection $b \colon S_0 \to \mathbb{R}$ with constant value λ carries the diffeomorphism $z \mapsto exp_z(\lambda X) = e^\lambda z$. Hence $\Phi_0^0([(\lambda, 0)]$ is the class of $y \mapsto e^\lambda y$ in $GermAut_{\mathcal{F}}(S_0, S_0)/exp(I_0 X)$.

(ii) Let again $M = \mathbb{R}$ and but now let $\mathcal{F}$ be generated by $X := z^2\partial_z$. A slice S_0 is just a neighborhood of 0 in M. Consider the bi-submersions $U \subset M \times \mathbb{R} \rightrightarrows M$ defined by the generator X and fix $u := (\lambda, 0) \in U$ where λ is a real number. The bisection $b \colon S_0 \to \mathbb{R}$ with constant value λ carries the diffeomorphism

$$z \mapsto exp_z(\lambda X) = \frac{z}{1 - \lambda z} = z + \lambda z^2 + \lambda^2 z^3 + \cdots .$$

Hence $\Phi_0^0[(\lambda, 0)]$ is the class of this diffeomorphism in $GermAut_{\mathcal{F}}(S_0, S_0)/exp(I_0 X)$.

(iii) We consider the S^1-action on $M = \mathbb{R}^2$ by rotations. Let $U \subset M \times \mathbb{R}$ be the bi-submersion generated by $y\partial_x - x\partial_y$. Any bisection through the origin is given by a map $\lambda \colon \mathbb{R}^2 \to \mathbb{R}$ (a section of the source map $\mathbf{s}$). It induces the diffeomorphism of $\mathbb{R}^2$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} cos(\lambda(x,y)) & -sin(\lambda(x,y)) \\ sin(\lambda(x,y)) & cos(\lambda(x,y)) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which represents the class $\Phi_0^0[(\lambda_0, 0)]$ for $\lambda_0 := \lambda(0, 0)$.

Above we fixed a choice of transversals S_x and S_y. This choice is immaterial due to the next lemma, whose proof uses technicalities that we give in Appendix A.2.

Lemma 2.12. *Let x, y be points in a foliated manifold $(M, \mathcal{F})$ lying in the same leaf. Choose two transversals S_x^i, at x and two transversals S_y^i at y $(i = 1, 2)$. Then the maps*

$${}^i\Phi_x^y \colon H_x^y \to GermAut_{\mathcal{F}}(S_x^i, S_y^i)/exp(I_x\mathcal{F})|_{S_x^i}$$

defined in Theorem 2.7 coincide upon the canonical identification given by eq. (A.7).

Proof. Let $h \in H_x^y$, and take any bi-submersion $(U, \mathbf{t}, \mathbf{s})$ with a point $u \in U$ satisfying $[u] = h$. For $i = 1, 2$ take any section $\bar{b}^i \colon S_x^i \to U$ through u of $\mathbf{s}$ such that $(\mathbf{t} \circ \bar{b}^i)(S_x^i) \subset S_y^i$, and extend it to a bisection b^i of U such that $(d_x b^i)(T_x L) = 0$, where L denotes the leaf through x. Define the local diffeomorphisms $\phi^i := \mathbf{t} \circ b^i$ from a neighborhood of x to a neighborhood of y. By Lemma A.7 there exists $\psi_y^{21} \in exp(I_y\mathcal{F})$ mapping S_y^1 to S_y^2. Define ψ_x^{12} by requiring

$$\phi^2 = \psi_y^{21} \circ \phi^1 \circ \psi_x^{12}. \tag{2.4}$$

This equation can be reformulated as

$$(\phi^2)^{-1} \circ \left(\phi^1 \circ (\phi^1)^{-1}\right) \circ \psi_y^{21} \circ \phi^1 = (\psi_x^{12})^{-1}. \tag{2.5}$$

Now $(\phi^2)^{-1} \circ \phi^1 \in exp(I_x\mathcal{F})$ by the claim in the proof of Theorem 2.7 (*cf.* Appendix A.1). Further $(\phi^1)^{-1} \circ \psi_y^{21} \circ \phi^1$ also lies in $exp(I_x\mathcal{F})$. Indeed, it is the time-1 flow of the pushforward by $(\phi^1)^{-1}$ of the time-dependent vector field in $I_y\mathcal{F}$ defining ψ_y^{21}, and the pushforward by $(\phi^1)^{-1}$ maps $I_y\mathcal{F}$ to $I_x\mathcal{F}$. Hence we conclude that $(\psi_x^{12})^{-1}$, and therefore ψ_x^{12}, lies in $exp(I_x\mathcal{F})$.

Hence we found ψ_x^{12} mapping S_x^2 to S_x^1 lying in $exp(I_x\mathcal{F})$, and ψ_y^{21} mapping S_y^1 to S_y^2 lying in $exp(I_y\mathcal{F})$, satisfying eq. (2.4). From Lemma A.8 it is clear that, under the identification given by eq. (A.7), the class $[\phi^1]$ is identified with the class $[\phi^2]$. $\square$

2.4 Injectivity

Let $(M, \mathcal{F})$ be a singular foliation. We conjecture that the morphism of groupoids Φ defined in Cor. 2.8 is injective. Equivalently:

Conjecture 2.13. *For all x, y lying in the same leaf, the map Φ_x^y defined in eq. (2.3) is injective.*

We are able to prove Conjecture 2.13 in two extreme cases: at a point where the foliation vanishes (Prop. 2.15) and in the case of regular foliations (Prop. 2.18).

2.4.1 The case of points where the foliation vanishes

Lemma 2.14. *Let $x \in (M, \mathcal{F})$ and ψ a local diffeomorphism in $exp(I_x \mathcal{F})$. Let $(U, \mathbf{t}, \mathbf{s})$ be a path holonomy bi-submersion at x (whence $U \subseteq M \times \mathbb{R}^n$). Then the element $(x, 0) \in U$ carries the diffeomorphism ψ.*

Proof. Let $X_1, \ldots, X_n \in \mathcal{F}$ such that $[X_1], \ldots, [X_n]$ are a basis of $\mathcal{F}_x$. Let W be a neighborhood of x in M such that $\mathcal{F}_W = < X_1, \ldots, X_n >$. Then there exist a family of functions $f_1^t, \ldots, f_n^t \in I_x$, depending smoothly on $t \in [0, 1]$, such that $\psi = exp(\sum f_i X_i)$ (the flow of the time-dependent vector field $\{\sum f_i^t X_i\}_{t \in [0,1]}$). Let $(U, \mathbf{t}, \mathbf{s})$ be the path holonomy bi-submersion defined by $X_1, \ldots, X_n$ at x, and choose $Y_1, \ldots, Y_n \in C_c^\infty(U; \ker d\mathbf{s})$ (linearly independent) such that $d\mathbf{t}(Y_i) = X_i$. Define a section of $\mathbf{s}$ by $b^\psi \colon W \to U$, where

$$b^\psi(y) = exp_{(y,0)}\left(\sum_{i=1}^n (f_i \circ \mathbf{t}) Y_i\right).$$

Then $\mathbf{t} \circ b^\psi = exp(\sum f_i X_i) = \psi$, so b^ψ is a bisection carrying ψ. This bisection passes through $(x, 0)$ as all f_i^t lie in I_x. $\qquad\square$

We prove Conjecture 2.13 at a point where the foliation vanishes.

Proposition 2.15. *Let $x \in M$ a point where the foliation $\mathcal{F}$ vanishes. Then the map $\Phi_x^x \colon H_x^x \to \dfrac{GermAut_\mathcal{F}(S_x, S_x)}{exp(I_x \mathcal{F})|_{S_x}}$ is injective.*

Proof. Since Φ is a groupoid morphism by Cor. 2.8, it is enough to show for all $h \in H_x^x$: if $\Phi(h)$ is the class of Id_{S_x}, then $h = 1_x$. Let U a bi-submersion in the path-holonomy atlas and $u \in U$ with $[u] = h$. By the definition of Φ, the germ $\Phi(h)$ lies in the class of Id_{S_x} iff there is a section $b \colon S_x \to U$ of $\mathbf{s}$ through u such that $\psi := \mathbf{t} \circ b \in exp(I_x \mathcal{F})|_{S_x}$. Since the slice S_x is an open neighborhood of x in M, this means that u carries a diffeomorphism $\psi \in exp(I_x \mathcal{F})$. By Lemma 2.14, the point $(x, 0) \in U$ also carries ψ. Hence by [AnSk06, Cor. 2.11] there exists a morphism of bi-submersions $U \to U$ with $(x, 0) \mapsto u$, and therefore $h = [u] = [(x, 0)] = 1_x$. $\qquad\square$

2.4.2 The regular case

Lemma 2.16. *Let $(M, \mathcal{F})$ be a singular foliation and $x \in M$. Let $U \subset M \times \mathbb{R}^n$ be path-holonomy bi-submersion and $u \in U_x$.*

1) Let b be a bisection through u, and c a bisection through $(x, 0)$, and denote the diffeomorphisms they carry by ϕ_b resp. ϕ_c. Then there exists a bisection of U through u carrying $\phi_b \circ \phi_c$.

2) If $u \in U_x^x$, then there exits a bisection of U through u carrying an orientation-preserving local diffeomorphism of a neighborhood $W \subset M$ of x.

Proof. 1) There exits a morphism of bi-submersions $\alpha \colon U \circ U \to U$ with $(u, (x, 0)) \to u$, since $(x, 0)$ carries the identity and by [AnSk06, Cor. 2.11]. The image of the bisection (b, c) under α is the a bisection with the required properties.

2) Apply Prop. 1.2, so $W \cong (L \cap W) \times S_x$, where S_x is a slice at x. Denote by $s_1, \ldots, s_{dim(M)}$ a set of adapted coordinates on W, so that there exists a set of generators $\{X_i\}_{i \leq n}$ of $\mathcal{F}|_W$ with $X_1 = \partial_{s_1}, \ldots, X_k = \partial_{s_k}$. The corresponding path-holonomy bi-submersion is isomorphic to U by [AnZa11, Lemma 2.6], so we may assume that it is U. The bisection

$$c \colon W \cong (L \cap W) \times S_x \to U, s \mapsto (-2s_1, 0, \ldots, 0)$$

carries the diffeomorphism $s \mapsto exp_s(-2s_1 \partial_{s_1}) = (-s_1, s_2, \ldots, s_{dim(M)})$, which clearly is orientation-reversing.

Now let b be an arbitrary bisection through u. If the local diffeomorphism carried by b (which fixes x) is not orientation-preserving, by part 1) the local diffeomorphism carried by $b \circ c$ will be. $\square$

Lemma 2.17. *Let B be an open neighborhood of the origin 0 in $\mathbb{R}^k$, and ϕ an orientation-preserving diffeomorphism of B fixing 0. Then, shrinking B if necessary, we can write $\phi = exp(X)$ where $\{X_t\}_{t \in [0,1]}$ is a time-dependent vector field on B vanishing at 0.*

Proof. It suffices to show that ϕ is isotopic to Id_B by diffeomorphisms fixing 0. A concrete isotopy from ϕ to its derivative $d_0\phi$ (restricted to B) is given by their convex linear combination. Now $d_0\phi$ lies in $GL_+(\mathbb{R}^k)$, hence it can be connected to $Id_{\mathbb{R}^k}$ by a path in $GL_+(\mathbb{R}^k)$. $\square$

Until the end of this section we consider a regular foliation $(M, \mathcal{F})$. We prove Conjecture 2.13 in the regular case:

Proposition 2.18. *Let $(M, \mathcal{F})$ be a manifold with a regular foliation. Fix points x, y lying in the same leaf, and slices S_x, S_y there. Then the map $\Phi_x^y \colon H_x^y \to GermAut_{\mathcal{F}}(S_x, S_y)$ is injective.*

Proof. First notice that the target of the map Φ_x^y of Thm. 2.7 is really as above, due to Lemma 2.6.

Since Φ is a groupoid morphism by Cor. 2.8, it is enough to show that if $h \in U_x^x$ satisfies $\Phi(h) = Id_{S_x}$, then $h = 1_x$. Let U a bi-submersion in the path-holonomy atlas and $u \in U$ with $[u] = h$. There is a bisection $b \colon W \to U$ through u, defined on an open neighborhood $W \subset M$, which carries an orientation-preserving diffeomorphism ϕ (the bi-submersions U is a composition of path-holonomy bi-submersions, so just apply Lemma 2.16 to each of them). Since $\phi|_{S_x} = \Phi(h) = Id_{S_x}$, we deduce that that $\phi|_{(L \cap W) \times \{p\}}$ is orientation-preserving for all $p \in S_x$. By Lemma 2.17, applied smoothly on each (portion of) leaf $(L \cap W) \times \{p\}$, we have $\phi = exp(X)$ where $\{X_t\}_{t \in [0,1]}$ is a time-dependent vector field on W, tangent to the leaves and vanishing on the slice S_x. In particular, $X_t \in I_x\mathcal{F}$ for all t. Hence we showed that $u \in U$ carries a diffeomorphism ϕ that lies in $exp(I_x\mathcal{F})$.

By Lemma 2.14, the point $(x, 0) \in U$ also carries ϕ. Hence by [AnSk06, Cor. 2.11] there exists a morphism of bi-submersions $U \to U$ with $(x, 0) \mapsto u$, and therefore $h = [u] = [(x, 0)] = 1_x$. $\qquad\square$

Consider the holonomy groupoid of a regular foliation as defined by Winkelnkemper:

$$\mathcal{H} := \{\text{paths lying in leaves}\}/\text{holonomy}$$

where "holonomy" is meant in the sense of §2.1. In [AnSk06, Cor. 3.10] it was shown that H and $\mathcal{H}$ are canonically isomorphic. We end this section recovering this bijection in an alternative way.

Make a smooth choice of slices S_x at every $x \in M$. We denote by $GermAut(S, S)$ the (groupoid over M of) germs to automorphisms between such slices. $\mathcal{H}$ can be identified with the image of the map $hol \colon \{\text{paths lying in leaves}\} \to GermAut(S, S)$ introduced in §2.1, so we can regard $\mathcal{H}$ as a subset of $GermAut(S, S)$.

Corollary 2.19. $\Phi \colon H \to GermAut(S, S)$ *induces a bijection* $H \cong \mathcal{H}$.

Proof. To show $\mathcal{H} \subset \Phi(H)$, we let γ be a path in a leaf of $(M, \mathcal{F})$, and need to find $h \in H$ such that $\Phi(h) = hol(\gamma)$. Cover the image of γ by open subsets $W_1, \ldots, W_N$ such that on each W_α the foliation $\mathcal{F}$ is generated by vector fields $\{X_1^\alpha, \ldots, X_n^\alpha\}$. Denote by $U_\alpha \subset M \times \mathbb{R}^n$ the corresponding the path-holonomy bi-submersion. We may assume that γ is defined on the interval $[0, N]$ and that the velocity of γ equals X_1^α when $t \in [\alpha - 1, \alpha]$, for all $\alpha = 1, \ldots, N$. Consider the bi-submersion $U := U_N \circ \cdots \circ U_1$, and its point

$$u = (u_N, \ldots, u_1) \text{ where } u_\alpha := (\gamma(\alpha - 1), (1, 0, \ldots, 0)) \in U_\alpha, \forall \alpha = 1, \ldots, N.$$

Then $h := [u] \in H$ satisfies $\Phi(h) = hol(\gamma)$.

We show $\Phi(H) \subset \mathcal{H}$. Given an element $h \in H$, take a representative $u \in U$, where U is a bi-submersion in the path-holonomy atlas. Therefore $U = U_N \circ \cdots \circ U_1$ for path-holonomy bi-submersions U_α $(\alpha = 1, \ldots, N)$, and $u = (u_N, \ldots, u_1)$. Notice that $U_\alpha \subset M \times \mathbb{R}^{dim(\mathcal{F}_{\mathbf{s}(h)})}$, so we can rescale its elements by defining $t \cdot (z, \lambda) := (z, t\lambda)$ for all $t \in [0, 1]$ and $(z, \lambda) \in U_\alpha$. We now define a path from $\mathbf{s}(h)$ to $\mathbf{t}(h)$ lying in a leaf of the foliation, by

$$\gamma \colon [0, N] \to M, \quad \gamma(t) = \mathbf{t}_{U_\alpha}((t - \alpha - 1) \cdot u_\alpha) \text{ when } t \in [\alpha - 1, \alpha].$$

From the definition of Φ and hol it follows that $hol(\gamma) = \Phi(h)$, so $\Phi(h) \in \mathcal{H}$.

Hence Φ maps H surjectively onto $\mathcal{H}$. Since Φ is injective by Prop. 2.18, we obtain the desired bijection. $\qquad\square$

3 Linear Holonomy Transformations

In this section we consider two notions obtained differentiating, over a leaf L, the map $\Phi \colon H \to \{\text{holonomy transformations}\}$ defined in Thm. 2.7. In §3.1 we consider $\Phi|_{H_L}$ as a groupoid action and linearize it, obtaining a groupoid representation of H_L on NL (first order holonomy). In §3.2 we differentiate the latter to a representation of the Lie algebroid A_L on NL.

3.1 Lie groupoid representations by Linear Holonomy Transformations

Here we linearise the holonomy transformations arising in Theorem 2.7. The result, on every leaf L, can be phrased as a representation (linear groupoid action) of H_L on the vector bundle $NL := \cup_{x \in L} N_x L$. We also give a linear groupoid action of H_L on the Lie algebra bundle $\mathfrak{g}_L = \cup_{x \in L} \mathfrak{g}_x$ (*cf.* [AnZa11, §1.3]) which we interpret as the adjoint representation.

Proposition 3.1. *Let x, y be points of $(M, \mathcal{F})$ lying in the same leaf L.*

1) There is a canonical map

$$\Psi_x^y \colon H_x^y \to Iso(N_x L, N_y L).$$

defined as follows:

- *Given $h \in H_x^y$ take any bi-submersion $(U, \mathbf{t}, \mathbf{s})$ in the path-holonomy atlas with a point $u \in U$ satisfying $[u] = h$,*

- *define*
  $$\Psi_x^y(h) \colon N_x L \to N_y L, \ [v] \mapsto [\mathbf{t}_*(\tilde{v})]$$
  where $\tilde{v} \in T_u U$ is any $\mathbf{s}_*$-lift of $v \in T_x M$.

2) We have $\Psi_x^y(h) = d_x \tau$, where the diffeomorphism τ is chosen as in Theorem 2.7, for any choice of transversals S_x, S_y and using the canonical identifications $N_x L \cong T_x S_x$, $N_y L \cong T_y S_y$.

Proof. 1) Fixing $u \in U$ as above, the map $N_x L \to N_y L$, $[v] \mapsto [\mathbf{t}_*(\tilde{v})]$ is well-defined since $\mathbf{t}(\mathbf{s}^{-1}(x)) = L$ implies that $\mathbf{t}_*(Ker(d_u \mathbf{s})) = T_y L$. The map $\Psi_x^y(h)$ is independent of the choice of bi-submersion U and of u since, for any other choice U' and u', there exists by definition a morphism of bi-submersions $U \to U'$ mapping u to u'.

2) The diffeomorphism $\tau \colon S_x \to S_y$ can be described as follows: take any (local) $\mathbf{s}$-section $b \colon M \to U$ through u such that the local diffeomorphism $\mathbf{t} \circ b$ of M maps S_x into S_y, and restrict $\mathbf{t} \circ b$ to the slice S_x. Now choose $v \in T_x S_x$. Then $\tilde{v} := b_*(v) \in T_u U$ is an $\mathbf{s}_*$-lift of v, so $\mathbf{t}_*(\tilde{v}) = d_x \tau(v)$. $\square$

Remark 3.2. Prop. 3.1 2) says that $\Psi_x^y(h)$ can be regarded as the "derivative at x" of $\Phi_x^y \in \dfrac{GermAut_{\mathcal{F}}(S_x, S_y)}{exp(I_x \mathcal{F})|_{S_x}}$. (Even though $\Phi_x^y(h)$ is an equivalence class of maps, its derivative at x is well-defined by Lemma A.9.)

Corollary 3.3. *For every leaf L the map*

$$\boxed{\Psi_L \colon H_L \to Iso(NL, NL)}$$

obtained assembling the maps Ψ_x^y for all $x, y \in L$, is a groupoid morphism. When H_L is smooth, Ψ_L is a Lie groupoid morphism, i.e. a representation of H_L on the vector bundle NL.

Proof. We first show that Ψ_L is a morphism of set-theoretic groupoids. One way to deduce this is to use the fact that Φ is a groupoid morphism (Cor. 2.8) and then Prop. 3.1 2).

An alternative, direct proof is as follows. Let g, h be composable elements of H_L and $\xi \in T_{\mathbf{s}(h)}M$. Let u be a point of a bi-submersion $(U, \mathbf{t}^U, \mathbf{s}^U)$ (of the path-holonomy atlas) representing g, and similarly let $v \in V$ represent h. Then $gh \in H$ is represented by $(u, v) \in U \circ V$. We have

$$\Psi(gh)[\xi] = [\mathbf{t}_*^{U \circ V}(X)] = [\mathbf{t}_*^U(Y)]$$

where $X \in T_{(u,v)}(U \circ V)$ is a $\mathbf{s}_*^{U \circ V}$-lift of ξ, hence $X = (Y, Z)$ where Z is a $\mathbf{s}_*^V$-lift of ξ and $\mathbf{s}_*^U Y = \mathbf{t}_*^V Z$. On the other hand,

$$\Psi(g)(\Psi(h)[\xi]) = [\mathbf{t}_*^U(\text{a } \mathbf{s}_*^U\text{-lift of } \mathbf{t}_*^V Z)] = [\mathbf{t}_*^U(Y)].$$

Now assume that H_L is smooth. For the smoothness of the map Ψ_L we argue as follows. For every bi-submersion U, the following locally defined map is smooth: $U_L \times NL \to NL, (u, [v]) \mapsto [\mathbf{t}_*^U(\tilde{v})]$, where $\tilde{v} \in T_u U$ is any $\mathbf{s}_*$-lift of $v \in T_{\mathbf{s}(u)}M$. We conclude recalling that the differentiable structure on H_L is induced by the bi-submersions, see [AnZa11, §2.4]. $\square$

Remark 3.4. Notice [We, §4][CrStr11, §1.2] that given any Lie groupoid Γ over a manifold M_Γ and a leaf $L_\Gamma \subset M_\Gamma$, the restriction of Γ to the leaf acts[4] linearly on NL_Γ. However this can not used to obtain the action of H_L on NL described in Cor. 3.3, since H is usually not a Lie groupoid.

Examples 3.5. We calculate the linear holonomy at the origin for Examples 2.11, by taking the derivative at the origin of the holonomy transformations obtained there.

(i) Let $M = \mathbb{R}$ and $\mathcal{F}$ be generated by $X := z\partial_z$. Then $\Psi_0^0[(\lambda, 0)] \in Iso(T_0M, T_0M)$ is $e^\lambda Id_{T_0M}$.

(ii) Let $M = \mathbb{R}$ and $\mathcal{F}$ be generated by $X := z^2\partial_z$. Then $\Psi_0^0[(\lambda, 0)] \in Iso(T_0M, T_0M)$ is given by Id_{T_0M}.

(iii) We consider the S^1-action on $M = \mathbb{R}^2$ by rotation. Then $\Psi_0^0[(\lambda_0, 0)] \in Iso(\mathbb{R}^2, \mathbb{R}^2)$ is given by $\begin{pmatrix} cos(\lambda_0) & -sin(\lambda_0) \\ sin(\lambda_0) & cos(\lambda_0) \end{pmatrix}$.

3.1.1 Adjoint representations

The holonomy groupoid H also acts on the isotropy Lie algebras of the foliation. We show that when the essential isotropy group (see §1.3) of a leaf L is discrete , then the restriction to L of this action is the usual adjoint representation of the Lie groupoid H_L (Cor. 3.9).

The action of H on the isotropy Lie algebras is described in the next proposition, which will be proven in appendix A.1. Notice that when the foliation $\mathcal{F}$ is regular, we have $\mathfrak{g}_x = \{0\}$ for all $x \in M$, so the proposition is vacuous.

Proposition 3.6. *Let x, y belong to the same leaf of a singular foliation. There is a canonical, well-defined map*

$$\widehat{\Psi}_x^y \colon H_x^y \to Iso(\mathfrak{g}_x, \mathfrak{g}_y). \tag{3.1}$$

Under the identification $\mathfrak{g}_x \cong \mathcal{F}_{S_x}/I_x\mathcal{F}_{S_x}$ given by any slice S_x at x (see [AnZa11, Rem. 1.6]) and an analogue identification at y, the element $h \in H_x^y$ acts by

$$[Y] \in \mathcal{F}_{S_x}/I_x\mathcal{F}_{S_x} \mapsto [\tau_* Y] \in \mathcal{F}_{S_y}/I_y\mathcal{F}_{S_y}, \tag{3.2}$$

where the diffeomorphism $\tau \colon S_x \to S_y$ associated to h is chosen as in Theorem 2.7.

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[4]See eq. (4.3). The map in Prop. 3.1 1) is obtained adapting the definition of this action to the setting of bi-submersions.

Remark 3.7. Let L be a leaf of $(M, \mathcal{F})$ and S_x a transversal to this leaf at x. In §4.1 we introduce the linearisation of the transversal foliation $\mathcal{F}_{S_x}$; it is a foliation $\mathcal{F}_{lin}|_{N_x L}$ defined on the normal space $N_x L$. For any $h \in H$, writing $x = \mathbf{s}(h)$ and $y = \mathbf{t}(h)$, the isomorphism $\Psi_x^y \colon N_x L \to N_y L$ maps the linearised foliation $\mathcal{F}_{lin}|_{N_x L}$ to $\mathcal{F}_{lin}|_{N_y L}$. This follows from the fact that the diffeomorphisms $\tau \colon S_x \to S_y$ map the foliation $\mathcal{F}_{S_x}$ to $\mathcal{F}_{S_y}$. As we show in lemma 4.2, the Lie algebra $\mathfrak{g}_x$ carries more information than the foliation $\mathcal{F}_{lin}|_{N_x L}$.

For every leaf L, assembling the maps constructed in Prop. 3.6 we obtain a map $\widehat{\Psi}_L \colon H_L \to Iso(\mathfrak{g}_L)$. It is a groupoid morphism. (This follows from the fact that if τ_i is a diffeomorphism associated to h_i as in Theorem 2.7, $i = 1, 2$, then $\tau_1 \circ \tau_2$ is a diffeomorphism associated to $h_1 h_2$.)

Recall that given a transitive Lie groupoid Γ over a manifold L and points $x, y \in L$, there is a map $\Gamma_x^y \to Hom(\Gamma_x^x, \Gamma_y^y)$ given by conjugation: an element $h \in \Gamma_x^y$ is mapped to the homomorphism $I_h \colon \gamma \mapsto h\gamma h^{-1}$ (where $\gamma \in \Gamma_x^x$). Hence by differentiation we obtain a map

$$\Gamma_x^y \to Iso(T_x \Gamma_x^x, T_y \Gamma_y^y), \quad h \mapsto d_x(I_h).$$

We refer to the resulting representation of Γ on $\cup_{x \in L} T_x \Gamma_x^x$ as the *adjoint representation of* Γ. The next two statements show that the representation defined in Prop. 3.6 is equivalent to the adjoint representation of H_L, provided the leaf L satisfies regularity conditions.

Proposition 3.8. *Consider the connected and simply connected Lie group G_x integrating $\mathfrak{g}_x$. Let $E \colon \mathfrak{g}_x \to H_x^x$ the composition of the exponential map $exp \colon \mathfrak{g}_x \to G_x$ with the map $\varepsilon \colon G_x \to H_x^x$ discussed in §1.3. For every $h \in H_x^y$ put $I_h(\gamma) = h\gamma h^{-1}$. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g}_x & \xrightarrow{\widehat{\Psi}_x^y(h)} & \mathfrak{g}_y \\ \scriptstyle E \downarrow & & \downarrow \scriptstyle E \\ H_x^x & \xrightarrow{I_h} & H_y^y \end{array} \tag{3.3}$$

Corollary 3.9. *Let $(M, \mathcal{F})$ be a singular foliation and L a leaf whose essential isotropy is discrete. Fix $x, y \in L$. Then the Lie groupoid representation $\widehat{\Psi}_L \colon H_L \to Iso(\mathfrak{g}_L)$ is equivalent to the adjoint representation of H_L.*

Proof of Cor. 3.9. Recall that H_L is a Lie groupoid by Thm. 1.3. We use the notation of Prop. 3.8, with $x, y \in L$. By assumption, the map $\varepsilon \colon G_x \to H_x^x$ has discrete kernel, and further $d_0 exp = Id_{\mathfrak{g}_x}$. Hence $d_0 E \colon \mathfrak{g}_x \to T_x H_x^x$ is an isomorphism. Taking derivatives in diagram (3.3) we obtain a commuting diagram

$$\begin{array}{ccc} \mathfrak{g}_x & \xrightarrow{\widehat{\Psi}_x^y(h)} & \mathfrak{g}_y \\ \scriptstyle d_0 E \downarrow & & \downarrow \scriptstyle d_0 E \\ T_x H_x^x & \xrightarrow{d_x(I_h)} & T_y H_y^y. \end{array}$$

As this holds for all $h \in H_L$, we are done. $\qquad\square$

Until the end of this section, we turn to the proof of Prop. 3.8. It suffices to prove the commutativity of diagram (3.3) locally, namely in small neighborhoods $\tilde{\mathfrak{g}}_x, \tilde{\mathfrak{g}}_y$ of the origin. Indeed, any $w \in \tilde{\mathfrak{g}}_x$ can be written as nv for some natural number n and $v \in \tilde{\mathfrak{g}}_x$. We have $I_h E(nv) = (I_h E(v))^n$ since

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ε and I_h are group homomorphisms, and we have $[E\widehat{\Psi}_x^y(h)](nv) = ([E\widehat{\Psi}_x^y(h)](v))^n$ since $\widehat{\Psi}_x^y(h)$ is linear and ε is a group homomorphisms.

For the convenience of the reader, before the proof we recall some facts from §1.3 (see also [AnZa11, §3]) that will be used in the proof: let G_x be the connected and simply connected Lie group integrating $\mathfrak{g}_x$ and $\tilde{G}_x$ be a neighborhood of the identity where the exponential map $exp\colon \tilde{\mathfrak{g}}_x \to \tilde{G}_x$ is a diffeomorphism. Let $X_1, \ldots, X_n \in \mathcal{F}$ induce a basis of $\mathcal{F}_x$, let $(U, \mathbf{t}, \mathbf{s})$ be the corresponding path holonomy bi-submersion at the point x, minimal at $(x, 0)$, and let $Y_1, \ldots, Y_n \in C^\infty(U, \ker d\mathbf{s})$ such that $d\mathbf{t}(Y_i) = X_i$ for every $1 \le i \le n$.

Recall that ε is an extension of the map $\tilde{\varepsilon}\colon \tilde{G}_x \to H_x^x$ defined as $\tilde{\varepsilon} = \sharp \circ \Delta$, where $\Delta\colon \tilde{G}_x \to U_x^x$ is a diffeomorphism and $\sharp\colon U_x^x \to H_x^x$ is the quotient map. Hence the restriction of $E = \varepsilon \circ exp$ to the neighborhood $\tilde{\mathfrak{g}}_x$ is given explicitly by

$$E|_{\tilde{\mathfrak{g}}_x}\colon \tilde{\mathfrak{g}}_x \to H_x^x, \quad \sum_{i=1}^l k_i[X_i] \mapsto \sharp(exp_{(x,0)}(\sum_{i=1}^l k_i Y_i)),$$

where the $\{X_i\}$ are chosen so that the first l of them vanish at x, for $l = dim(\mathfrak{g}_x)$.

Proof of Prop. 3.8. Let $h \in H_x^y$ be the class of some w in a bi-submersion $(W, \mathbf{t}_W, \mathbf{s}_W)$, so that $\mathbf{s}_W(w) = x$, $\mathbf{t}_W(w) = y$. Let S_x, S_y be slices, $\tau\colon S_x \to S_y$ be a diffeomorphism chosen as in Theorem 2.7 using the bi-submersion W. Identifying $\mathfrak{g}_p$ with $\mathcal{F}_{S_p}/I_p\mathcal{F}_{S_p}$ as in [AnZa11, Rem. 1.6] ($p \in \{x, y\}$), the map $\widehat{\Psi}_x^y(h)\colon \mathfrak{g}_x \to \mathfrak{g}_y$ is given by $[Y] \mapsto [\tau_*(Y)]$ (see Prop. 3.6). Let ϕ be a local diffeomorphism extending τ arising from a bisection b_W of $(W, \mathbf{t}_W, \mathbf{s}_W)$ through w.

Choose vector fields $X_1, \ldots, X_n \in \mathcal{F}$ inducing a basis of $\mathcal{F}_x$ so that the first l of them are tangent to S_x, and let $(U, \mathbf{t}, \mathbf{s})$ be the path-holonomy bi-submersion they define. Choose lifts $Y_1, \ldots, Y_n \in C^\infty(U(x); \ker d\mathbf{s})$ via $\mathbf{t}$ of $X_1, \ldots, X_n$ respectively. Put $X_i' = \phi_* X_i \in \mathcal{F}$, $i = 1, \ldots, n$. Then $X_1', \ldots, X_n'$ induce a basis of $\mathcal{F}_y$. Let $(U', \mathbf{t}', \mathbf{s}')$ be the path-holonomy bi-submersion they define and choose lifts $Y_1', \ldots, Y_n' \in C^\infty(U; \ker d\mathbf{s}')$ via $\mathbf{t}'$ likewise.

Now pick an element $v = \sum_{i=1}^l k_i[X_i]$ in $\widetilde{\mathfrak{g}}_x$. We must show that $\left(E \circ \widehat{\Psi}_x^y(h) \right)(v) = (I_h \circ E)(v)$. The left-hand side of this expression, since $\widehat{\Psi}_x^y(h)[X_i] = [X_i']$, is the class of the element

$$v_1 := exp_{(y,0)}(\sum_{i=1}^l k_i Y_i') \in U'.$$

The right-hand side $I_h(\sharp(exp_{(x,0)}(\sum_{i=1}^l k_i Y_i)))$ is the class of

$$v_2 := (w, exp_{(x,0)}(\sum_{i=1}^l k_i Y_i), w) \ \in \ W \circ U \circ \overline{W},$$

where $\overline{W}$ is the inverse bi-submersion of $(W, \mathbf{t}_W, \mathbf{s}_W)$. We will show that the elements v_1 and v_2 carry the same local diffeomorphism, whence by [AnSk06, Cor. 2.11] they quotient to the same element in H_y^y.

To this end, let $\mathcal{O}_x$ be an open neighborhood of x in M and consider the bisection $b := \{exp_{(m,0)}(\sum_{i=1}^{l} k_i Y_i) \colon m \in \mathcal{O}_x\}$ of $(U, \mathbf{t}, \mathbf{s})$ at the point $exp_{(x,0)}(\sum_{i=1}^{l} k_i Y_i)$. Then the bisection $b_W \times b \times b_W$ of $W \circ U \circ \overline{W}$ contains v_2 and corresponds to the local diffeomorphism of $\mathcal{O}_y := \phi(\mathcal{O}_x)$ given by

$$z \mapsto \phi(exp_{\phi^{-1}(z)} \sum_{i=1}^{l} k_i X_i) = exp_z \sum_{i=1}^{l} k_i \phi_* X_i.$$

On the other hand, the bisection $b' := \{exp_{(z,0)}(\sum_{i=1}^{l} k_i Y_i') \colon z \in \mathcal{O}_y\}$ contains v_1 and carries the same local diffeomorphism, since $\phi_* X_i = X_i'$. Therefore the elements v_1 and v_2 induce the same element in H_y^y. $\qquad\square$

3.2 Lie algebroid representations by Linear Holonomy Transformations

Here we differentiate the previous holonomy transformations to representations of the transitive Lie algebroid A_L over a leaf L. For the sake of simplicity of exposition, we consider only the case of *embedded* leaves, even though the results hold for arbitrary immersed leaves.

We consider the $C^\infty(M)$-module $\mathcal{N} = \mathfrak{X}(M)/\mathcal{F}$ (see Appendix A.3 for more details about $\mathcal{N}$). Notice that the Lie bracket of vector fields descends to a map $\mathcal{F} \times \mathcal{N} \to \mathcal{N}$.

Now fix an embedded leaf L. We consider the Lie algebroid $A_L \to L$ introduced in §1.1, whose sections are given by $\mathcal{F}/I_L\mathcal{F}$, as well as the normal bundle $NL = \dfrac{T_L M}{TL} \to L$, whose space of sections is $\mathcal{N}/I_L\mathcal{N}$ (see Lemma A.11). The Lie bracket induces the *Bott connection on NL*

$$\nabla^{L,\perp} \colon C^\infty(L; A_L) \times C^\infty(L; NL) \to C^\infty(L; NL), \qquad \nabla_{\langle X \rangle}^{L,\perp} \langle Y \rangle = \langle [X, Y] \rangle.$$

The map $\nabla^{L,\perp}$ is a Lie algebroid representation of A_L on NL, that is, a flat Lie algebroid connection, and hence can equivalently be regarded as a Lie algebroid morphism

$$\boxed{\nabla^{L,\perp} \colon A_L \to Der(NL)}$$

Here $Der(NL)$ denotes the Lie algebroid over L whose sections are given by $CDO(NL)$, the first order differential operators $D \colon C^\infty(L; NL) \to C^\infty(L; NL)$ such that there exists a vector field $\sigma_D \in \mathfrak{X}(M)$ with $D(fX) = fD(X) + \sigma_D(X)(f)X$ (*cf.* [Ma05]).

Examples 3.10. Restricting the Lie algebroid representation $\nabla^{L,\perp}$ to the isotropy Lie algebra $\mathfrak{g}_x$ at some point $x \in L$ we obtain a Lie algebra representation of $\mathfrak{g}_x$ on the vector space $N_x L$. Explicitly, $[X] \in \mathfrak{g}_x \subset \mathcal{F}/I_x\mathcal{F}$ sends $\langle Y \rangle \in N_x L = \mathfrak{X}(M)/(\mathcal{F} + I_x\mathfrak{X}(M))$ to $\langle [X, Y] \rangle \in N_x L$. We present 3 examples of this Lie algebra representation. In all 3 cases $L = \{0\}$.

   (i) As in Ex. 3.5, consider $M = \mathbb{R}$ and $\mathcal{F}$ generated by $x\partial_x$. Then $\mathfrak{g}_0$ is the one-dimensional Lie algebra with basis $[x\partial_x]$ and $N_0 L = T_0 M$. Since $\mathcal{L}_{x\partial_x}\partial_x = -\partial_x$ we conclude that the element $[x\partial_x] \in \mathfrak{g}_0$ acts by $-Id_{N_0 L}$.

(ii) As in Ex. 3.5, we consider the foliation $\mathcal{F}$ generated by the vector field $x\partial_y - y\partial_x$. Then $\mathfrak{g}_0$ is the one-dimensional Lie algebra with basis $[x\partial_y - y\partial_x]$ and $N_0L = T_0M$ has basis $\{\partial_x|_0, \partial_y|_0\}$. The Lie algebra element $[x\partial_y - y\partial_x] \in \mathfrak{g}_0$ acts on N_0L by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(iii) Consider $M = \mathbb{R}^2$ and $\mathcal{F}$ generated by the Euler vector field $x\partial_x + y\partial_y$. Then $\mathfrak{g}_0$ is the one-dimensional Lie algebra with basis $[x\partial_x + y\partial_y]$ and $N_0L = T_0M$. The Lie algebra element $[x\partial_x + y\partial_y]$ acts by $-Id|_{N_0L}$.

We show that, under regularity conditions on the leaf L, the Lie groupoid representation Ψ_L of H_L on NL (see Cor. 3.3) differentiates to the Lie algebroid representation $\nabla^{L,\perp}$ of A_L. Before that, we show that in the case of regular foliations this holds not just over individual leaves but over the whole of M.

Recall that if $B \to M$ is a vector bundle, then $Iso(B)$[5] is the transitive Lie groupoid consisting of linear isomorphisms between fibers of $B \to M$.

Lemma 3.11. *Let $\mathcal{F}$ be a regular foliation on M, so $\mathcal{F} = C^\infty(M; F)$ for an involutive distribution F. Assume that all leaves of $\mathcal{F}$ be embedded. Let $N := \cup_L NL \to M$ where L ranges over all leaves, so that $\mathcal{N} = C^\infty(M; N)$.*

Then the Lie groupoid representation $\Psi \colon H \to Iso(N)$ (see Cor. 3.3) differentiates to the Lie algebroid representation $F \to Der(N)$ which, at the level of sections, is given by the map $\mathcal{F} \times \mathcal{N} \to \mathcal{N}$ induced by the Lie bracket.

Proof. We will work directly with sections of the Lie algebroids F and $Der(N)$ and with **s**-sections of the corresponding Lie groupoids H and $Iso(N)$. Given $X \in \mathcal{F}$, we construct a path $\gamma(\epsilon)$ of sections of the source map of H with $\dot{\gamma}(0) = X$ (recall that $\mathcal{F}$ is the space of sections of the Lie algebroid F of H). Specifically, for each ϵ, let $\gamma(\epsilon) \colon M \to H$ be defined as follows: $(\gamma(\epsilon))(x)$ is the holonomy class of the curve $[0,1] \to M, t \mapsto \phi_X^{t\epsilon}(x)$, where ϕ_X^t is the time-t flow of X. The map Φ of Thm. 2.7 is given by holonomy along paths (see Cor. 2.19), so $\Phi(\gamma(\epsilon))$ is the restriction of ϕ_X^ϵ to suitably chosen slices.

Hence $\Psi(\gamma(\epsilon))$ is the vector bundle isomorphism of N given by $(\phi_X^\epsilon)_*$ (more precisely, the map it induces on $N = TM/F$). Therefore $\Psi_*(X) = \frac{d}{d\epsilon}|_{\epsilon=0}(\Psi(\gamma(-\epsilon)))$ is the first order differential operator which acts on $\langle Z \rangle \in C^\infty(M; N)$ (the class of $Z \in \mathfrak{X}(M)$) by

$$\langle Z \rangle \mapsto \frac{d}{d\epsilon}|_{\epsilon=0}(\phi_X^{-\epsilon})_*\langle Z \rangle = \langle \mathcal{L}_X Z \rangle = \langle [X,Z] \rangle.$$

$\square$

Proposition 3.12. *Let $(M, \mathcal{F})$ be a singular foliation and L an embedded leaf whose essential isotropy is discrete, so that by Thm. 1.3 the Lie groupoid H_L integrates the Lie algebroid A_L.*

Then the Lie groupoid representation $\Psi_L \colon H_L \to Iso(NL)$ of Cor. 3.3 differentiates to the Lie algebroid representation $\nabla^{L,\perp} \colon A_L \to Der(NL)$.

Proof. Given $\tilde{X} \in \mathcal{F}/I_L\mathcal{F} = C^\infty(L; A_L)$ defined in a neighborhood of $x \in L$, we construct a path $\gamma(\epsilon)$ of sections of the source map of H_L with $\dot{\gamma}(0) = \tilde{X}$. We may assume that $(\tilde{X} \mod I_x\mathcal{F}) \in \mathcal{F}_x$ is non-zero, for otherwise it acts trivially. Choose a lift $X \in \mathcal{F}$. Starting from a basis of $\mathcal{F}_x$,

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[5]This is called *frame groupoid* in [Ma05, Ex. 1.1.12].

we can construct generators $\{X_i\}_{i \le n}$ of $\mathcal{F}$ in a neighborhood M_0, with $X_1 = X$. Denote by U the corresponding path-holonomy bi-submersion, and fix vertical lifts $\{Y_i\}_{i \le n}$ of the X_i w.r.t. the target map $\mathbf{t}$. For every ϵ sufficiently close to zero, the map

$$\hat{\gamma}(\epsilon) \colon M_0 \to U, y \mapsto exp_{(y,0)}(\epsilon Y_1)$$

is a bisection of U carrying ϕ_X^ϵ, the time-ϵ flow of X. The path $\epsilon \mapsto \gamma(\epsilon) := \sharp(\hat{\gamma}(\epsilon))|_{M_0 \cap L}$ is a path of sections of the source map of H_L with $\dot{\gamma}(0) = \tilde{X}$.

Using Prop. 3.1 2) we obtain that $\Psi_L(\gamma(\epsilon))$ is the vector bundle isomorphism of NL given by $(\phi_X^\epsilon)_*$ (or rather the map it induces on NL). Therefore $(\Psi_L)_*(\tilde{X}) = \dfrac{d}{d\epsilon}|_{\epsilon=0}(\Psi_L(\gamma(-\epsilon)))$ is the first order differential operator which acts on $\langle Z \rangle \in C^\infty(L; NL) = \mathfrak{X}(M)/(\mathcal{F} + I_L \mathfrak{X}(M))$ (see Lemma A.11) by

$$\langle Z \rangle \mapsto \frac{d}{d\epsilon}|_{\epsilon=0}(\phi_X^{-\epsilon})_* \langle Z \rangle = \langle \mathcal{L}_X Z \rangle = \langle [X, Z] \rangle = \nabla_{\tilde{X}}^{L, \perp} \langle Z \rangle.$$

$\square$

Finally, recall that every transitive Lie algebroid acts by the bracket on its isotropy bundle (the kernel of the anchor map), see [Ma05, Ex. 3.3.15]. In the case of A_L this representation is

$$\widehat{\nabla} \colon A_L \to Der(\mathfrak{g}_L), \quad \widehat{\nabla}_X(V) = [X, V]_{A_L},$$

where $\mathfrak{g}_L$ denotes the isotropy bundle of A_L (a bundle of Lie algebras whose fiber over $x \in L$ is canonically isomorphic to $\mathfrak{g}_x$). By $Der(\mathfrak{g}_L)$ we denote the Lie algebroid over L whose sections are covariant differential operators on the vector bundle $\mathfrak{g}_L$ which are derivations of the bracket on the fibres of $\mathfrak{g}_L$.

Assume that the essential isotropy group of the embedded leaf L is discrete. Then the Lie groupoid representation $\widehat{\Psi}_L \colon H_L \to Iso(\mathfrak{g}_L)$ (see Prop. 3.6) differentiates to the representation $\widehat{\nabla} \colon A_L \to Der(\mathfrak{g}_L)$. This follows immediately from Cor. 3.9 and [Ma05, Prop. 3.7.4].

4 The linearized foliation near a leaf

In this section, for any leaf L, we show that there is well-defined notion of "linearization of $\mathcal{F}$ at L." In §4.1 we realize the linearized foliation as a foliation on NL, and show that it is induced by a Lie groupoid action, provided L is embedded and has discrete essential isotropy group. In §4.2 we give an alternative description of the linearized foliation as a foliation on $(H_x \times N_x L)/H_x^x$, under the same assumptions. In these two subsections L is taken to be embedded (and not immersed) just to simplify the exposition. Finally, in §4.3, we make some comments on the linearization problem: under what conditions is $\mathcal{F}$ isomorphic to its linearization nearby a leaf? Notice that an answer to this question would constitute a version of the Reeb Stability Theorem for singular foliations. We also briefly discuss the relation to singular Riemannian foliations.

4.1 The linearized foliation on the normal bundle

Let L be an embedded leaf of the singular foliation $(M, \mathcal{F})$. There is a canonical identification

$$I_L/I_L^2 \cong \Gamma(N^*L) = C_{lin}^\infty(NL), \ [f] \mapsto df|_L,$$

where I_L denotes the functions on M that vanish on L, $C^\infty_{lin}(NL)$ the fiberwise linear functions on the normal bundle, and $[f] := (f \mod I_L^2)$ for $f \in I_L$.

Given a vector field Y on M tangent to L, we denote by Y_{lin} the vector field on NL which acts as $Y|_L$ on the fiberwise constant functions, and as follows on $C^\infty_{lin}(NL) \cong I_L/I_L^2$: for all $f \in I_L$,

$$Y_{lin}[f] := [Y(f)].$$

We obtain a bracket-preserving assignment

$$lin \colon \mathcal{F} \to \chi_{lin}(NL), \quad Y \mapsto Y_{lin},$$

where $\chi_{lin}(NL)$ denotes the vector fields on NL which preserve the fiberwise constant functions and $C^\infty_{lin}(NL)$. Notice that lin factors to a map $\mathcal{F}/I_L\mathcal{F} \to \chi_{lin}(NL)$.

Definition 4.1. The *linearization of $\mathcal{F}$* is the foliation $\mathcal{F}_{lin}$ on NL generated (as a $C^\infty(NL)$-module) by $\{Y_{lin} : Y \in \mathcal{F}\}$.

Remark 4.2. The linearization procedure, obviously, is not injective. This is already apparent if one considers a fiber $N_x L$ of the normal bundle. More precisely, by Lemma 4.3 it is clear that restricting the map (obtained factoring) lin we obtain a canonical surjective Lie algebra morphism $\mathfrak{g}_x \to \mathcal{F}_{lin}|_{N_x L} := \{Z|_{N_x L} : Z \in \mathcal{F}_{lin} \text{ is tangent to } N_x L\}$. Explicitly, it is given by

$$\mathfrak{g}_x \to \mathcal{F}_{lin}|_{N_x L}, \ \langle Y \rangle \mapsto (Y_{lin})|_{N_x L}.$$

This map is usually not injective (take for instance $(M, \mathcal{F}) = (\mathbb{R}, x^2 \partial_x)$ and $x = 0$).

Recall that we defined the Lie algebroid morphism $\nabla^{L,\perp} \colon A_L \to Der(NL)$ in §3.2. At the level of section it induces $\nabla^{L,\perp} \colon \mathcal{F}/I_L\mathcal{F} \to CDO(NL)$, which is essentially given by the Lie bracket of vector fields.

Lemma 4.3. *The following diagram commutes:*

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\ lin\ } & \chi_{lin}(NL) \\
\downarrow & & \downarrow{\scriptstyle \cong} \\
\mathcal{F}/I_L\mathcal{F} & \xrightarrow{\nabla^{L,\perp}} & CDO(NL).
\end{array}
$$

Here the right arrow is the Lie algebra isomorphism given by

$$\chi_{lin}(NL) \cong CDO(NL), Y \mapsto (a \mapsto [Y, a]_{\chi(NL)}), \tag{4.1}$$

where $a \in \Gamma(NL)$ is also interpreted as a vertical (constant) vector field on NL.

Proof. Let $Y \in \mathcal{F}$. We have to check that $[Y_{lin}, \cdot]_{\chi(NL)} = [Y, \cdot]|_L \mod TL$ when acting on elements of $\Gamma(NL)$, i.e. that

$$[Y_{lin}, Z_{ver}]_{\chi(NL)} = ([Y, Z])_{ver} \tag{4.2}$$

for all $Z \in \chi(M)$, where $Z_{ver} := (Z|_L \mod TL) \in \Gamma(NL)$.

To this aim, take $f \in I_L$. For the r.h.s. we have

$$([Y, Z])_{ver}[f] = [Y, Z](f)|_L = \big(Y(Z(f)) - Z(Y(f))\big)|_L.$$

For the l.h.s.

$$[Y_{lin}, Z_{ver}]_{\chi(NL)}[f] = Y_{lin}(Z_{ver}[f]) - Z_{ver}(Y_{lin}[f])$$
$$= Y_{lin}(Z(f)|_L) - Z_{ver}[Y(f)]$$
$$= (Y(Z(f)))|_L - (Z(Y(f)))|_L$$

where we used $Z_{ver}[f] = (Z(f))|_L \in C^\infty(L)$, $Y_{lin}[f] = [Y(f)] \in C_{lin}(NL)$ and $Y_{lin}(g|_L) = Y(g)|_L \in C^\infty(L)$ (for $g := Z(f)$).

Both sides of eq. (4.2) are vertical (constant) vector fields on NL, and as we just showed that their action on fiberwise linear functions agree, we conclude that they are equal. $\qquad\square$

From Lemma 4.3 we obtain immediately:

Corollary 4.4. *The subset $\{Y_{lin} : Y \in \mathcal{F}\}$ of $\chi_{lin}(NL)$ agrees with the image of $\nabla^{L,\perp} \colon \mathcal{F}/I_L\mathcal{F} \to CDO(NL)$, under the identification (4.1).*

Given an action of a Lie groupoid $G \rightrightarrows M$ on a map $\pi \colon N \to M$, the *induced foliation* on N is the one generated (as a $C^\infty(N)$-module) by the image of the corresponding infinitesimal action $\Gamma(A) \to \chi(N)$, where A is the Lie algebroid of G.

Corollary 4.5. *Let $(M, \mathcal{F})$ be a singular foliation and L an embedded leaf whose essential isotropy is discrete. Then the linearized foliation $\mathcal{F}_{lin}$ is the foliation induced by the Lie groupoid action Ψ_L of H_L on NL (see Cor. 3.3).*

Proof. By Prop. 3.12 the above Lie groupoid action differentiates to the Lie algebroid action of A_L on NL which, under the identification (4.1), is given by $\nabla^{L,\perp}$. Now use Cor. 4.4. $\qquad\square$

4.2 The linearized foliation on $(H_x \times N_x L)/H_x^x$.

We give an alternative description of the linearization of $\mathcal{F}$ at L defined in Def. 4.1, in the case that L is an embedded leaf whose essential isotropy group is discrete. Fix a point x in L. Since H_L is a Lie groupoid (Thm. 1.3), $H_x \to L$ is a principal H_x^x-bundle. There is a linear action of H_x^x on $N_x L$ given by Ψ (see Prop. 3.1). We consider the associated vector bundle, that is,

$$Q := \frac{H_x \times N_x L}{H_x^x},$$

where the action of H_x^x on $H_x \times N_x L$ is given by $g \cdot (h, \xi) = (hg^{-1}, \Psi(g)\xi)$.

Lemma 4.6. *There is a canonical diffeomorphism*

$$\Upsilon \colon Q \to NL, \quad [(h, \xi)] \mapsto \Psi(h)\xi.$$

Proof. Υ is well-defined, since for all $g \in H_x^x$ we have $\Psi(hg^{-1})(\Psi(g)\xi) = \Psi(h)\xi$ as a consequence of the fact that Ψ is a groupoid morphism, see Cor. 3.3. Υ is surjective: given $\eta \in N_y L$, pick any $h \in H_x^y$ and define $\xi := (\Psi(h))^{-1}\eta$. Then $\Upsilon[(h, \xi)] = \eta$. Using the fact that Ψ is a groupoid morphism, it is straightforward to check that Υ is injective. $\qquad\square$

There is a map
$$\Gamma(A_L) = \mathcal{F}/I_L\mathcal{F} \to \chi(Q)$$
defined as follows. Given $Y \in \Gamma(A_L) = \Gamma(ker(\mathbf{s}_*|_M))$, consider the vector field $(R_*Y, 0)$ on $H_x \times N_x L$, defined to be $((R_h)_*Y, 0_\xi)$ at the point (h, ξ). This vector field is H_x^x-invariant, and hence projects to a vector field on Q, which we denote by $[(R_*Y, 0)]$.

Proposition 4.7. *Let L be an embedded leaf of $(M, \mathcal{F})$ whose essential isotropy group is discrete. Under the canonical diffeomorphism Υ:*

1) the action Ψ_L of H_L on NL corresponds to the action of H_L on Q induced by left groupoid multiplication on H_x

2) $\{Y_{lin} : Y \in \mathcal{F}\}$ on NL corresponds to $\{[(R_*Y, 0)] : Y \in \Gamma(A_L)\}$ on Q.

Proof. 1) follows from the definition of Υ.

2) follows from 1), from Cor. 4.5, and from the fact that the infinitesimal generators of left groupoid multiplication are the right-invariant vector fields. □

Remark 4.8. Consider the case when the foliation $\mathcal{F}$ is regular. Then the leaves of the (regular) foliation $\mathcal{F}_{lin}$ on NL are given by pasting together the flat sections of the flat connection $\nabla^{L, \perp}$ (see Prop. 4.5). Further, the leaves of the above (regular) foliation on Q are obtained projecting the foliation on $H_x \times N_x L$ by horizontal copies of H_x.

4.3 Remarks on the linearization problem

Consider a singular foliation $(M, \mathcal{F})$ and an embedded leaf L.

Definition 4.9. We say that $\mathcal{F}$ *is linearizable about L* if there exists a tubular neighborhoods $U \subset M$ of L and $V \subset NL$ of the zero section, and a diffeomorphism $U \to V$ mapping L to the zero section inducing an isomorphism (of $C^\infty(U)$-modules) between the foliation $\mathcal{F}|_U$ and $\mathcal{F}_{lin}|_V$, where $\mathcal{F}_{lin}$ is the linearized foliation on NL.

Notice that it does not make sense to extend Def. 4.9 to immersed leaves L, as the zero section is always an embedded submanifold of NL.

Remark 4.10. Let $\mathcal{F}$ be generated by just one vector field X vanishing at a point x. Then $\mathcal{F}$ is linearizable about $\{x\}$ iff there exists a smooth, nowhere vanishing function f defined on $V \subset T_x M$ and a diffeomorphism $U \to V$ that maps X to $f \cdot X_{lin}$. In particular, the linearizability of the foliation $\mathcal{F}$ about $\{x\}$ is a weaker condition than the linearizability of the vector field X about x.

Remark 4.11. It is known that not every singular foliation comes from a Lie algebroid [AnZa11, Prop. 1.3]. (The local version of the question is still open.) However if an embedded leaf L of $\mathcal{F}$ has discrete essential isotropy group and $\mathcal{F}$ is linearizable about L, then $\mathcal{F}|_U$ comes from a Lie algebroid, where U is some tubular neighbourhood of L. This follows from Cor. 4.5. A refinement of this statement will be given in Prop. 4.12 (see also Remark 4.13 a).

Recall that a Lie groupoid $G \rightrightarrows U$ is *proper* if its target-source map $(\mathbf{t}, \mathbf{s}) \colon G \to U \times U$ is proper (i.e., preimages of compact sets are compact). Properness imples that G is *proper at y*, for all $y \in U$ (see [CrStr11, §1]).

Proposition 4.12. *Let $(M, \mathcal{F})$ be a singular foliation, and x a point of an embedded leaf L whose essential isotropy is discrete. The following are equivalent:*

a) *$\mathcal{F}$ is linearizable about L and H_x^x is compact*

b) *there exists a tubular neighborhood U of L and a Lie groupoid $G \rightrightarrows U$, proper at x, inducing the foliation $\mathcal{F}|_U$.*

Proof. $a) \Rightarrow b)$. By Cor. 4.5, the linearized foliation $\mathcal{F}_{lin}$ is induced by a Lie groupoid action of H_L on NL. In other words, $\mathcal{F}_{lin}$ is the foliation induced by the transformation groupoid $H_L \ltimes NL \rightrightarrows NL$. This transformation groupoid is proper by Lemma 4.14 below (applied to H_L), so in particular it is proper at $x \in L \subset NL$. Now use the assumption that $\mathcal{F}$ is linearizable.

$b) \Rightarrow a)$. The linearization theorem for proper groupoids, in the version of Crainic-Struchiner [CrStr11, Thm. 1], implies that there is an isomorphism from G to $G_{lin}|_V$, where we shrink U if necessary and V is a neighborhood of the zero section of NL. Here the Lie groupoid $G_{lin} \rightrightarrows NL$ is the linearization of G about L, which induces the foliation $\mathcal{F}_{lin}$ on NL by Lemma 4.15 below. In particular, the base map of the above isomorphism is a diffeomorphism $U \to V$ that sends $\mathcal{F}|_U$ to $\mathcal{F}_{lin}|_V$, hence $\mathcal{F}$ is linearizable about L. Further, since G is proper at x, its isotropy group G_x^x is compact. Hence H_x^x, which is a quotient of G_x^x due to [AnSk06, Ex. 3.4(4)], is also compact. $\qquad\square$

Remarks 4.13. a) In general, a foliation linearizable about a leaf L does not come from a Lie groupoid proper at x (where $x \in L$). For instance, the foliation on $\mathbb{R}^2$ induced by the action of $SL(2, \mathbb{R})$ on $\mathbb{R}^2$ (see [AnSk06], [AnZa11]) is linearizable around zero (as the action is linear). However this foliation does not arise from a groupoid G proper at 0: if it did, H_0^0 would be compact (being a quotient of G_0^0), contradicting the non-compactness of $H_0^0 = SL(2, \mathbb{R})$. The same applies, of course, if we replace $SL(2, \mathbb{R})$ by $GL(2, \mathbb{R})$, in fact for any foliation defined by a linear action of a non-compact group (see [AnZa11, Ex. 4.6]).

The situation is different if one assumes the set-up of Prop. 4.12. More precisely: under the assumptions that the embedded leaf L has discrete essential isotropy group and H_x^x is compact, the foliation is linearizable at L iff it comes from a Lie groupoid proper at x.

b) Obtaining linearization conditions for an arbitrary singular foliation is a problem of different order, which deserves to be addressed in a separate article. A way to approach such linearization results is by understanding certain averaging processes. However, averaging here is impossible because, unlike the regular case, the holonomy groupoid of a singular foliation not only has pathological topology in general, but its source-fibers (when smooth) have jumping dimensions. A way out of this would be to work on the level of bi-submersions instead; but there the notion of properness which allows averaging becomes irrelevant: We may have foliations arising from a proper Lie groupoid whose path holonomy bi-submersions are not proper in any sense.

c) Notice that a linearization theorem for singular foliations would extend the Reeb stability theorem for regular foliations (*cf.* [MoMr03, Thm. 2.9]). Instances of the linearization theorem in which additional structure is present were recently established: for the symplectic foliation of a Poisson manifold by Crainic-Marcuţ in [CrMa10], and for orbits of a proper Lie groupoid by Crainic-Struchiner in [CrStr11].

In fact, in view of the previous remarks, it seems reasonable to think that the compactness of H_x^x might be one of the sufficient conditions for the singular version of the Reeb stability theorem.

We end with two Lemmas needed in the proof of Prop. 4.12.

Lemma 4.14. *Let $\Gamma \rightrightarrows L$ be a transitive Lie groupoid such that for some $x \in L$ the isotropy group Γ_x^x is compact. Then, for any action of Γ on a surjective submersion $\pi \colon N \to L$, the corresponding transformation groupoid $\Gamma \ltimes N \rightrightarrows N$ is proper.*

Proof. First we point out that Γ is a proper groupoid. Indeed Γ is isomorphic to the gauge groupoid for the principal Γ_x^x-bundle $\Gamma_x \to L$, whose fiber is compact, so that the target-source map for the gauge groupoid is proper.

Given any compact subsets $K_1, K_2 \subset N$, consider the preimage $\psi^{-1}(K_1 \times K_2)$ under the target-source map of the transformation groupoid, that is, under $\psi \colon (\Gamma \ltimes N) \to N \times N, (g, \xi) \mapsto (g\xi, \xi)$. Clearly

$$\psi^{-1}(K_1 \times K_2) = \{(g, \xi) \in \Gamma \ltimes N : g\xi \in K_1, \xi \in K_2\}$$

is contained in $(\mathbf{t}^{-1}(\pi(K_1)) \cap \mathbf{s}^{-1}(\pi(K_2))) \times K_2$, which is compact since Γ is a proper groupoid. (Here $\mathbf{t}, \mathbf{s}$ are the target and source maps of Γ.) Hence $\psi^{-1}(K_1 \times K_2)$, being a closed subset of a compact one, is compact. $\qquad\square$

Let $G \rightrightarrows M$ be a Lie groupoid. Let L be a leaf, and denote by G_L the restriction of G to L. As we pointed out in Rem. 3.4, there is an action of G_L on the normal bundle $NL = TM|_L/TL$. Given a bisection $\gamma \colon L \to G_L$ (viewed as a section of the source map of G_L), its action on NL is given by the bundle automorphism

$$(\mathbf{t} \circ b)_*|_L, \tag{4.3}$$

where $b \colon M \to G$ is any bisection of G extending γ. The *linearization of G about L*, denoted by G_{lin}, is the transformation groupoid $G_L \ltimes NL \rightrightarrows NL$ of this action [CrStr11, §1.2]. The following lemma states that the foliation of the linearized groupoid is the linearization of the groupoid foliation.

Lemma 4.15. *Let $G \rightrightarrows M$ be a Lie groupoid, and denote by $\mathcal{F}$ the induced foliation. Let L be an embedded leaf. Denote by G_{lin} the linearization of G about L, and denote by $\mathcal{F}(G_{lin})$ the foliation on NL it induces. Then $\mathcal{F}_{lin} = \mathcal{F}(G_{lin})$.*

Proof. First observe that, for any family ϕ_ϵ of diffeomorphisms of M preserving L with $\phi_0 = Id_M$, the linearization of the vector field $\frac{d}{d\epsilon}|_0\phi_\epsilon$ is obtained applying $\frac{d}{d\epsilon}|_0$ to the linearization of the ϕ_ϵ's, i.e. to $(\phi_\epsilon)_*|_L$ (seen as an automorphism of NL). In particular, given a family of bisections $b_\epsilon \colon M \to G$ with $b_0 = Id_M$, the linearization of the vector field $\frac{d}{d\epsilon}|_0(\mathbf{t} \circ b_\epsilon)$ is exactly $\frac{d}{d\epsilon}|_0(\mathbf{t} \circ b_\epsilon)_*|_L$.

Both the foliations $\mathcal{F}_{lin}$ and $\mathcal{F}(G_{lin})$ are generated by linear vector fields on NL (i.e. elements of $\chi_{lin}(NL)$): $\mathcal{F}_{lin}$ by its very definition, and $\mathcal{F}(G_{lin})$ because G_{lin} is the transformation groupoid of an action by vector bundle automorphisms of NL. So it suffices to consider linear vector fields.

We have

$$\mathcal{F} = \{\frac{d}{d\epsilon}|_0(\mathbf{t} \circ b_\epsilon)\},$$

where b ranges over all families $b_\epsilon \colon M \to G$ of bisections of G with $b_0 = Id_M$.

The linear vector fields in $\mathcal{F}(G_{lin})$ are the infinitesimal generators of the G_L action on NL. By eq. (4.3) they are exactly

$$\{\frac{d}{d\epsilon}|_0(\mathbf{t} \circ b_\epsilon)_*|_L\},$$

where for any family $\gamma_\epsilon \colon L \to G_L$ of bisections of G_L with $\gamma_0 = Id_L$ we choose a family of bisections $b_\epsilon \colon M \to G$ of G with $b_0 = Id_M$ extending it.

From the above observation it is clear that the set of linear vector fields in $\mathcal{F}_{lin}$ and $\mathcal{F}(G_{lin})$ agree. □

4.3.1 Riemannian foliations

The following definition, when specialized to the case of regular foliations, reduces to the notion of Riemannian foliation given in the Introduction.

Definition 4.16. Let $(M, \mathcal{F})$ be a foliated manifold. We say that a Riemannian metric g on M is *adapted* if, for all leaves L, the bundle metric on NL (induced by $NL \cong (TL)^\perp$) is preserved by the action Ψ_L of H_L defined in Cor. 3.3.

Not every foliation admits an adapted Riemannian metric, but foliations arising from proper Lie groupoids do, by the work of Pflaum-Posthuma-Tang [PPT]. It would be interesting to study the geometric properties of (singular) foliations with an adapted metric, as done in [PPT] for those arising from proper Lie groupoids. Here we present just one result, which relies on [PPT].

Recall that a *singular Riemannian foliation* in Molino's sense ([Mo88, §6], see also [ABT, §1]) consists of manifold M with a suitable[6] partition into immersed submanifolds (leaves), together with a complete Riemannan metric such that if a geodesic intersects perpendicularly a leaf, it intersects perpendicularly every leaf it meets.

Proposition 4.17. *Let $(M, \mathcal{F})$ be a singular foliation and let L be an embedded leaf whose essential isotropy is discrete. Assume that $\mathcal{F}$ is linearizable about L and that the Lie group H_x^x is compact, where $x \in L$.*

Then there is a tubular neighborhood U of L such that $(U, \mathcal{F}|_U)$ admits an adapted, complete Riemannian metric which makes it into a singular Riemannian foliation.

Proof. By (the proof of) Prop. 4.12, there exists a neighborhood U and a proper Lie groupoid Γ over U inducing the foliation $\mathcal{F}|_U$. Hence U admits a complete Riemannian metric g whose transversal component is preserved by the action of Γ, by [PPT, Prop. 3.13] (cf. also [PPT, Def. 3.11]). The metric g is adapted (in the sense of Def. 4.16), as a consequence of the fact that $H|_U$ is a quotient of Γ [AnSk06, Ex. 3.4.(4)]. Further, g makes the foliation $\mathcal{F}|_U$ into singular Riemannian foliation by [PPT, Prop. 6.4]. □

5 Deformations of singular foliations

As a further application of the Bott connection $\nabla^{L,\perp}$ introduced in §3.2, we look at deformations of singular foliations. In §5.1 we consider deformations of singular foliations (keeping the underlying $C^\infty(M)$-module structure fixed). In §5.2 we attach to them a class in foliated cohomology.

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[6]More precisely, it is required that every tangent vector to a leaf can be extended to a vector field on M tangent to the leaves.

5.1   Deformations

In this subsection we view singular foliations as Lie-Rinehart algebras. We study the resulting notion of deformation for a singular foliation, which requires that the underlying $C^\infty(M)$-module structure be kept fixed.

A regular foliation $F \subset TM$ on a manifold M can be equivalently described as a Lie algebroid over M with injective anchor[7]. Since injectivity is an open condition, a smooth family of foliations of M parametrized by $t \in (-\epsilon, \epsilon)$ which agrees with F at $t = 0$ is the same thing as a smooth family of Lie algebroids over M which agrees with F at $t = 0$. We now consider the case of singular foliations.

Recall that a *Lie-Rinehart algebra* (an algebraic version of the notion of Lie algebroid) over a unital commutative algebra $\mathcal{C}$ consists of a unital left $\mathcal{C}$-module $\mathcal{M}$, a Lie algebra structure $[\cdot, \cdot]$ on $\mathcal{M}$, and a map $\rho \colon \mathcal{M} \to Der(\mathcal{C})$ which is a $\mathcal{C}$-module morphism, a Lie algebra morphism, and satisfies the Leibniz rule (*cf.* [Hu04, §1]).An isomorphism between Lie-Rinehart algebras $(\mathcal{C}, \mathcal{M})$ and $(\mathcal{C}', \mathcal{M}')$ consists of an algebra isomorphism $\mathcal{C} \to \mathcal{C}'$ and a Lie algebra isomorphism $\mathcal{M} \to \mathcal{M}'$ which intertwine both the module structures and ρ, ρ'.

Following [CrMo08, Def. 1] we define:

Definition 5.1. Let M a manifold, $\mathcal{M}$ a fixed $C^\infty(M)$-module and $I = (-\epsilon, \epsilon)$ an interval.

  a) A family of Lie-Rinehart algebras over I is a smoothly varying family $(\mathcal{A}_t)_{t \in I} = (\mathcal{M}, [\cdot, \cdot]_t, \rho_t)$.

  b) The families $(\mathcal{A}_t)_{t \in I}$ and $(\mathcal{A}'_t)_{t \in I}$ are equivalent if there exists a family of Lie-Rinehart algebra isomorphisms $h_t : \mathcal{A}_t \to \mathcal{A}'_t$ varying smoothly with respect to t.

  c) A deformation of a Lie-Rinehart algebra $\mathcal{A} = (\mathcal{M}, \rho, [\cdot, \cdot])$ is a family $(\mathcal{A}_t)_{t \in I}$ of Lie-Rinehart algebras such that $\mathcal{A}_0 = \mathcal{A}$.

  d) Two deformations $(\mathcal{A}_t)_{t \in I}$ and $(\mathcal{A}'_t)_{t \in I}$ of $(\mathcal{A}, \rho, [\cdot, \cdot])$ are equivalent if there exists an equivalence $h_t : \mathcal{A}_t \to \mathcal{A}'_t$ with $h_0 = id$.

Similarly to the regular case, a singular foliation $\mathcal{F} \subset \mathfrak{X}_c(M)$ can be equivalently described as a Lie-Rinehart algebra $(\mathcal{M}, [\cdot, \cdot], \rho)$ over $C^\infty(M)$ for which $\rho \colon \mathcal{M} \to Der(C^\infty(M)) = \mathcal{X}(M)$ is injective and takes values in compactly supported vector fields.

Denote by $\mathcal{F}_{mod}$ the $C^\infty(M)$-module underlying $\mathcal{F}$. Applying Def. 5.1, we obtain that a *deformation of a foliation* $(M, \mathcal{F})$ consists of Lie brackets $[\cdot, \cdot]_t$ and maps ρ_t, agreeing with the Lie bracket of vector fields $[\cdot, \cdot]_{\mathcal{X}(M)}$ and the inclusion into $\mathcal{X}(M)$ for $t = 0$, such that

$$(\mathcal{F}_{mod}, [\cdot, \cdot]_t, \rho_t)$$

is a family of Lie-Rinehart algebras. Notice that for every t the module $\{\rho_t(X) : X \in \mathcal{F}\}$ is a foliation, and ρ_t (as long as it is injective) determines $[\cdot, \cdot]_t$ by virtue of $\rho_t([X, Y]_t) = [\rho_t(X), \rho_t(Y)]_{\mathcal{X}(M)}$.

Example 5.2. Consider a foliation $(M, \mathcal{F})$ and let $\{\varphi^t\}_{t \in I}$ be a one-parameter family of diffeomorphisms of M. Denote by $[\cdot, \cdot]$ the Lie bracket of vector fields in $\mathcal{F}$. Then $(\mathcal{F}_{mod}, [\cdot, \cdot], (\varphi^t)_*)$ is a deformation as above. Further it is equivalent to the constant deformation (i.e., it is a trivial deformation), an equivalence being given by $\{(\varphi^t)_*\}_{t \in I}$. All trivial deformations of $(M, \mathcal{F})$ are of this kind, as a consequence of the fact that any algebra automorphism of $C^\infty(M)$ is the pullback of functions by a diffeomorphism of M.

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[7]More precisely: if $F \subset TM$ is a regular foliation, then the restriction of the Lie bracket of vector fields $[\cdot, \cdot]_{\mathcal{X}(M)}$ and the inclusion into TM make F into a Lie algebroid with injective anchor. If $(A, [\cdot, \cdot]_A, \rho_A)$ is a Lie algebroid and ρ_A is injective, then it is canonically isomorphic to $(\rho_A(A), [\cdot, \cdot]_{\mathcal{X}(M)}$, inclusion) and $\rho_A(A)$ is a regular foliation.

The fact that in our notion of deformation of a foliation $(M, \mathcal{F})$ we insist on fixing the $C^\infty(M)$-module structure makes our notion of deformation somewhat restrictive. We now see to what extent.

Lemma 5.3. *Let $\mathcal{F}_1, \mathcal{F}_2$ be foliations on M. If they are isomorphic as $C^\infty(M)$-modules then there exists a diffeomorphism ϕ of M such that for all $x \in M$ the vector spaces $(\mathcal{F}_1)_x$ and $(\mathcal{F}_2)_{\phi(x)}$ are isomorphic.*

Proof. Notice that any algebra automorphism of $C^\infty(M)$ is the pullback by a diffeomorphism of M. Hence a $C^\infty(M)$-module isomorphism consist of a linear isomorphism $\Phi \colon \mathcal{F}_1 \to \mathcal{F}_2$ and a diffeomorphism ϕ of M such that $\Phi(\phi^*(f)X) = f\Phi(X)$ for all $f \in C^\infty(M)$ and $X \in \mathcal{F}_1$. Hence, for every x, Φ maps $I_x \mathcal{F}_1$ isomorphically onto $I_{\phi(x)}\mathcal{F}_2$, so the linear map $(\mathcal{F}_1)_x \to (\mathcal{F}_2)_{\phi(x)}, [X] \to [\Phi(X)]$ is an isomorphism. □

For foliations generated by one vector field we obtain a full characterization of when they are isomorphic as modules:

Lemma 5.4. *Let X and Y be vector fields on M. The $C^\infty(M)$-modules $\langle X \rangle$ and $\langle Y \rangle$ are isomorphic iff there is a diffeomorphism ϕ of M such that*

$$\phi(Int(\{X = 0\})) = Int(\{Y = 0\}).$$

Here $Int(\{X = 0\})$ denotes the interior of set of zeros of X.

Proof. Suppose that $\langle X \rangle$ and $\langle Y \rangle$ are isomorphic. By [AnZa11, Ex. 1.2 i)] we know that $\langle X \rangle_x$ is trivial exactly when $x \in Int(\{X = 0\})$, and similarly for $\langle Y \rangle$. Hence from Lemma 5.3 there is a diffeomorphism of M mapping $Int(\{X = 0\})$ to $Int(\{Y = 0\})$, proving the claim.

For the converse, assume the existence of a diffeomorphism ϕ as in the statement, i.e. with $Int(\{\phi_* X = 0\}) = Int(\{Y = 0\})$. As $\langle X \rangle \cong \langle \phi_* X \rangle$ as $C^\infty(M)$-modules (via the map ϕ_*), it is enough to show that there is a module isomorphism $\langle Y \rangle \to \langle \phi_* X \rangle$. The module morphism determined by $Y \mapsto \phi_* X$ is an isomorphism: it is injective because if $h\phi_* X = 0$ for some $h \in C^\infty(M)$ then h vanishes on $\overline{\{\phi_* X \neq 0\}} = \overline{\{Y \neq 0\}}$, so $hY = 0$. □

Example 5.5. If $\{X_t\}$ is a smooth family of vector fields, then $\mathcal{F}_t := \langle X_t \rangle$ is usually not a deformation of $\mathcal{F}_0$ in the above sense: take for example $X_0 \equiv 0$ and X_t non-zero vector fields for $t \neq 0$, and apply Lemma 5.4 [8]. Making additional assumptions on X_0 one can assure that $\{\mathcal{F}_t\}$ is a deformation of $\mathcal{F}_0$. For instance one can assume that X_0 is a vector field in general position, that is, with isolated zeros and so that X is transverse to the zero section of TM.

5.2 Deformation cocycles

In [He73] deformations of regular foliations were shown to be controlled by "foliated cohomology", whereas in [CrMo08] by "deformation cohomology". Let us explain how these cohomologies are defined in our framework, i.e. when $\mathcal{F}$ is a singular foliation:

---

[8]This shows that if we deform a Lie algebroid in the sense of [CrMo08], then we don't get a deformation of the corresponding singular foliation

1) $H^*_{def}(\mathcal{F})$ is the *deformation cohomology* defined exactly as in the beginning of [CrMo08, §2], replacing the $C^\infty(M)$-module of Lie algebroid sections with $\mathcal{F}$ (notice that the definition given there holds for any Lie-Rinehart $C^\infty(M)$-algebra.). An n-cochain is a multilinear and antisymmetric map $D : \underbrace{\mathcal{F} \otimes \cdots \otimes \mathcal{F}}_{n\text{-times}} \to \mathcal{F}$ which is a multiderivation. Namely, there exists a $C^\infty(M)$-multilinear map $\sigma_D : \underbrace{\mathcal{F} \otimes \cdots \otimes \mathcal{F}}_{n-1\text{-times}} \to \mathcal{X}(M)$, called the *symbol* of D, such that

$$D(X_1, \ldots, fX_n) = fD(X_1, \ldots, X_n) + \sigma_D(X_1, \ldots, X_{n-1})(f)X_n$$

for all $X_1, \ldots, X_n \in \mathcal{F}$ and $f \in C^\infty(M)$. Notice that the above expression determines σ_D uniquely. The boundary map $\delta : C^n_{def}(\mathcal{F}) \to C^{n+1}_{def}(\mathcal{F})$ is given by a suitable Eilenberg-Maclane formula (see [CrMo08, §2]). 0

2) $H^*(\mathcal{F}; \mathcal{N})$ is the *foliated cohomology* defined as in [He73, §1], but using the map $\mathcal{F} \times \mathcal{N} \to \mathcal{N}$ induced by the Lie bracket of vector fields (see §3.2), which makes $\mathcal{N}$ into a representation of $\mathcal{F}$. An n-cochain is a map of $C^\infty(M)$-modules $\underbrace{\mathcal{F} \wedge \cdots \wedge \mathcal{F}}_{n\text{-times}} \to \mathcal{N}$.

3) The two cohomologies are related via a canonical map

$$H^n_{def}(\mathcal{F}) \to H^{n-1}(\mathcal{F}; \mathcal{N}) \tag{5.1}$$

defined as follows: Given a cochain $D \in C^n_{def}(\mathcal{F})$ consider its symbol σ_D which is an $\mathcal{X}(M)$-valued map. The assignment $D \to \natural \circ \sigma_D$ where $\natural : \mathcal{X}(M) \to \mathcal{X}(M)/\mathcal{F} = \mathcal{N}$ is the quotient map, induces a well defined map in cohomology. When $\mathcal{F}$ is a regular foliation it was shown in [CrMo08] that this map is an isomorphism.

The following result and its proof is an immediate adaptation to the context of foliations (or Lie-Rinehart $C^\infty(M)$-algebras) of Prop. 2 and Thm. 2 of [CrMo08].

A) Given a deformation of $\mathcal{F}$, consider the cocycles in $C^2_{def}(\mathcal{F})$ given by

$$c_t(a, b) = \frac{d}{ds}|_{s=t}([a, b]_s)$$

for all t, and assume that M is compact. Then the deformation is trivial (that is, equivalent to the constant deformation of $\mathcal{F}$) iff there is a smooth family $D_t \in C^1_{def}(\mathcal{F})$ with $\delta(D_t) = c_t$.

When $\mathcal{F}$ is regular, say $\mathcal{F} = C^\infty_c(M, F)$ for an involutive regular distribution F on M, and $F_t \subset TM$ is a smooth deformation of F by involutive regular distributions, then Heitsch [He73] gave a geometric construction of a class in $H^1(\mathcal{F}; \mathcal{N})$, using orthogonal projections $TM \to F_t$. In [CrMo08] it was shown that Heitsch's class corresponds to the class $[c_0] \in H^2_{def}(\mathcal{F})$ via the canonical isomorphism (5.1).

Given a deformation of a singular foliation now, it is impossible to construct geometrically a class in $H^1(\mathcal{F}; \mathcal{N})$ as in [He73] (the orthogonal projections used in [He73] are no longer smooth in the singular case). We do have the deformation class $[c_0] \in H^2_{def}(\mathcal{F})$ defined in A) though, which we can carry to a class in $H^1(\mathcal{F}; \mathcal{N})$ using the map (5.1). It would be interesting to investigate what this class is the obstruction to. In the regular case it is exactly Heitsch's class [He73].

A Appendix

A.1 Proofs of Theorem 2.7 and Proposition 3.6

Let $(M, \mathcal{F})$ be a foliation. If $W \subset \mathbb{R}^n \times M$ is a minimal bi-submersion obtained from a basis of $\mathcal{F}_x$ and $b \colon M_0 \to \mathbb{R}^n$ a bisection defined in a neighborhood of x in M, then in general it is not true that tb is a bisection at x for all $t \in [0, 1]$, as the following example shows.

Example A.1. If $M = \mathbb{R}$ and $\mathcal{F} = \langle \partial_x \rangle$, then $b = -2x \colon M \to \mathbb{R}$ is a bisection. For any t, the map $\mathbf{t}(tb) \colon M \to M$ is $x(1 - 2t)$, so for $t = \dfrac{1}{2}$ it is not a diffeomorphism. Even more is true: since the diffeomorphism carried by b is orientation-reversing, it is not isotopic to the identity.

To prove Theorem 2.7 we will need bisections b such that tb is a bisection at x for all $t \in [0, 1]$. Their existence is guaranteed by the following two lemmas.

Lemma A.2. *Let $x \in (M, \mathcal{F})$, $\{X_i\}_{i \leq n}$ generators for $\mathcal{F}$ in a small neighborhood M_0 of x defining a basis of $\mathcal{F}_x$. Let $W \subset \mathbb{R}^n \times M_0$ be the corresponding bi-submersions, and denote by L the leaf through x.*

If b is a section of $\mathbf{s} \colon M_0 \to W$ such that $(d_x b)(T_x L) = \{0\} \times T_x L$ then the image of b is a bisection near x. Further, $\mathrm{image}(tb)$ is a bisection at x for all $t \in [0, 1]$.

Proof. Let $y := \mathbf{t}(b(x)) \in L$. Clearly the source $\mathbf{s}$ maps $\mathbf{t}^{-1}(y)$ into L. Hence $d_{b(x)} \mathbf{s}$ maps $T_{b(x)} \mathbf{t}^{-1}(y)$ into $T_x L$. So

$$
\begin{aligned}
&T_{b(x)}(\mathrm{image}(b)) \cap T_{b(x)} \mathbf{t}^{-1}(y) \\
={}&T_{b(x)}(\mathrm{image}(b)) \cap T_{b(x)} \mathbf{t}^{-1}(y) \cap (d_{b(x)} \mathbf{s})^{-1}(T_x L) \\
={}&T_{b(x)}(\mathrm{image}(b|_L)) \cap T_{b(x)} \mathbf{t}^{-1}(y) \\
={}&\{0\}.
\end{aligned}
$$

The last equality holds for the following reason: By our assumption $T_{b(x)}(\mathrm{image}(b|_L)) = \{0\} \times T_x L = T_{b(x)}(\mathrm{graph}(\hat{b}|_L))$ where $\hat{b} \colon M_0 \to \mathbb{R}^n$ is the constant map with value $b(x)$. The latter defines a bisection as $\mathbf{t} \circ (\hat{b} \times Id_{M_0})$ is a diffeomorphism, namely $exp(\sum_{i \leq n} b(x)_i X_i)$.

The last statement of the lemma holds simply because $d_x(tb) = \{0\} \times T_x L$. $\qquad \square$

Remark A.3. We make a brief comment about the composition of path-holonomy bi-submersions. For $\alpha = 1, 2$ let $x^\alpha \in M$, $\{X_i^\alpha\}_{i \leq n}$ generators for $\mathcal{F}$ defining a basis of $\mathcal{F}_{x^\alpha}$, and denote by W^α the corresponding bi-submersions. By [AnSk06, Prop. 2.10 a)] W^α is a neighborhood of $(0, x^\alpha)$ in $\mathbb{R}^n \times M$. The composition[9] of the two bi-submersions [AnSk06, Prop. 2.4 b))] is

$$
W^2 \circ W^1 = \{(\lambda^2, y^2), (\lambda^1, y^1) : exp_{y^1}(\textstyle\sum \lambda_i^1 X_i^1) = y^2, (\lambda^\alpha, y^\alpha) \in W^\alpha\} \subset (\mathbb{R}^n \times M) \times (\mathbb{R}^n \times M).
$$

As the component y^2 is determined by the other three, $W_2 \circ W_1$ can be identified canonically with

$$
\{(\lambda^2, \lambda^1, y^1) : (\lambda^1, y^1) \in W^1, (\lambda^2, exp_{y^1}(\textstyle\sum \lambda_i^1 X_i^1)) \in W^2\} \subset \mathbb{R}^n \times \mathbb{R}^n \times M.
$$

In the following we will freely switch from viewing an N-fold composition of minimal bi-submersions as an open subset of $(\mathbb{R}^n \times M) \times \cdots \times (\mathbb{R}^n \times M)$ to viewing it as an open subset of $\mathbb{R}^n \times \cdots \times \mathbb{R}^n \times M$.

---

[9]Notice that this might be empty, for instance if the supports of the $\{X_i^2\}$ are disjoint from the support of the $\{X_i^1\}$.

Lemma A.4. *Let W be any bi-submersion in the path-holonomy atlas containing the point x, and M_0 a neighborhood of x in M. Then W is isomorphic to a finite composition of path-holonomy bi-submersions, so we may assume that $W \subset \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times M$ is such a composition.*

The graph of any map $b \colon M_0 \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ satisfying $(d_x b)(T_x L) = 0$ can be canonically deformed to the bisection $\{0\} \times M_0$ by a path of bisections of W.

Proof. Every bi-submersion W in the path-holonomy atlas is by definition the composition of path-holonomy bi-submersions and their inverses. If $U = \mathbb{R}^n \times M \rightrightarrows M$ is the path-holonomy bi-submersion defined by a choice of local generators $X_1, \ldots, X_n$ of $\mathcal{F}$, then its inverse U^{-1} is obtained by switching the roles of $\mathbf{s}$ and $\mathbf{t}$, and the map $U \to U^{-1}, (\lambda, y) \to (-\lambda, exp_y(\sum \lambda_i X_i))$ provides an isomorphism of bi-submersions. This proves the first statement of the lemma.

Assume now that the bi-submersion W is a composition $W = W_N \circ \cdots \circ W_1$ where $W_\alpha \subset \mathbb{R}^n \times M$ is a minimal bi-submersion ($\alpha = 1, \ldots, N$). If $b = (b^N, \ldots, b^1) \colon M_0 \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ satisfies $(d_x b)(T_x L) = 0$, then the path defined on $[0, N]$ by

$$b_t := (0, 0, \ldots, 0, 0, tb^1) \text{ for } t \in [0, 1], \tag{A.1}$$
$$b_t := (0, 0, \cdots, 0, (t-1)b^2, b^1) \text{ for } t \in [1, 2],$$
$$\cdots$$
$$b_t := ((t-N+1)b^N, b^{N-1}, \cdots, b^2, b^1) \text{ for } t \in [N-1, N]$$

consists of maps whose graphs are bisections of W. This is seen applying repeatedly Lemma A.2. Notice that when $t = 0$ we obtain the zero map, and when $t = N$ we obtain the map b. $\square$

Lemma A.5. *Let U be any bi-submersion, $p \in U$. Let S_x and S_y be slices transverse to the foliation at $x := \mathbf{s}(p)$ and $y := \mathbf{t}(p)$ respectively. Then there exists a bisection of U through p whose corresponding diffeomorphism maps S_x into S_y.*

Proof. Claim: The map $(\mathbf{s}, \mathbf{t}) \colon U \to M \times M$ is transverse at p to the submanifold $S_x \times S_y$.

We have to show that $Im(d_p(\mathbf{s}, \mathbf{t})) + (T_x S_x \times T_y S_y) = T_{x,y}(M \times M)$. This follows from the fact that $(T_x L \times T_y L) \subset Im(d_p(\mathbf{s}, \mathbf{t}))$, which in turn holds for the following reason: Given $X \in T_x L, Y \in T_y L$, since $d_p \mathbf{t}(Ker(d_p \mathbf{s})) = T_y L$ by [AnSk06, Def. 2.1 ii)], there exists $Z' \in Ker(d_p \mathbf{s})$ with $d_p \mathbf{t}(Z') = Y$. Similarly, there exists $Z'' \in Ker(d_p \mathbf{t})$ with $d_p \mathbf{s}(Z'') = X$. Hence $Z' + Z''$ maps to (X, Y) under $d_p(\mathbf{s}, \mathbf{t})$.

From the above claim it follows that, shrinking the transversals if necessary, $C := \mathbf{s}^{-1}(S_x) \cap \mathbf{t}^{-1}(S_y)$ is a submanifold of U of codimension $2dim(L)$.

Claim: Denote by $\sigma \colon C \to S_x$ the restriction of $\mathbf{s}$. Then $d_p \sigma$ is surjective with kernel $Ker(d_p \mathbf{s}) \cap Ker(d_p \mathbf{t})$.

Let $Z \in ker(d_p \sigma)$. In particular $Z \in ker(d_p \mathbf{s})$, so $d_p \mathbf{t}(Z) \in T_y L$. Since at the same time $d_p \mathbf{t}(Z) \in T_y S_y$ we conclude that $Z \in ker(d_p \mathbf{t})$. So $ker(d_p \sigma) \subset Ker(d_p \mathbf{s}) \cap Ker(d_p \mathbf{t})$. The latter is the kernel of the surjective linear map $d_p \mathbf{t} \colon Ker(d_p \mathbf{s}) \to T_y L$, hence it has dimension $dim(U) - dim(M) - dim(L)$. Hence

$$dim(Im(d_p \sigma)) = dim(C) - dim(ker(d_p \sigma)) \geq dim(M) - dim(L) = dim(S_x),$$

proving both statements of the claim.

The last claim has several consequences. As σ is a submersion near p, we can find a $\beta\colon S_x \to$ $\mathbf{s}^{-1}(S_x) \cap \mathbf{t}^{-1}(S_y)$ with $\sigma \circ \beta = Id_{S_x}$. Further, the intersection of $Ker(d_p\mathbf{t})$ with T_pC is $Ker(d_p\mathbf{s}) \cap Ker(d_p\mathbf{t})$. As the image of β is transversal to the latter, it follows that $\mathbf{t} \circ \beta\colon S_x \to S_y$ is a diffeomorphism onto its image. In particular, we can find an extension of β to a section b of $\mathbf{s}$ such that $T_p(image(b))$ has trivial intersection with $Ker(d_p\mathbf{t})$, so b will be a bisection of U. $\qquad\square$

Proof of Theorem 2.7. We show that the map Φ_x^y is well-defined, i.e. that it does not depend on the choices of bi-submersion U and of map $\bar{b}\colon S_x \to U$.

Fix a point h of H, let $x := \mathbf{s}(h), y := \mathbf{t}(h)$. Let U be a bi-submersion in the path-holonomy atlas, let $u \in U$ such that $[u] = h$, and $\bar{b}\colon S_x \to U$ a section of $\mathbf{s}$ through u such that $\mathbf{t} \circ \bar{b}$ maps S_x to S_y (it exists as a consequence of Lemma A.5). We may assume that U is a composition of N path-holonomy bi-submersions (see Lemma A.4). Take $\bar{b}$ and extend it to a bisection b of U such that $(d_x b)(T_x L) = \{0\} \times T_x L$, where L denotes the leaf through x. By Lemma A.4, the bisection b can be deformed canonically to the zero-bisection by a path of bisections.

Similarly, choose another bi-submersion U' in the path-holonomy atlas, a point u' with $[u'] = h$ and $\bar{b}'\colon S_x \to U'$ a section of $\mathbf{s}'$ through u' such that $\mathbf{t}' \circ \bar{b}'$ maps S_x to S_y. Since u and u' represent the same point $h \in H$, by definition there exists a morphism of bi-submersions $\sigma\colon U' \to U$ with $u' \mapsto u$. Then $\sigma \circ \bar{b}'\colon S_x \to U$ is a section of $\mathbf{s}$ through u carrying the same diffeomorphism as $\bar{b}'$. Take $\sigma \circ \bar{b}'$ and extend it to a bisection b' of U, defined in some neighborhood M_0 of x, such that $(d_x b')(T_x L) = \{0\} \times T_x L$. By Lemma A.4, b' is a bisection of U which can be deformed canonically to the zero-bisection by a path of bisections.

Our aim is to compare the diffeomorphisms $S_x \to S_y$ induced by $\bar{b} = b|_{S_x}$ and by $\bar{b}' = b'|_{S_x}$. Denote by ϕ_t (resp. ϕ_t') the local diffeomorphisms of M carried by tb (resp tb'), for $t \in [0, N]$. To finish the proof, it is enough to prove the following claim.

Claim: $\{\phi_t^{-1} \circ \phi_t'\}_{t \in [0,N]}$ is the flow of a time dependent vector field that lies in $I_x\mathcal{F}$.

To prove the claim we proceed as follows. Recall from Lemma A.4 that U is a product of path-holonomy bi-submersions: $U = W^N \circ \cdots \circ W^1$. Define $\{Z_t\}$ to be the time-dependent vector field corresponding to the 1-parameter family of diffeomorphisms $\{\phi_t\}$. Fix $t \in [0, N]$, and $\alpha \in \{1, \ldots, N\}$ so that $t \in [\alpha - 1, \alpha]$. To write down explicitly Z_t, denote by $\{X_i^\alpha\}_{i \le n}$ the vector fields in $\mathcal{F}$ that give rise to W^α. Using eq. (A.1), we have for all $z \in M_0$:

$$Z_t(\phi_t(z)) = \frac{d}{dt}\phi_t(z) = \sum_i b_i^\alpha(z) \cdot X_i^\alpha(\phi_t(z)),$$

or equivalently[10]

$$Z_t = \sum_i ((\phi_t^{-1})^* b_i^\alpha) \cdot X_i^\alpha. \tag{A.2}$$

Similarly, the time-dependent vector field corresponding to ϕ_t' is $Z_t' = \sum_i ((\phi_t'^{-1})^* b_i'^\alpha) \cdot X_i^\alpha$. Now eq. (2.1), which gives a relation between the flows of any two time-dependent vector fields, implies

$$\phi_t' = \phi_t \circ (\text{time-}t \text{ flow of } \{(\phi_s)_*^{-1}(Z_s' - Z_s)\}_{s \in [0,N]}). \tag{A.3}$$

---

[10]If tb was not a bisection, ϕ_t would not be invertible and we could not define the vector field Z_t.

By eq. (A.2), for all $s \in [\beta - 1, \beta]$ where $\beta \in \{1, \ldots, N\}$, we have $Z'_s - Z_s = \sum_i g_{i,s} X_i^\beta$ where the functions $g_{i,s}$ (defined in a neighborhood of $\phi_s(x)$) are given by

$$g_{i,s} := (\phi_s^{-1})^* b_i^\beta - (\phi'_s{}^{-1})^* b'_i{}^\beta.$$

Notice that $g_{i,s}$ vanishes at $\phi_s(x)$ as a consequence of the fact that $b(x) = b'(x) = u$ and $\phi_s(x) = \phi'_s(x)$. Equivalently, $\phi_s^* g_{i,s} \in I_x$. Hence

$$(\phi_s)_*^{-1}(Z'_s - Z_s) = \sum_i (\phi_s^* g_{i,s}) \cdot (\phi_s)_*^{-1} X_i^\beta \in I_x \mathcal{F}.$$

From eq. (A.3) it follows that $\phi_t^{-1} \circ \phi'_t$ is the time-t flow of a time dependent vector field that lies in $I_x \mathcal{F}$. $\square$

For the proof of Prop. 3.6 we need one more lemma:

Lemma A.6. *Let $x \in (M, \mathcal{F})$, $Y \in \mathcal{F}(x)$ and $\psi \in exp(I_x \mathcal{F})$. Then $Y - \psi_* Y \in I_x \mathcal{F}$.*

Proof. For any time-dependent vector field X whose time 1 flow is ψ, we have

$$Y - \psi_* Y = \psi_*(\psi_*^{-1} Y - Y) = \psi_* \left[\int_0^1 \frac{d}{dt} ((\psi_t^{-1})_* Y) dt \right] = \psi_* \left[\int_0^1 (\psi_t^{-1})_* ([X_t, Y]) \, dt \right], \qquad \text{(A.4)}$$

where ψ_t is the time-t flow of $\{X_t\}$. The last equation holds because the integrands are equal, see for example [Le03].

Since $\psi \in exp(I_x \mathcal{F})$, we can choose X so that $X_t \in I_x \mathcal{F}$ for all t. From this and $Y(x) = 0$, using the Leibniz rule one shows that $[X_t, Y] \in I_x \mathcal{F}$. Since $\psi_t(x) = x$ for all t, we are done. $\square$

Proof of Prop. 3.6. Fix $h \in H_x^y$ and slices S_x and S_y. Any diffeomorphism τ chosen as in Theorem 2.7 maps x to y and maps the foliation $\mathcal{F}_{S_x}$ to $\mathcal{F}_{S_y}$, hence gives a map $\mathfrak{g}_x \to \mathfrak{g}_y$ as in eq. (3.2).

First we show that the map (3.2) is independent of the choice of the diffeomorphism τ. Recall that τ arises from a bi-submersion U in the path-holonomy atlas with a point u representing h and a section $\bar{b} \colon S_x \to U$ through u. As in the proof of Thm. 2.7 let ϕ be a local diffeomorphism extending τ defined in a neighborhood of x in M, arising from a bisection b of U such that $(d_x b)(T_x L) = \{0\} \times T_x L$, where L denotes the leaf through x. Under the canonical isomorphism $\mathfrak{g}_x \cong \mathcal{F}_{S_x} / I_x \mathcal{F}_{S_x}$ (see [AnZa11, Rem. 1.6]), the map (3.2) is[11]

$$\mathfrak{g}_x \to \mathfrak{g}_y, [Y] \mapsto [\phi_*(Y)], \qquad \text{(A.5)}$$

where $Y \in \mathcal{F}(x)$. Let τ' be another diffeomorphism associated to h as in Theorem 2.7, and let ϕ' be a local diffeomorphism arising from a bisection of U as above[12]. We have to show that $\mathfrak{g}_x \to \mathfrak{g}_y, [Y] \mapsto [\phi'_*(Y)]$ agrees with the map (A.5). That is, given $Y \in \mathcal{F}(x)$, we have to show that $\phi_* Y - \phi'_* Y \in I_y \mathcal{F}$. This goes as follows: we have $\phi_* Y - \phi'_* Y = \phi_*(Y - \psi_* Y)$ where $\psi := \phi_*^{-1} \circ \phi'_* \in$

---

[11]The reason why we do not use this map in the statement of Prop. 3.6 is that a priory we do not know that it is independent of the choice of ϕ, see Remark 2.10c).

[12]By Thm. 2.7 we know that $\tau^{-1} \circ \tau'$ is the the restriction to S_x of the time-1 flow of a time dependent vector field $\{X_t\}$ lying in $I_x \mathcal{F}$. However the X_t are usually not tangent to S_x, and for this reason in the sequel we work with local diffeomorphisms ϕ, ϕ' of M rather than with the local diffeomorphisms τ, τ' of S_x.

$exp(I_x\mathcal{F})$ by the claim in Theorem 2.7. Hence Lemma A.6 implies that $Y - \psi_* Y \in I_x\mathcal{F}$, and with $\phi_*(I_x\mathcal{F}) = I_y\mathcal{F}$ we are done.

Last, we show that the map (3.2) is independent of the choice of slices, and hence canonical. Denote $S_x^1 := S_x$ and let S_x^2 be another slice at x. Fix $\psi_x^{12} \in exp(I_x\mathcal{F})$ mapping S_x^2 to S_x^1. Notice that the isomorphism

$$\mathcal{F}_{S_x^2}/I_x\mathcal{F}_{S_x^2} \to \mathcal{F}_{S_x^1}/I_x\mathcal{F}_{S_x^1} \tag{A.6}$$

induced by $(\psi_x^{12})_*$ coincides with the one obtained by the canonical identification of both sides of eq. (A.6) with $\mathfrak{g}_x$. This is a consequence of the fact that the automorphism of $\mathfrak{g}_x$ induced by $(\psi_x^{12})_*$ is $Id_{\mathfrak{g}_x}$, by Lemma A.6. Similarly, denote $S_y^1 := S_y$ and let S_y^2 be another slice at y, and $\psi_x^{21} \in exp(I_y\mathcal{F})$ mapping S_y^1 to S_y^2. We have to show that the diagram of isomorphisms

$$
\begin{array}{ccc}
\mathcal{F}_{S_x^2}/I_x\mathcal{F}_{S_x^2} & \longrightarrow & \mathcal{F}_{S_y^2}/I_y\mathcal{F}_{S_y^2} \\
\downarrow & & \uparrow \\
\mathcal{F}_{S_x^1}/I_x\mathcal{F}_{S_x^1} & \longrightarrow & \mathcal{F}_{S_y^1}/I_y\mathcal{F}_{S_y^1}
\end{array}
$$

commutes, where the horizontal maps are given by (3.2) (applied to the two choices of slices) and the vertical maps are induced by $(\psi_x^{12})_*$ and $(\psi_y^{21})_*$ respectively, as in eq. (A.6). We have

$$\Phi_x^y(h) = \langle \tau \rangle \in GermAut_{\mathcal{F}}(S_x^1, S_y^1)/exp(I_x\mathcal{F})|_{S_x^1},$$

hence by Lemma 2.12 we have

$$\Phi_x^y(h) = \langle \psi_y^{21} \circ \tau \circ \psi_x^{12} \rangle \in GermAut_{\mathcal{F}}(S_x^2, S_y^2)/exp(I_x\mathcal{F})|_{S_x^2},$$

showing that the above diagram commutes. $\qquad\square$

A.2 Changing transversals

We put here some technical results regarding different choices of transversals which are used in §2.

Lemma A.7. *Let x be a point in a foliated manifold $(M, \mathcal{F})$, and S_x^1, S_x^2 two transversals at x. Then there exists $\psi \in exp(I_x\mathcal{F})$ mapping S_x^1 to S_x^2.*

Proof. By the splitting theorem Prop. 1.2 there exists a neighborhood W of x in M and a diffeomorphism of foliated manifolds that identifies $(W, \mathcal{F}_W)$ with the product of the foliated manifolds $(S_x^1, \mathcal{F}_{S_x^1})$ and I^k endowed with the foliation consisting of just one leaf, where $I := (-1, 1)$. We use this identification and denote by $\pi \colon S_x^1 \times I^k \to S_x^1$ the natural projection. As $\pi^{-1}(x, 0)$ is given by the leaf of $\mathcal{F}$ through x, it is clear that π maps S_x^2 diffeomorphically onto S_x^1. Hence there is a unique map $\theta \colon S_x^1 \to I^k$ whose graph is S_x^2. Denote by $(s_1, \ldots, s_k)$ the standard coordinates on I^k, and consider the vector field $\sum_{i=1}^k \pi^* \theta_i \cdot \partial_{s_i}$ on $S_x^1 \times I^k$, which lies in $I_x\mathcal{F}$ since $\theta(x) = 0$. Its time-one flow takes S_x^1 to S_x^2. $\qquad\square$

Lemma A.8. *Let x, y be points in a foliated manifold $(M, \mathcal{F})$ lying in the same leaf and S_x^i, S_y^i transversals at x and y respectively, $i = 1, 2$. There is an identification*

$$GermAut_{\mathcal{F}}(S_x^1, S_y^1)/exp(I_x\mathcal{F})|_{S_x^1} \to GermAut_{\mathcal{F}}(S_x^2, S_y^2)/exp(I_x\mathcal{F})|_{S_x^2}, \quad [\tau] \mapsto [\tilde{\psi}_y^{21} \circ \tau \circ \tilde{\psi}_x^{12}] \tag{A.7}$$

where $\psi_x^{12} \in exp(I_x\mathcal{F})$ maps S_x^2 to S_x^1, $\psi_y^{21} \in exp(I_x\mathcal{F})$ maps S_y^1 to S_y^2, and $\tilde{\psi}_x^{12}$ (resp. $\tilde{\psi}_y^{21}$) is the restriction to S_x^2 (resp. S_y^1).

Further, this identification is canonical, namely independent of the choice of ψ_x^{12} and ψ_y^{21}.

Proof. Make a choice for ψ_x^{12} and ψ_y^{21} (this is possible by Lemma A.7). To show that the map (A.7) is well-defined, we take $\tau, \hat{\tau} \in GermAut_{\mathcal{F}}(S_x^1, S_y^1)$ with $\hat{\tau}^{-1} \circ \tau \in exp(I_x\mathcal{F})|_{S_x^1}$ and need to check that

$$(\tilde{\psi}_y^{21} \circ \hat{\tau} \circ \tilde{\psi}_x^{12})^{-1} \circ (\tilde{\psi}_y^{21} \circ \tau \circ \tilde{\psi}_x^{12}) \in exp(I_x\mathcal{F})|_{S_x^2}.$$

This is done using that for any time-dependent vector field X in $\mathcal{F}$ and $f \in I_x$, we have

$$(\psi_x^{12})^{-1} \circ exp(fX) \circ \psi_x^{12} = exp((\psi_x^{12})_*^{-1}(fX)) \in exp(I_x\mathcal{F}).$$

Now we show that the map is canonical: Let $\hat{\psi}_x^{12}$ and $\hat{\psi}_y^{21}$ be as above; we have to show that

$$(\tilde{\psi}_y^{21} \circ \tau \circ \tilde{\psi}_x^{12})^{-1} \circ (\tilde{\hat{\psi}}_y^{21} \circ \tau \circ \tilde{\hat{\psi}}_x^{12}) \in exp(I_x\mathcal{F})|_{S_x^2},$$

which follows from a computation similar to the above. $\qquad\square$

Lemma A.9. *Let x be a point in a foliated manifold $(M, \mathcal{F})$, lying in the leaf L. If $\psi \in exp(I_x\mathcal{F})$ then $d_x\psi$ induces the identity on $N_xL := T_xM/T_xL$.*

Proof. Denote by $\{X_t\} \subset I_x\mathcal{F}$ the time-dependent vector field whose time-1 flow is ψ. As X_t is tangent to L and vanishes at x, it is clear that $d_x\psi$ maps T_xL to itself, so it induces an endomorphism of N_xL.

To show that this endomorphism is the identity we proceed as follows: Let $Y_x \in T_xM$, and extend it to a vector field Y defined near x. Since $X_t \in I_x\mathcal{F}$, using the Leibniz rule one shows that $[X_t, Y]|_x \in T_xL$. Hence $(\psi_t^{-1})_*([X_t, Y])|_x \in T_xL$ for all t, so from eq. (A.4) we conclude that $Y_x - \psi_* Y_x \in T_xL$. $\qquad\square$

A.3 The normal module $\mathcal{N}$

Let $(M, \mathcal{F})$ be a manifold with a foliation. Here we study the $C^\infty(M)$-module $\mathcal{N} = \mathfrak{X}(M)/\mathcal{F}$ which is needed in §3.2 and §5. Notice that $\mathcal{N}$ is locally finitely generated (since $\mathfrak{X}(M)$ is), and that it does not inherit the Lie bracket of $\mathfrak{X}(M)$.

For every $x \in M$ consider the vector space

$$\mathcal{N}_x := \mathcal{N}/I_x\mathcal{N} = \mathfrak{X}(M)/(\mathcal{F} + I_x\mathfrak{X}(M)).$$

Lemma A.10. *Let x belong to the leaf L. Then the evaluation map identifies $\mathcal{N}_x$ with $N_xL = \dfrac{T_xM}{T_xL}$.*

Proof. The map

$$\mathfrak{X}(M)/(\mathcal{F} + I_x\mathfrak{X}(M)) \to N_xL, \quad \langle X \rangle \mapsto X_x \bmod T_xL$$

is clearly well-defined and surjective. It is injective because if $X \in \mathfrak{X}(M)$ satisfies $X_x \in T_xL$, then there exists $Y \in \mathcal{F}$ with $Y_x = X_x$, and hence $X - Y \in I_x\mathfrak{X}(M)$. $\qquad\square$

The union $N = \cup_{x \in M} \mathcal{N}_x$ should be considered the normal bundle of the foliation $\mathcal{F}$. It is a field of vector spaces of varying dimensions over M. Given a leaf L we now interpret $\mathcal{N}/I_L\mathcal{N}$, which is the same as $\mathfrak{X}(M)/(\mathcal{F} + I_L\mathfrak{X}(M))$.

Lemma A.11. *Let L be an embedded[13] leaf of $(M, \mathcal{F})$, and let $NL := \dfrac{T_L M}{TL}$ be its normal bundle. The evaluation map identifies $\mathcal{N}/I_L\mathcal{N}$ with $C^\infty(L; NL)$. Hence $\mathcal{N}/I_L\mathcal{N}$ is a projective $C^\infty(L)$-module.*

Proof. We parallel the proof of Lemma A.10. The map

$$q \colon \mathfrak{X}(M)/(\mathcal{F} + I_L\mathfrak{X}(M)) \to C^\infty\left(L; \frac{T_L M}{TL}\right), \quad \langle X \rangle \mapsto X \mid_L \mod TL$$

is well defined and surjective. We show its injectivity. If $X \in \mathfrak{X}(M)$ satisfies $X|_L \subset TL$, then there exists $Y \in \mathcal{F}$ with $Y_L = X_L$ (this is clear locally by Prop. 1.2, and holds on the whole of L by a partition of unity argument). Hence $X - Y \in I_L\mathfrak{X}(M)$. $\qquad\square$

Remark A.12. (Higher order holonomy transformations)

The normal bundle N carries the first order transversal information of the foliation $(M, \mathcal{F})$, whence the correct way to think of the action of the holonomy groupoid H on N is as linearized holonomy transformations. However, higher order holonomy is interesting as well (*cf.* Dufour et al [DuWa06, Du08]). In the regular case, this higher order holonomy is described by an action on $\mathcal{N}$ of the k-th jet prolongation $J^k H$ of the (smooth) holonomy groupoid, for every $k \in \mathbb{N}$ (*cf.* [EvLuWe99, App. A, B]). Roughly speaking, for any Lie groupoid G its k-th jet prolongation $J^k G$ is the Lie groupoid formed by the k-th tangent spaces of all its bisections. When $(M, \mathcal{F})$ is singular the holonomy groupoid is no longer smooth, so we cannot apply directly this definition. However, $J^k H$ can be defined as a quotient. Let us sketch this construction for $k = 1$:

Given a bi-submersion $(U, \mathbf{t}, \mathbf{s})$ of $(M, \mathcal{F})$, one can define its first jet prolongation $(J^1 U, j^1\mathbf{t}, j^1\mathbf{s})$. Elements of $J^1 U$ are of the form TV, where V is a bisection of U, and $j^1\mathbf{t} = \mathbf{t} \circ \pi$, $j^1\mathbf{s} = \mathbf{s} \circ \pi$, where $\pi : TV \to V$ is the bundle projection. One checks easily that it is also a bi-submersion of $(M, \mathcal{F})$. Furthermore, a morphism of bi-submersions $f : U_1 \to U_2$ can be prolonged to a morphism $j^1 f : J^1 U_1 \to J^2 U_2$. This way, given an atlas $\mathcal{U} = \{(U_i, \mathbf{t}_i, \mathbf{s}_i)\}_{i \in I}$ we define $J^1 H$ as the quotient of the atlas $J^1\mathcal{U} = \{(J^1 U_i, j^1\mathbf{t}_i, j^1\mathbf{s}_i)\}_{i \in I}$.

Now a bisection V of U corresponds to a local diffeomorphism ϕ^V of M such that $(\phi^V)_*\mathcal{F} \subset \mathcal{F}$. It follows that the class of TV in $J^1 H$ corresponds exactly to the germ of $(\phi^V)_* : \mathfrak{X}(M) \to \mathfrak{X}(M)$, which descends to $(\phi^V)_* : \mathcal{N} \to \mathcal{N}$. This formula defines the action of $J^1 H$ on $\mathcal{N}$.

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[13]If the leaf L is not embedded, then in order to realize $C^\infty(L; NL)$ from $\mathcal{N}$ one can proceed as in [AnSk06, Rem. 1.16] or [AnZa11, Rem. 1.8], replacing $\mathcal{F}$ there by $\mathcal{N}$.

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