STEFAN-SUSSMANN SINGULAR FOLIATIONS, SINGULAR SUBALGEBROIDS, AND THEIR ASSOCIATED SHEAVES

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Abstract. We explain and motivate Stefan-Sussmann singular foliations, and by replacing the tangent bundle of a manifold with an arbitrary Lie algebroid we introduce singular subalgebroids. Both notions are defined using compactly supported sections. The main results of this note are an equivalent characterization in which the compact support condition is removed, and an explicit description of the sheaf associated to any Stefan-Sussmann singular foliation or singular subalgebroid.

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1. Introduction

As an introduction we survey the case of foliations. We start recalling smooth foliations, understood as partitions of a manifold into immersed submanifolds; they clearly contain the class of regular foliations. Then we argue that, using submodules of sections (vector fields) instead of leaves, one obtains another extension of the notion of regular foliation which records more information than just a partition into leaves. Finally we explain briefly the purpose of the remainder of this note.

1.1. **Partitions into leaves.** A naive way to think of a foliation is as a partition of a manifold M to immersed and connected submanifolds (leaves). Here are some examples:

- Parallel lines in $\mathbb{R}^2$ with slope θ. When we pass to the torus, the nature of θ plays an important role: If $\theta \in \mathbb{Q}$ we obtain a partition of the torus to circles, whereas if $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the leaves are (infinite) lines wrapped around the torus in a dense way.

- The partition of $\mathbb{R}^2$ into the x-axis (a 1-dimensional leaf), the open upper halfplane and the open lower halfplane (2-dimensional leaves).
- The partition of $\mathbb{R}^2$ to the upper halfplane (a 2-dimensional leaf) and every point of the lower halfplane (each of which is a 0-dimensional leaf).

Notice that all of the above partitions are "smooth" in the following sense[1]: every vector $v \in TM$ which is tangent to the leaf through the footpoint of v can be extended to a vector field $X \in \mathfrak{X}(M)$ which at every point $p \in M$ is tangent to the leaf through p. In this note we refer to such partitions as *smooth foliations*. For instance in the first example X will be a multiple of the Kronecker vector field $\partial_x + \theta\partial_y$ by a smooth function $f \in C^\infty(\mathbb{R}^2)$. There are partitions which are not "smooth" in this sense, for example:

- The partition of $\mathbb{R}^2$ into the x-axis (a 1-dimensional leaf) and each of the remaining points of the plane (0-dimensional leaves). (The complement of the x-axis is dense in $\mathbb{R}^2$, so every vector field which vanishes there must vanish everywhere in $\mathbb{R}^2$ due to continuity.)

In the literature foliations are often required to be smooth in the above sense, (*e.g.* singular Riemannian foliations, see the survey [1]). Notice that in the above examples we allow the dimension of the leaves to vary. More precisely, given a point p, the dimensions of the leaves that intersect small enough neighbourhoods of p might drop (but not become bigger than the one of the leaf through p.)

By and large, the theory of smooth foliations has been developed under the assumption that all the leaves have equal dimension. In this case, the local structure of a foliated manifold M is as follows: there exists an atlas by smooth charts which are split into a longitudinal direction ℓ and a transversal one τ, and the coordinate changes are of the form $(\ell, \tau) \mapsto (f(\ell, \tau), g(\tau))$. In fact, this local description implies both the partition of M to leaves, as well as the smoothness; it is taken as the standard definition of a *regular foliation*.

However, partitions where the dimension of certain leaves drops are abundant in mathematics. Take for instance the orbits of the action of S^1 by rotations on $\mathbb{R}^2$, or on S^2: All the orbits have dimension 1, except for the fixed points. In fact, lots of smooth actions of Lie groups on a manifold M have orbits with varying dimension, and the induced partition of M to orbits is smooth (every vector tangent to an orbit is the value of a vector field defined by the infinitesimal generators of the action). For example, actions of orthogonal groups provide many examples of singular Riemannian foliations.

Other interesting foliations where the dimension of the leaves drops arise in Poisson geometry, as every Poisson manifold is endowed with a foliation by symplectic leaves. In fact, the Poisson structure at hand is completely determined by its symplectic foliation. The "dimension drop" phenomenon arises in lots of interesting Poisson structures, for instance linear Poisson structures on the dual $\mathfrak{g}^*$ of a Lie algebra. Take for example $\mathfrak{g}$ to be the Lie algebra of $SU(2, \mathbb{R})$: The leaves of the associated symplectic foliation (coadjoint orbits) are concentric spheres in $\mathbb{R}^3$, whose dimension drops at the origin (which is the fixed point of the coadjoint action).

While regular foliations are well-understood, there is a great deal of subtlety present when the "dimension drop" phenomenon occurs[2], which is not captured

[1] Another charaterization of smoothness, involving charts, is given in [7, §1.5].

[2] As an instance of this subtlety let us point out that a manifold M with a foliation exhibiting the dimension-drop phenomenon does admit local coordinates which split to a longitudinal and

by the notion of smooth foliation as we introduced it at the beginning of this subsection. In the remainder of this introduction we will describe another, much finer extension of the notion of regular foliation (see definition 1.4).

1.2. **Regular foliations.** Let us make a fresh start with regular foliations and try to give a more algebraic perspective in this setting.

Instead of looking at the leaves, we may look at the tangent spaces to the leaves. This way we obtain a constant rank distribution, in fact a vector subbundle $F \to M$ of the tangent bundle TM. Furthermore, the $C^\infty(M)$-module of sections ΓF is involutive, namely it is closed by the Lie bracket of vector fields. Moreover, the Serre-Swan theorem shows that the subbundle F can be recovered from ΓF: recall that for every $x \in M$ the vector space $\Gamma F/I_x(\Gamma F)$ is isomorphic to F_x (here $I_x \subset C^\infty(M)$ stands for the functions which vanish at x). The isomorphism is the evaluation map $ev_x : \Gamma F/I_x(\Gamma F) \to T_xM, ev_x([\xi]) := \xi(x)$.

Observe that the module $\Gamma_c F$ of compactly supported sections of F has the same properties, in particular the fiber $\Gamma_c F/I_x(\Gamma_c F)$ is isomorphic to F_x, so the module ΓF is not the only one with this property. In fact, instead of starting with the vector bundle F, we could start with the choice of a submodule $\mathcal{F}$ of vector fields tangent to the leaves such that the evaluation map $ev_x : \mathcal{F}/I_x\mathcal{F} \to T_xM$ maps isomorphically onto the tangent space F_x to the leaf. There can be several choices of such a module; for instance, for the regular foliation on a product manifold $M_1 \times M_2$ consisting of just one leaf, take the $C^\infty(M_1 \times M_2)$-submodule $\mathcal{F}_1$ generated by (the canonical lifts of) $\mathfrak{X}_c(M_1)$ and $\mathfrak{X}(M_2)$, or the submodule $\mathcal{F}_2$ generated by $\mathfrak{X}(M_1)$ and $\mathfrak{X}_c(M_2)$, etc. However, for every partition of a manifold M to leaves of constant dimension, there are always two extreme possibilities:

A) $\mathcal{F}$, the module generated by all vector fields tangent to the leaves. (Of course, this module is none other than ΓF.)
B) $\mathcal{F}_{comp}$, the module generated by all the compactly supported vector fields tangent to the leaves. (This module is $\Gamma_c F$.)

The classical Frobenius theorem says that, item (A) is equivalent to having a partition of M to leaves of constant dimension: if we start from a $C^\infty(M)$-module $\mathcal{F}$ which is involutive and can be realised as the module of sections of a vector subbundle F of TM, then there is a unique partition of M to leaves which determines $\mathcal{F}$ in the above way. As for item (B), the module $\mathcal{F}_{comp}$ is involutive and locally finitely generated. In general, it is not the module of sections of any vector subbundle of TM. But it is the module of *compactly supported* sections of a vector subbundle of TM.

On the other hand, thanks to the Serre-Swan theorem, the $C^\infty(M)$-submodules of $\mathfrak{X}(M)$ which can be realized as modules of sections of some vector bundle F are exactly the projective modules. This discussion leads to the following algebraic characterization of regular foliation, which we phrase as a definition.

Definition 1.1. Let M be a smooth manifold and $\mathfrak{X}(M)$ its $C^\infty(M)$-module of vector fields. A **regular foliation** on M is a $C^\infty(M)$-submodule $\mathcal{F}$ of $\mathfrak{X}(M)$ such that:

(1) $\mathcal{F}$ is locally finitely generated;

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a transversal direction (see [7]). However, due to the dimension drop phenomenon the change of coordinates resists a smooth formulation. So, in contrast with the regular case we do not have a good notion of foliation atlas.

(2) $\mathcal{F}$ is involutive, namely $[\mathcal{F},\mathcal{F}] \subseteq \mathcal{F}$;
(3) $\mathcal{F}$ is projective;
(4) for every $x \in M$ the evaluation map $ev_x : \mathcal{F}_x := \mathcal{F}/I_x\mathcal{F} \to T_xM$ is injective for every $x \in M$.

Remark 1.2. i) Item (4) in Definition 1.1 is absolutely necessary. Here is an example of a foliation which is not regular, and satisfies only the other items: Take $M = \mathbb{R}$ and consider the action of $\mathbb{R}$ defined by the flow of the vector field $x\partial_x$. In other words, the $C^\infty(\mathbb{R})$-module $\mathcal{F}^1$ is the one generated (only) by $x\partial_x$. At zero the fiber $(\mathcal{F}^1)_0$ is isomorphic to $\mathbb{R}$ and the evaluation map vanishes since zero is a fixed point. On the other hand the orbits of the action are $(-\infty,0), \{0\}$ and $(0,+\infty)$, and the induced partition of $\mathbb{R}$ is not a regular foliation (but it is a smooth foliation in the sense of §1.1).

ii) Consider the vector subbundle F of TM corresponding to $\mathcal{F}$ in definition 1.1 via the Serre-Swan theorem and notice that the module $\mathcal{F}_{comp} = \Gamma_c F$ also satisfies this definition. So we could have formulated this definition using a submodule of the $C^\infty(M)$-module of compactly supported vector fields $\mathfrak{X}_c(M)$. We will use this observation and elaborate on it in §1.3 below.

1.3. **Stefan-Sussmann singular foliations.** In definition 1.1 we characterized algebraically partitions of M into leaves of constant dimension. Now we discuss how to extend definition 1.1 to partitions of M into leaves whose dimension drops, obtaining as a result the notion of *Stefan-Sussmann singular foliation*. An essential requirement is that, given a regular foliation, one can regard it as a Stefan-Sussmann singular foliation in a *unique* way. From that respect, one is tempted to start by considering the module $\mathcal{F}$ of all vector fields which are tangent to the leaves (or the compactly supported ones having that property). But in the dimension-drop case this module may not be locally finitely generated: This is exactly the case for the foliation on $\mathbb{R}$ with leaves the half-line $(0,+\infty)$ (1-dimensional leaf) and all the remaining points (each one of which is a 0-dimensional leaf).

So when the dimension of the leaves drops, one is a priori forced to make a *choice* of a module $\mathcal{F}$ of vector fields tangent to the leaves (since there is no canonical such module). Lots of different choices of module appear, as the following examples exhibit this clearly. Notice that for regular foliation we obtained a unique module once we imposed that it consist of compactly supported vector fields (see §1.2), and imposing the same condition in these examples is not sufficient to obtain a preferred module.

Example 1.3. i) The partition of $\mathbb{R}$ to $(-\infty,0), \{0\}, (0,+\infty)$ we discussed in remark 1.2 arises also by the action of $\mathbb{R}$ defined by the flow of the vector field $x^k\partial_x$, for any $k \in \mathbb{N}$, $k \geq 2$. For each such k we obtain a different module $\mathcal{F}^k$.

ii) Consider the partition of $\mathbb{R}^2$ to $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$. These are the orbits of the linear action of $GL(2,\mathbb{R})$ or $SL(2,\mathbb{R})$ or $\mathbb{C}^*$ (by complex multiplication). The module $\mathcal{F}$ in each case is the one generated by the infinitesimal generators of the action.

Now let us see which items of definition 1.1 may survive when the dimension of the leaves is allowed to drop:

- Item (4) does not survive as we explained in remark 1.2 i).
- Item (3) does not survive either. All the modules $\mathcal{F}^k$ defined in example 1.3 i) are projective, but for the foliation of $\mathbb{R}^2$ we discussed in ii) we find the

following: the fiber $\mathcal{F}_0$ is $\mathbb{R}^4$ for the action of $GL(2,\mathbb{R})$, $\mathbb{R}^3$ for the action of $SL(2,\mathbb{R})$ and $\mathbb{R}^2$ for the action of $\mathbb{C}^*$, as was shown in [2, Prop. 1.4]. At every other point $x \neq 0$, the foliation is regular (with a single leaf), so $\mathcal{F}_x = \mathbb{R}^2$. A module $\mathcal{F}$ is projective iff the dimension of $\mathcal{F}/I_x\mathcal{F}$ is constant at all points x, as a consequence of the Serre-Swan theorem. So the module defined by the action of $GL(2,\mathbb{R})$ (or $SL(2,\mathbb{R})$) is not projective. However the module defined by the action of $\mathbb{C}^*$ is projective.

Whence, the only items that may survive from definition 1.1 in the dimension-drop case, are (1) and (2). These are enough to constitute an algebraic definition that allows to recover the dimension-drop phenomenon, as the generalization of the Frobenius theorem by Stefan and Sussmann shows [11, 12].

Definition 1.4. A **Stefan-Sussmann singular foliation** on M is a $C^\infty(M)$-submodule $\mathcal{F}$ of $\mathfrak{X}_c(M)$ such that:

(1) $\mathcal{F}$ is locally finitely generated;
(2) $\mathcal{F}$ is involutive, namely $[\mathcal{F},\mathcal{F}] \subseteq \mathcal{F}$;

Remark 1.5. i) In definition 1.4 we require $\mathcal{F}$ to consist of compactly supported vector fields. We will see in §5 that it is possible to give an equivalent definition, using modules of vector fields which are not necessarily compactly supported but which satisfy the partition of unity property.

ii) $\mathcal{F}$ being locally finitely generated means that for every point of M there is an open neighbourhood $i\colon U \hookrightarrow M$ with the following property: denote $i^*\mathcal{F} := \{X|_U : X \in \mathcal{F} \text{ has support in } U\}$ and $\widehat{i^*\mathcal{F}} := \{X \in \mathfrak{X}(U) : fX \in i^*\mathcal{F} \text{ for all } f \in C^\infty_c(U)\}$. Then there are finitely many $Y_1, \ldots, Y_n$ of $\widehat{i^*\mathcal{F}}$ such that every element of $i^*\mathcal{F}$ is a $C^\infty_c(U)$-linear combination of the Y_j's.

iii) Definition 1.4 satisfies the requirement that regular foliations can be regarded as Stefan-Sussmann singular foliations in a unique way. This is clear from definition 1.1 and remark 1.2 ii), in which a preference for possibility B) in §1.2 (compact support assumption) is made.

iv) The examples discussed right before definition 1.4 show that there is a hierarchy among foliations $(M,\mathcal{F})$. If the module $\mathcal{F}$ is projective, the foliation is called *almost regular*. This case was studied by Debord in [6]; The Serre-Swan theorem forces $\cup_{x\in M}\mathcal{F}_x$ to be a Lie algebroid whose anchor map is induced by the pointwise evaluation map ev_x. It turns out that this anchor map is injective in the dense and open subset of M which consists of the regular points. The almost regular case includes the regular one. The "truly" singular foliations are the ones where $\mathcal{F}$ is not projective.

v) Given a Stefan-Sussmann singular foliation $(M,\mathcal{F})$, the fiber $\mathcal{F}_x := \mathcal{F}/I_x\mathcal{F}$ at some $x \in M$ fits in an exact sequence

$$0 \to \mathfrak{g}_x \to \mathcal{F}_x \xrightarrow{ev_x} T_xL \to 0$$

where L is the leaf at x. The kernel $\mathfrak{g}_x$ is the quotient of $\mathcal{F}(x) = \{\xi \in \mathcal{F} : \xi(x) = 0\}$ by the maximal ideal $I_x\mathcal{F}$. It inherits the Lie bracket of vector fields and becomes a Lie algebra. The singular points of $(M,\mathcal{F})$ are exactly those where the kernel $\mathfrak{g}_x$ does not vanish.

Let us see how to obtain a module $\mathcal{F}$ as in definition 1.4 for some of the previous examples of smooth foliations with dimension-drop. Note that in the last two examples the module is projective.

Example 1.6. i) Let G be a Lie group acting smoothly on a manifold M. If $\mathfrak{g}$ is the Lie algebra of G and $\{X_1, \ldots, X_k\}$ a basis of $\mathfrak{g}$ then every X_i defines a vector field $X^\dagger \in \mathfrak{X}(M)$ (an infinitesimal generator of the action). Take $\mathcal{F}$ to be the $C_c^\infty(M)$-span of $X_1^\dagger, \ldots, X_k^\dagger$.

ii) Let $(M, \{\ ,\ \})$ be a Poisson manifold. Every $f \in C^\infty(M)$ defines a Hamiltonian vector field X_f; take $\mathcal{F}$ the $C_c^\infty(M)$-span of $\{X_f : f \in C^\infty(M)\}$. This $\mathcal{F}$ is locally finitely generated by the Hamiltonians of the coordinate functions, and involutive due to the well known equality $\{X_f, X_g\} = -X_{\{f,g\}}$.

iii) The foliation of $\mathbb{R}$ with a 1-dimensional leaf $(0, +\infty)$ and every non-positive real number as a 0-dimensional leaf: Take $\mathcal{F}$ the $C_c^\infty(M)$-span of $f\partial_x$, where $f \in C^\infty(\mathbb{R})$ is any (fixed) function which vanishes exactly on $(-\infty, 0]$.

iv) The foliation of $\mathbb{R}$ with leaves $(-\infty, 0), \{0\}, (0, +\infty)$: Take $\mathcal{F}^k$ the $C_c^\infty(M)$-span of $x^k\partial_x$ for any $k \in \mathbb{N}$.

Given a smooth foliation as in §1.1, the reader might wonder why we insist on considering modules of vector fields instead of just the partition to leaves. After all, in the regular case the two viewpoints are equivalent. The reason is that when the dimension-drop phenomenon occurs, the leaves alone do not contain enough information about the dynamics of the situation involved. For instance, the orbits of the action of $GL(2,\mathbb{R})$ on $\mathbb{R}^2$ are the same as the orbits of the action of $SL(2,\mathbb{R})$, but obviously the two actions are different. One would like this difference to be recorded, and Stefan-Sussmann singular foliations allow for this. In other words, the notion of Stefan-Sussmann singular foliation is a genuine generalization of the notion of regular foliation, which allows to record more information than the notion of smooth foliation described in §1.1.

At the level of global geometry, the above-mentioned difference is recorded by the holonomy groupoid associated in [2] to any Stefan-Sussmann singular foliation: the holonomy groupoids associated to the two actions are different. Moreover, a very important feature of the holonomy groupoid construction given in [2] is that locally it arises from a very simple and smooth structure, that of a bisubmersion, which makes it appropriate for the development of analytical tools along a Stefan-Sussmann singular foliation (pseudodifferential calculus, analytic index, etc). The introduction of bisubmersions is possible exactly because we view singular foliations as modules of vector fields; we elaborate more on this point in §3.

We return briefly to the relation between regular foliations and Stefan-Sussmann singular foliations, since we did not provide a proof for the equivalence stated just before Def. 1.1. Given a manifold M, there is an canonical injection

$$\{\text{Regular foliations in the sense of §1.1}\} \to \{\text{Stefan-Sussmann singular foliations}\},$$

mapping the regular foliation integrating (in the sense of Frobenius) the involutive distribution D to $\Gamma_c(D)$. The image admits a simple characterization: it consists of the Stefan-Sussmann singular foliations $\mathcal{F}$ with the property that their evaluation at points of M delivers a constant rank distribution D. This characterization follows from this known fact (see [2, Ex. 1.3(2)], whose proof we include for completeness:

Lemma 1.7. *Let $\mathcal{F}$ be a Stefan-Sussmann singular foliation whose evaluation at points of M delivers a constant rank distribution D. Then necessarily $\mathcal{F} = \Gamma_c(D)$.*

Proof. Denote by k the rank of D. First notice that for every point $p \in M$ there exists a subset $\mathbf{Y} := \{Y^1, \ldots, Y^k\} \subset \mathcal{F}$ whose evaluation at p delivers a basis of

D_p. Denote by V the open neighbourhood of p on M on which the set $\mathbf{Y}$ is linearly independent. Take a cover $\{V_\alpha\}_{\alpha\in A}$ of open sets as above. Now fix $X \in \Gamma_c(D)$; we have to show that $X \in \mathcal{F}$. There exist finitely many V_α's which cover $supp(X)$, since the latter is compact. So we may assume that the cover $\{V_\alpha\}_{\alpha\in A}$ is such that only finitely many V_α's intersect $supp(X)$. Let $\{\varphi_\alpha\}_{\alpha\in A}$ be a partition of unity on M such that $supp(\varphi_\alpha) \subset V_\alpha$. For every α, since $\varphi_\alpha X$ is supported on the open subset V_α, there are $h^i_\alpha \in C^\infty(M)$ such that $\varphi_\alpha X = \sum_{i=1}^k h^i_\alpha Y^i_\alpha \in \mathcal{F}$. Hence $X = \sum_{\alpha\in A} \varphi_\alpha X$, being effectively a finite sum of elements of $\mathcal{F}$, lies in $\mathcal{F}$. □

Purpose of this note: Motivated by the case of foliations discussed in this introduction, in §2 we discuss the generalization of the above ideas to a wider class of singular situations, called singular subalgebroids. These objects are formulated using submodules of sections of Lie algebroids and they appear naturally. Exactly as Stefan-Sussmann singular foliations, they give rise to smooth objects encoding them, as we explain in §3.

In the second half of this note we clarify why Stefan-Sussmann singular foliations and singular subalgebroids are defined in terms of compactly supported sections, and give equivalent characterizations. More precisely, we show in theorem 5.1 that there is a bijection between singular subalgebroids and the class of $C^\infty(M)$-submodules which *satisfy the partition of unity property.* The latter will be defined in §4, and for regular foliations this property specializes to possibility A) in §1.2. Hence this question, which we intend to answer in a forthcoming paper, is well-posed:

Compare the holonomy groupoid of Androulidakis-Skandalis [2] *for Stefan-Sussmann singular foliations to the one obtained, by a similar procedure, for submodules which satisfy the partition of unity property (and likewise for singular subalgebroids).*

The above bijection is an intermediate step toward an explicit description of the sheaf naturally associated to Stefan-Sussmann singular foliations. Being defined using compactly supported sections, they naturally give rise to a presheaf, and in §6 (theorem 6.3) we describe the natural sheaf associated to it via the sheafification procedure. The description in terms of sheaves is important because it is the appropriate setting to treat analytic or holomorphic foliations.

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2. Singular subalgebroids

In this section we introduce the notion of a singular subalgebroid and discuss a few motivating examples. To this end, we fix a Lie algebroid A with anchor map $\rho : A \to TM$ (i.e. A is a vector bundle over M, and $\Gamma(A)$ has a Lie algebra structure satisfying a Leibniz rule involving ρ when one of the entries is multiplied with a function on M). We will assume A is integrable and fix an integrating Lie groupoid $\mathcal{G} \rightrightarrows M$ with connected $\mathbf{s}$-fibers, and denote by $\mathbf{s}$ and $\mathbf{t}$ its source and target map respectively. Recall that a Lie groupoid is an extension of the notion of Lie group in which only selected pairs of elements can be multiplied, and that every Lie groupoid induces a Lie algebroid, which can be considered its infinitesimal counterpart.

Definition 2.1. A **singular subalgebroid** of A is an involutive, locally finitely generated $C^\infty(M)$-submodule $\mathcal{B}$ of $\Gamma_c(A)$.

Example 1. *i) The singular subalgebroids of $A = TM$ are exactly the Stefan-Sussmann singular foliations on M defined in definition 1.4 (see* [2]*). Note that TM can be integrated both by the pair groupoid $M \times M$ and by the fundamental groupoid $\Pi(M)$.*

ii) Let $F\colon D \to A$ a morphism of Lie algebroids covering a diffeomorphism between the base manifolds. We do not assume that F has constant rank. Then

$$\mathcal{B} := \{F(\mathbf{d}) : \mathbf{d} \in \Gamma_c(D)\}$$

is a singular subalgebroid of A. For instance, the image of the anchor map $\rho : A \to TM$ is the Stefan-Sussmann singular foliation underlying the Lie algebroid A. Let us point out that the Stefan-Sussmann singular foliations arising from a Lie algebroid in this way are a huge and very interesting class, however there do exist Stefan-Sussmann singular foliations which do not arise from any Lie algebroid [5, Prop. 2.3].

A singular subalgebroid $\mathcal{B}$ of A gives rise to a Stefan-Sussmann singular foliation on the Lie groupoid $\mathcal{G}$, and can be recovered from this Stefan-Sussmann singular foliation. Given a section $\boldsymbol{\alpha} \in \mathcal{B} \subset \Gamma_c(M, \ker(d\mathbf{s}) \mid_M)$, consider the associated right-invariant vector field $\overrightarrow{\boldsymbol{\alpha}} \in \Gamma(\mathcal{G}, \ker(d\mathbf{s})) \subset \mathfrak{X}(\mathcal{G})$ defined by $\overrightarrow{\boldsymbol{\alpha}}_g = (R_g)_*\boldsymbol{\alpha}_{\mathbf{t}(g)}$ for all $g \in \mathcal{G}$ (*cf.* [9, §3.5]). Put

$$\overrightarrow{\mathcal{B}} := \langle\ \{\overrightarrow{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathcal{B}\}\ \rangle, \tag{1}$$

the $C^\infty(\mathcal{G})$-module generated by $\{\overrightarrow{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathcal{B}\}$. Notice that $\text{support}(\overrightarrow{\boldsymbol{\alpha}}) = \mathbf{t}^{-1}(\text{support}(\boldsymbol{\alpha}))$, hence $\overrightarrow{\boldsymbol{\alpha}}$ is not necessarily compactly supported. The module $\overrightarrow{\mathcal{B}}$ is involutive as well as locally finitely generated, although technically it is not a Stefan-Sussmann singular foliation in the sense of definition 1.4 as it does not satisfy the compact support condition. Nevertheless, every $\overrightarrow{\boldsymbol{\alpha}} \in \overrightarrow{\mathcal{B}}$ has a time-1 flow:

Lemma 2.2. *For every section $\boldsymbol{\alpha} \in \mathcal{B}$ the vector field $\overrightarrow{\boldsymbol{\alpha}} \in \overrightarrow{\mathcal{B}}$ is complete.*

Proof. Identifying the anchor map $\rho : A \to TM$ with $d\mathbf{t}|_M : \ker(d\mathbf{s}) \mid_M \to TM$, we get that $\overrightarrow{\boldsymbol{\alpha}}$ is $\mathbf{t}$-related with the vector field $\rho(\boldsymbol{\alpha})$, which has the same support as $\boldsymbol{\alpha}$, whence it is complete. It follows from [9, Thm. 3.6.4] that $\overrightarrow{\boldsymbol{\alpha}}$ is complete as well. ☐

Of course $(\overrightarrow{\mathcal{B}})_c$, the module of compactly supported vector fields in $\overrightarrow{\mathcal{B}}$, is a Stefan-Sussmann singular foliation, but it is more convenient to work with $\overrightarrow{\mathcal{B}}$. Similarly to the above, we introduce the notation $\overleftarrow{\mathcal{B}}$ for the $C^\infty(\mathcal{G})$-submodule of $\Gamma(\mathcal{G}, \ker(d\mathbf{t}))$ generated by the left-invariant vector fields $\overleftarrow{\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha} \in \mathcal{B}$.

3. Bisubmersions and the holonomy groupoid

The integration of singular subalgebroids is dealt with in [4, 3], by modifying the procedure introduced in [2] to integrate Stefan-Sussmann singular foliations. In [2] it is shown that the object of integration for a Stefan-Sussmann singular foliation

$(M,\mathcal{F})$ is a (highly pathological) topological groupoid $H(\mathcal{F}) \rightrightarrows M$ called the *holonomy groupoid*[3] of the foliation. The building blocks for the construction of $H(\mathcal{F})$ are *bisubmersions*. The latter are smooth manifolds endowed with certain smooth maps that "desingularize" the Stefan-Sussmann singular foliation. In this section we explain how the notion of bisubmersion is formulated in the context of singular subalgebroids. We do so to emphasize that, even though singular subalgebroids are defined in terms of modules of sections, they are encoded by differential-geometric objects.

Let us start by recalling the following:

(1) Let $\varphi\colon U \to V$ be a smooth map between manifolds and $X \in \mathfrak{X}(U)$ and $Y \in \mathfrak{X}(V)$. We say that X is **φ-related** to Y iff $\varphi_*(X(p)) = Y(\varphi(p))$ for all $p \in U$.

(2) Let $\mathcal{F}$ be a $C^\infty(V)$-submodule of $\mathfrak{X}(V)$. Define the pullback

$$\varphi^{-1}(\mathcal{F}) := \{X \in \mathfrak{X}_c(U) : d\varphi(X) = \sum_{i=1}^{n} f_i(Y_i \circ \varphi) \text{ for } f_i \in C_c^\infty(U),\, Y_i \in \mathcal{F}\}.$$

Here $d\varphi\colon TU \to \varphi^*(TV)$ is a vector bundle map covering Id_U, where $\varphi^*(TV)$ denotes the pullback vector bundle. $\varphi^{-1}(\mathcal{F})$ is a $C^\infty(U)$-submodule of $\mathfrak{X}_c(U)$.

Remark 3.1. The pullbacks of $\mathcal{F}$ and of the module $\mathcal{F}_c$ of compactly supported elements of $\mathcal{F}$ are the same.

Definition 3.2. Let $\mathcal{B}$ be a singular subalgebroid of A. A **bisubmersion** for $\mathcal{B}$ is a smooth map $\varphi\colon U \to \mathcal{G}$, where U is a manifold, such that

(1) $\mathbf{s}_U := \mathbf{s} \circ \varphi$ and $\mathbf{t}_U := \mathbf{t} \circ \varphi\colon U \to M$ are submersions,

(2) for every $\boldsymbol{\alpha} \in \mathcal{B}$, there is $Z \in \mathfrak{X}(U)$ which is φ-related to $\overrightarrow{\boldsymbol{\alpha}}$ and $W \in \mathfrak{X}(U)$ which is φ-related to $\overleftarrow{\boldsymbol{\alpha}}$,

(3) $\varphi^{-1}(\overrightarrow{\mathcal{B}}) = \Gamma_c(U, \ker d\mathbf{s}_U)$ and $\varphi^{-1}(\overleftarrow{\mathcal{B}}) = \Gamma_c(U, \ker d\mathbf{t}_U)$.

We denote a bisubmersion of $\mathcal{B}$ by $(U, \varphi, \mathcal{G})$.

Now let us give an overview of some examples of bisubmersions, which are elaborated further in [4].

Example 2. *i) Let $(M,\mathcal{F})$ be a Stefan-Sussmann singular foliation. Recall from* [2] *that a bisubmersion of $(M,\mathcal{F})$ is a triple $(U, \mathbf{t}_U, \mathbf{s}_U)$ consisting of a manifold U with two submersions $\mathbf{t}_U$ and $\mathbf{s}_U$ to M, such that*

$$\mathbf{t}_U^{-1}(\mathcal{F}) = \mathbf{s}_U^{-1}(\mathcal{F}) = \Gamma_c(U, \ker d\mathbf{t}_U) + \Gamma_c(U, \ker d\mathbf{s}_U). \tag{2}$$

Such a triple is equivalent to a bisubmersion in the sense of definition 3.2 once we choose the pair groupoid $M \times M$ as an integration of TM; then put φ the map $(\mathbf{t}_U, \mathbf{s}_U) : U \to M \times M$

ii) The class of path-holonomy[4] *bisubmersions can be constructed explicitly for any singular subalgebroid $\mathcal{B}$ of A. Let $x \in M$ and put $I_x \subset C^\infty(M)$ the functions which vanish at x. Let $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n \in \mathcal{B}$ such that $[\boldsymbol{\alpha}_1], \ldots, [\boldsymbol{\alpha}_n]$ span $\mathcal{B}/I_x\mathcal{B}$. The associated path-holonomy bisubmersion is the map*

$$\varphi\colon U \to \mathcal{G}, (\lambda, y) \mapsto \exp_y \sum \lambda_i \overrightarrow{\boldsymbol{\alpha}}_i,$$

[3]For regular foliations $H(\mathcal{F})$ coincides with the usual holonomy groupoid. The same happens in the case of almost regular foliations studied by Debord in [6].

[4]We call a bisubmersion as above *minimal* if $[\boldsymbol{\alpha}_1], \ldots, [\boldsymbol{\alpha}_n]$ are a basis of $\mathcal{B}/I_x\mathcal{B}$.

where U is a neighborhood of $(0,x)$ in $\mathbb{R}^n \times M$. Recall that the time-1 flow of $\vec{\boldsymbol{\alpha}}_i$ is defined due to Lemma 2.2.

The explicit construction of the holonomy groupoid $H(\mathcal{B}) \rightrightarrows M$ is given in [4]. We will not be concerned with this groupoid in this note, nevertheless let us briefly overview that $H(\mathcal{B})$ is a quotient of the collection (*atlas*) of minimal path-holonomy bisubmersions arising by applying the above construction at every $x \in M$. The groupoid structure is the one arising by the following operations on bisubmersions:

- The inverse of $(U, \phi, \mathcal{G})$ is $i \circ \phi : U \to \mathcal{G}$, where $i : \mathcal{G} \to \mathcal{G}$ is the groupoid inversion;
- The product of $(U_j, \phi_j, \mathcal{G})$, for $j = 1, 2$ is $(U_1 \times_{\mathbf{s},\mathbf{t}} U_2, m \circ (\phi_1 \times \phi_2), \mathcal{G})$ where $m : \mathcal{G} \times_{\mathbf{s},\mathbf{t}} \mathcal{G} \to \mathcal{G}$ is the groupoid multiplication.

4. Global hulls

We defined singular subalgebroids in definition 2.1 using compactly supported sections. A singular subalgebroid immediately gives rise to a presheaf, and the sheafification procedure associates to it canonically a *sheaf.* In the remainder of this note we describe explicitly this sheaf. As an intermediate step, of independent geometric interest, we show that one might work by replacing the compact support condition with a "partition of unity" condition (see definition 4.3), and that doing so leads to an equivalent theory. We carry this out for submodules of any vector bundle, not necessarily a Lie algebroid.

Let us make a fresh start and fix a vector bundle A over M. Associated to any submodule of $\Gamma_c(A)$ there is a canonical natural submodule of $\Gamma(A)$, which contains non-compactly supported sections too when M is not compact. It was already defined in[5] [2, §1.1]:

Definition 4.1. For any submodule $\mathcal{E}$ of $\Gamma_c(A)$, the **global hull** of $\mathcal{E}$ is the following $C^\infty(M)$-submodule of $\Gamma(A)$:

$$\widehat{\mathcal{E}} := \{\boldsymbol{\alpha} \in \Gamma(A) : f\boldsymbol{\alpha} \in \mathcal{E} \text{ for all } f \in C^\infty_c(M)\}.$$

Clearly $\mathcal{E} \subset \widehat{\mathcal{E}}$. In proposition 5.1 we will see that $\mathcal{E}$ consists exactly of the compactly supported sections of $\widehat{\mathcal{E}}$.

Remark 4.2. Def. 4.1 extends mutatis mutandis to the case when $\mathcal{E}$ is a submodule of $\Gamma(A)$ rather than of $\Gamma_c(A)$. We will use this in §6.

Example 3. *i) If $\mathcal{E} = \Gamma_c(A)$ then obviously $\widehat{\mathcal{E}} = \Gamma(A)$.*

ii) More generally, if $F \colon D \to A$ is vector bundle map covering the identity on M and $\mathcal{E} = F(\Gamma_c(D))$, then we have $\widehat{\mathcal{E}} = F(\Gamma(D))$. The inclusion "$\supset$" is clear. For the other inclusion, choose a partition of unity $\{\varphi_i\}_{i \in I}$ on M by compactly supported functions. Given $a \in \widehat{\mathcal{E}}$, we have $a = \sum_{i\in I} \varphi_i a$ and each summand, lying in $\mathcal{E}$, is of the form $F(\boldsymbol{d}_i)$ for some $\boldsymbol{d}_i \in \Gamma_c(B)$. So $a = F(\sum_{i\in I} \boldsymbol{d}_i)$.

iii) Let $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n \in \Gamma(A)$. Consider the submodule $\mathcal{E}$ of $\Gamma_c(A)$ consisting of the finite $C^\infty_c(M)$-linear combinations of the $\boldsymbol{\alpha}_k$'s. Then

$$\widehat{\mathcal{E}} = \left\{ \sum_{k=1}^{n} f_k \boldsymbol{\alpha}_k : f_k \in C^\infty(M) \right\},$$

[5] There it was called "submodule of global sections".

i.e. $\widehat{\mathcal{E}}$ *consists of the finite* $C^\infty(M)$*-linear combinations of the* $\boldsymbol{\alpha}_k$*'s. The inclusion "* $\supset$ *" holds by definition 4.1. To show the other inclusion choose a partition of unity* $\{\varphi_i\}_{i\in I}$ *by functions with compact support. Then for any* $\boldsymbol{\alpha} \in \widehat{\mathcal{E}}$ *we have* $\boldsymbol{\alpha} = \sum_{i\in I} \varphi_i \boldsymbol{\alpha}$*, and every summand* $\varphi_i\boldsymbol{\alpha}$ *lies in* $\mathcal{E}$*, so it can be written as* $\sum_k f^i_k \boldsymbol{\alpha}_k$ *where* $f^i_k \in C^\infty_c(M)$*. Further, we may arrange that* $supp(f^i_k) \subset supp(\varphi_i)$*, hence* $F_k := \sum_{i\in I} f^i_k$ *is well-defined for every* k *since* $\{supp(\varphi_i)\}_{i\in I}$ *is locally finite. Therefore* $\boldsymbol{\alpha} = \sum_k F_k \boldsymbol{\alpha}_k$ *with* $F_k \in C^\infty(M)$*.*

Global hulls satisfy a property, which we now introduce. Recall [13, §13.2] that a partition of unity on a manifold M is a family of functions $\varphi_i \colon M \to \mathbb{R}_{\geq 0}$ such that

(1) $\{supp(\varphi_i)\}_{i\in I}$ is locally finite, *i.e.* , any point of M has a neighborhood that meets finitely many of the $supp(\varphi_i)$,
(2) $\sum_{i\in I} \varphi_i = 1$.

Definition 4.3. We say that a submodule $\mathcal{M}$ of $\Gamma(A)$ satisfies the **partition of unity property** iff the following is satisfied: for any partition of unity $\{\varphi_i\}_{i\in I}$,

for any family $\{a_i\}_{i\in I}$ of elements of $\mathcal{M}$, the sum $\sum_{i\in I} \varphi_i a_i$ lies in $\mathcal{M}$. $(\star)$

Remark 4.4. Given $\mathcal{M}$, the *existence* of a partition of unity $\{\varphi_i\}_{i\in I}$ by functions with *compact support* satisfying $(\star)$ implies that $\mathcal{M}$ satisfies the partition of unity property.

Indeed, given any partition of unity $\{\phi_j\}_{j\in J}$ and any family $\{a_j\}_{j\in J}$ of elements of $\mathcal{M}$, we have

$$\sum_{j\in J} \phi_j a_j = \sum_{i\in I} (\varphi_i \sum_{j\in J} \phi_j a_j). \tag{3}$$

For each i, the sum $\sum_{j\in J} \phi_j a_j$ can be replaced by the sum over the finite subset of indices j such that $supp(\phi_j)$ intersects the compact set $supp(\varphi_i)$. The sum, being finite, lies in $\mathcal{M}$, hence $\{\varphi_i\}_{i\in I}$ satisfying $(\star)$ implies that (3) lies in $\mathcal{M}$.

Here we used the fact that, for any compact $K \subset M$, the subset of indices j such that $supp(\phi_j) \cap K \neq \emptyset$ is finite. To prove this, cover K by finitely many open sets, each of which meets finitely many of the $supp(\phi_j)$. This is possible since $\{supp(\phi_j)\}_{j\in J}$ is locally finite.

Example 4. *An example of submodule* $\mathcal{M}$ *of* $\Gamma(A)$ *which* does not *satisfy the partition of unity property is the following. Let* $X \in \Gamma(A)$ *be a section with non-compact support and* $\mathcal{M} = C^\infty_c(M)X$*. Let* $\{\varphi_i\}_{i\in I}$ *be a partition of unity by functions with compact support. In particular,* $\varphi_i X \in \mathcal{M}$ *for every* i*. Then* $\sum_{i\in I} \varphi_i(\varphi_i X) \in \Gamma(A)$ *has non-compact support (since the function* $\sum_{i\in I} \varphi_i^2$ *is no-where vanishing), hence it does not belong to* $\mathcal{M}$*.*

To obtain an example where the submodule does not consist of compactly supported sections, we can take a variation of the above: on $M = \mathbb{R}$*, take the* $C^\infty(M)$*-submodule of* $\mathfrak{X}(M)$ *generated by* $C^\infty_c(M)\frac{\partial}{\partial x}$ *and* $C^\infty(M)x\frac{\partial}{\partial x}$*.*

Proposition 1. *For any submodule* $\mathcal{E}$ *of* $\Gamma_c(A)$*, the global hull* $\widehat{\mathcal{E}}$ *satisfies the partition of unity property.*

We will show in remark 5.2 that $\widehat{\mathcal{E}}$ is the smallest submodule containing $\mathcal{E}$ among those satisfying the partition of unity property.

Proof. Take a partition of unity $\{\varphi_i\}_{i\in I}$, and a family $\{a_i\}_{i\in I}$ of elements of $\widehat{\mathcal{E}}$. For any $f \in C_c^\infty(M)$, only finitely many $i \in I$ satisfy $supp(f) \cap supp(\varphi_i) \neq \emptyset$ (see the last paragraph of remark 4.4). Hence the sum $f(\sum_{i\in I} \varphi_i a_i) = \sum_{i\in I} f\varphi_i a_i$ is a finite sum. Each summand $f\varphi_i a_i$ lies in $\mathcal{E}$ since $f\varphi_i \in C_c^\infty(M)$, so the whole sum lies in $\mathcal{E}$. Therefore $\sum_{i\in I} \varphi_i a_i \in \widehat{\mathcal{E}}$. □

We end this section introducing an operation which, contrary to the global hull (definition 4.1), takes arbitrary submodules to compactly supported submodules.

Definition 4.5. Given any submodule $\mathcal{M}$ of $\Gamma(A)$, the **submodule of compactly supported sections** is

$$\mathcal{M}_c := \{fa : f \in C_c^\infty(M), a \in \mathcal{M}\} \subset \Gamma_c(A).$$

Remark 4.6. Notice that

$$\mathcal{M}_c = \{\text{compactly supported sections of } \mathcal{M}\}.$$

Indeed, the inclusion "$\subset$" is obvious, and if a is a compactly supported section of $\mathcal{M}$, choosing $\chi \in C_c^\infty(M)$ with $\chi|_{supp(a)} = 1$, we have $a = \chi a \in \mathcal{M}_c$. This justifies the name given in Def. 4.5 and shows that $\mathcal{M}_c$ is a submodule of $\Gamma_c(A)$.

5. The bijection between submodules of compactly supported sections and their global hulls

In this section we show that passing from a submodule of compactly supported sections $\mathcal{E}$ to its global hull $\widehat{\mathcal{E}}$ we do not lose any information. Hence one can choose freely with which submodule to work. This applies in particular to singular subalgebroids of a Lie algebroid.

Theorem 5.1. *Let A be a vector bundle. Let $\mathcal{SUBMOD}_c$ be the collection of submodules of $\Gamma_c(A)$ and $\mathcal{SUBMOD}_{pu}$ the collection of submodules of $\Gamma(A)$ satisfying the partition of unity property (definition 4.3).*

1. *The map $\mathcal{SUBMOD}_c \to \mathcal{SUBMOD}_{pu}$, $\mathcal{E} \mapsto \widehat{\mathcal{E}}$ is a bijection. The inverse map is $\mathcal{M} \mapsto \mathcal{M}_c$.*
2. *$\mathcal{E}$ is locally finitely generated iff $\widehat{\mathcal{E}}$ is.*
3. *When A is a Lie algebroud: $\mathcal{E}$ is involutive iff $\widehat{\mathcal{E}}$ is.*

Proof. (1) Let $\mathcal{E}$ be a submodule of $\Gamma_c(A)$. Then $(\widehat{\mathcal{E}})_c = \mathcal{E}$: the inclusion "$\subset$" is clear by the definition of $\widehat{\mathcal{E}}$ (definition 4.1), the opposite inclusion holds by Remark 4.6, since $\mathcal{E}$ is contained in $\widehat{\mathcal{E}}$ and $\mathcal{E}$ consists of compactly supported sections.

Let $\mathcal{M}$ be a submodule of $\Gamma(A)$ satisfying the partition of unity property. Then $\widehat{(\mathcal{M}_c)} = \mathcal{M}$. The inclusion "$\supset$" is clear by the definition of $\mathcal{M}_c$ (definition 4.5). For the other inclusion, take a partition of unity $\{\varphi_i\}_{i\in I}$ on M by functions with compact support, and for every $i \in I$ take $\chi_i \in C_c^\infty(M)$ with $\chi_i|_{supp(\varphi_i)} = 1$, *i.e.* so that $\varphi_i = \chi_i\varphi_i$. Take an element of $\widehat{(\mathcal{M}_c)}$, i.e. $a \in \Gamma(A)$ such that $fa \in \mathcal{M}_c$ for all $f \in C_c^\infty(M)$. Then $a = (\sum_{i\in I} \varphi_i)a = \sum_{i\in I} \varphi_i(\chi_i a)$. Since $\chi_i a \in \mathcal{M}_c \subset \mathcal{M}$, and since $\mathcal{M}$ satisfies the partition of unity property, the whole sum lies in $\mathcal{M}$, *i.e.* $a \in \mathcal{M}$.

(2) "⇒": Let $a \in \widehat{\mathcal{E}}$. Since $\mathcal{E}$ is locally finitely generated (see Remark 1.5), for any $x \in M$ there are a neighborhood U of x in M and finitely many $a_j \in \widehat{i^*\mathcal{E}}$ such that, for any $g \in C^\infty_c(U)$, the element $ga \in i^*\mathcal{E}$ can be written as $\sum_j g_j a_j$ where $g_j \in C^\infty_c(U)$. Choose g so that it equals one in a neighborhood $V \subset U$ of x. Then $a|_V = \sum_j g_j|_V a_j|_V$. Since $a_j \in \widehat{i^*\mathcal{E}}$ and V is relatively compact, $a_j|_V$ is the restriction to an element of $\mathcal{E}$, and this shows that $\widehat{\mathcal{E}}$ is locally finitely generated.

"⇐": Let $a \in \mathcal{E}$ and $x \in M$. Since in particular $a \in \widehat{\mathcal{E}}$, there are a neighborhood U of x and finitely many $a_j \in \widehat{\mathcal{E}}$ such that $a|_U = \sum_j f_j a_j|_U$ with $f_j \in C^\infty(U)$. For any $g \in C^\infty_c(U)$ we have $ga|_U = \sum_j (gf_j)a_j|_U$. Notice that $gf_j \in C^\infty_c(U)$ and $a_j|_U \in \widehat{i^*\mathcal{E}}$, showing that $\mathcal{E}$ is locally finitely generated.

(3) "⇒": let $a, b \in \widehat{\mathcal{E}}$. We want to show that for all $f \in C^\infty_c(M)$ one has $f[a,b] \in \mathcal{E}$. Let $\chi \in C^\infty_c(M)$ with $\chi|_{supp(f)} = 1$, so that $f\chi = f$. We know by assumption that the bracket $[fa, \chi b]$ lies in $\mathcal{E}$, and applying the Leibniz rule we can write it as $f[a,b]$ plus $C^\infty_c(M)$-multiples of a and b. Hence $f[a,b] \in \mathcal{E}$.

"⇐": follows since, by Remark 4.6, $\mathcal{E}$ consists exactly of the compactly supported sections of $\widehat{\mathcal{E}}$.

□

Remark 5.2. Let $\mathcal{E}$ be any $C^\infty(M)$-submodule of $\Gamma_c(A)$. We can now show that $\widehat{\mathcal{E}}$ is the smallest submodule containing $\mathcal{E}$ among those satisfying the partition of unity property. Indeed, let $\mathcal{M}$ be a submodule of $\Gamma(A)$ satisfying the partition of unity property with $\mathcal{E} \subset \mathcal{M}$. Then $\mathcal{E} \subset \mathcal{M} \cap \Gamma_c(A) = \mathcal{M}_c$. Taking global hulls on both sides we obtain $\widehat{\mathcal{E}} \subset \widehat{\mathcal{M}_c} = \mathcal{M}$, where the last equality holds by Thm. 5.1 (1).

6. Associated sheaves

In this section we will associate canonically a sheaf to any submodule of $\Gamma_c(A)$, hence in particular to any singular subalgebroid and Stefan-Sussmann singular foliation. The explicit description of the sheaf makes use of the global hull introduced in §4.

Recall (see for example [8, Appendix B]) that a *presheaf* $\mathcal{P}$ of C^∞-modules on M consists of a $C^\infty(U)$-module $\mathcal{P}(U)$ for every open $U \subset M$, and of a module homomorphism $r_{U,V} : \mathcal{P}(V) \to \mathcal{P}(U)$ (called restriction) for any pair of open subsets $U \subset V$ of M, such that $r_{U,U} = Id_{\mathcal{P}(U)}$ and $r_{U,W} = r_{U,V} \circ r_{V,W}$ whenever $U \subset V \subset W$.

$\mathcal{P}$ is a *sheaf* if additionally it satisfies, for any collection of open subsets $\{U_i\}_{i\in I}$ of M (denoting $U := \cup_{i\in I} U_i$):

(1) existence of gluing: given $f_i \in \mathcal{P}(U_i)$ for all $i \in I$ such that $r_{U_i\cap U_j, U_i}(f_i) = r_{U_i\cap U_j, U_j}(f_j)$ for all $i, j \in I$, there exists $f \in \mathcal{P}(U)$ such that $r_{U_i,U}(f) = f_i$ for all i,

(2) uniqueness of gluing: if $f, g \in \mathcal{P}(U)$ satisfy $r_{U_i,U}(f) = r_{U_i,U}(g)$ for all i, then $f = g$.

Let A be a vector bundle over M. As earlier, let $\mathcal{E}$ be a $C^\infty(M)$-submodule of $\Gamma_c(A)$. It gives rise to a presheaf of C^∞-modules on M, which we also denote by $\mathcal{E}$, which by definition associates to any open $U \subset M$ the following[6] $C^\infty(U)$-module:

$$\mathcal{E}(U) := C^\infty(U)\{a|_U : a \in \mathcal{E}\}. \tag{4}$$

Clearly $\mathcal{E}$ is not a sheaf in general, since "glueing" compactly supported sections of A one might obtain a section with non-compact support.

Proposition 2. *Let $\mathcal{E}$ be any $C^\infty(M)$-submodule of $\Gamma_c(A)$. Then the presheaf $\mathcal{S}$ defined by*

$$\mathcal{S}(U) := \widehat{\mathcal{E}(U)}$$

is a sheaf.

Remark 6.1. Recall that $\widehat{\mathcal{E}(U)}$ is defined in Def. 4.1 and Remark 4.2. Explicitly,

$$\widehat{\mathcal{E}(U)} = \{\boldsymbol{\alpha} \in \Gamma(A|_U) : f\boldsymbol{\alpha} \in \mathcal{E} \text{ for all } f \in C_c^\infty(U)\},$$

where $f\boldsymbol{\alpha}$ is viewed as an element of $\Gamma_c(A)$ which vanishes on $M \setminus U$.

Proof. $\mathcal{S}$ is clearly a presheaf: given open subsets $U \subset V$, the restriction of sections of A gives a well-defined map $r_{U,V} : \widehat{\mathcal{E}(V)} \to \widehat{\mathcal{E}(U)}$. We have to show that $\mathcal{S}$ satisfies the existence and uniqueness property for glueing required in the definition of sheaf. The uniqueness property is satisfied as $\Gamma(A)$ defines a sheaf (via $U \mapsto \Gamma(A|_U)$).

For the existence property, let $U = \cup_{i\in I} U_i$ be an arbitrary union of open subsets of M, and for every $i \in I$ take $a_i \in \widehat{\mathcal{E}(U_i)}$ so that they agree on double overlaps $U_i \cap U_j$. Since $\Gamma(A)$ defines a sheaf, there is a unique $a \in \Gamma(A|_U)$ such that $a|_{U_i} = a_i|_{U_i}$ for all $i \in I$. We have to show that $a \in \widehat{\mathcal{E}(U)}$, *i.e.* , for any $f \in C_c^\infty(U)$ we have to show that $fa \in \mathcal{E}(U)$.

As *supp*(f) is compact, we can select a *finite* subset $K \subset I$ such that *supp*$(f) \subset \cup_{i\in K} U_i$. There exists a partition of unity $\{\varphi_i\}_{i\in K}$ on $\cup_{i\in K} U_i$ such that *supp*$(\varphi_i) \subset U_i$ for all $i \in K$ [13, Thm. 13.7]. Since $f = f \sum_{i\in K} \varphi_i$, we have

$$fa = \sum_{i\in K} f\varphi_i a = \sum_{i\in K} f\varphi_i a_i,$$

using in the second equality that $\varphi_i a = \varphi_i a_i$ since *supp*$(\varphi_i) \subset U_i$ and $a|_{U_i} = a_i|_{U_i}$. Now the support of $f\varphi_i$ is compact (since the same holds for the support of f) and contained in U_i, so $f\varphi_i \in C_c^\infty(U_i)$. Together with $a_i \in \widehat{\mathcal{E}(U_i)}$, this implies that $f\varphi_i a_i$ lies in $\mathcal{E}(U_i)$, and its trivial extension to a section on U lies in $\mathcal{E}(U)$. Hence fa, which is given by the finite sum above, lies in $\mathcal{E}(U)$. □

Example 6.2. Let $\mathcal{E} = \Gamma_c(A)$. For every open subset U we have $\widehat{\mathcal{E}(U)} = \Gamma(A|_U)$.

Recall [8, Def. B.0.25][10, III 1] that given a presheaf $\mathcal{P}$, there is a smallest sheaf containing $\mathcal{P}$, namely the *sheafification* of $\mathcal{P}$, denoted $\mathcal{P}^+$. To every open subset U it associates

$$\begin{aligned}\mathcal{P}^+(U) :=& \{s : U \to \cup_{x\in U} \mathcal{P}_x \text{ with } s(x) \in \mathcal{P}_x \text{ such that} \\ & \qquad \forall x \in U\ \exists \text{ open } V \subset U,\ \exists t \in \mathcal{P}(V) \text{ with } s(y) = t_y \text{ for all } y \in V\}.\end{aligned}$$

Here $\mathcal{P}_x$ denotes the *stalk* of $\mathcal{P}$ at x, consisting of the germs of elements of $\mathcal{P}(W)$ as W varies among the open neighbourhoods of x in M. Further, $t_y \in \mathcal{P}_y$ denotes the germ at y of $t \in \mathcal{P}(V)$.

[6]Notice that if instead to U we associated $\{f \cdot a|_U : f \in C_c^\infty(U), a \in \mathcal{E}\}$, we would not obtain a presheaf, since the restricton maps $r_{U,V}$ would not be well-defined.

Theorem 6.3. *Let $\mathcal{E}$ the presheaf defined in eq.* (4)*. Its sheafification $\mathcal{E}^+$ is the sheaf $\mathcal{S}$ defined in proposition 2.*

Proof. It suffices to prove that $\mathcal{S} \subset \mathcal{E}^+$, because $\mathcal{E}^+$ is the smallest sheaf containing $\mathcal{E}$ and the sheaf $\mathcal{S}$ contains $\mathcal{E}$. Fix an open subset U of M. Any $b \in \mathcal{S}(U) = \widehat{\mathcal{E}(U)}$ gives rise to an element of $\mathcal{E}^+(U)$, namely $s\colon U \to \cup_{x \in U} \mathcal{E}_x$ where $s(x) = b_x$, the germ of b at x. The element s is well-defined: $s(x) \in \mathcal{E}_x$, since it agrees with the germ of an element of $\mathcal{E}$, namely the multiplication of b with any compactly supported function which is 1 in a neighbourhood of x. The element s really belongs to $\mathcal{E}^+(U)$: given $x \in U$, one can choose V to be an open set with compact closure and take $t \in \mathcal{E}(V)$ to be $b|_V$. This provides an embedding of $\widehat{\mathcal{E}(U)}$ in $\mathcal{E}^+(U)$, finishing the proof. □

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