

Smoothness of holonomy covers for singular foliations and essential isotropy ¹

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Abstract

Given a singular foliation, we attach an “essential isotropy” group to each of its leaves, and show that its discreteness is the integrability obstruction of a natural Lie algebroid over the leaf. We show that a condition ensuring discreteness is the local surjectivity of a transversal exponential map associated with the maximal ideal of vector fields prescribed to be tangent to the foliation.

The essential isotropy group is also shown to control the smoothness of the holonomy cover of the leaf (the associated fiber of the holonomy groupoid), as well as the smoothness of the associated isotropy group. Namely, the (topological) closeness of the essential isotropy group is a necessary and sufficient condition for the holonomy cover to be a smooth (finite-dimensional) manifold and the isotropy group to be a Lie group.

These results are useful towards understanding the normal form of a singular foliation around a compact leaf. At the end of this article we briefly outline work of ours on this normal form, to be presented in a subsequent paper.

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Introduction

Background and motivations.

An important tool for the study of regular foliations is the notion of holonomy. In particular, the holonomy groupoid of a regular foliation is the starting point for many constructions in foliation theory. In this paper we will consider the much larger class of *singular* foliations, by which we mean a choice of $C^\infty(M)$ -module \mathcal{F} of the vector fields of M , which is locally finitely generated and stable by the Lie bracket². Recall that the holonomy of singular foliations was first studied by Dazord in [Da84], [Da85]. For a particular case of “quasi-regular” foliations Debord [De1] [De2] constructed the holonomy groupoid and showed that it is smooth.

A different construction of the holonomy groupoid for *any* singular foliation was given by the first author and Skandalis in [AnSk06]. It realizes the groupoid as a quotient of certain (smooth) covers of its open sets (“bi-submersions”). It is a topological groupoid, and its topology is usually quite “bad”, reflecting the fact that it arises from a singular foliation. The key feature of this construction, distinguishing it from the previous ones, is that the holonomy covers of a leaf that it induces keep track of the dynamics of the foliation, namely the choice of module \mathcal{F} , rather than the partition of M to leaves (note that in the singular case it is possible for two different modules of vector fields as above to induce the same partition to leaves). The construction of [AnSk06] was motivated by and allows the generalization of Noncommutative Geometry results to the singular case: In [AnSk09] a longitudinal pseudodifferential calculus was developed, and in [AnSk10] its associated index theory was given.

In the current article, motivated by questions on the topology of a singular foliation, we prove a few smoothness results about the holonomy groupoid. The main topological invariant we would like to understand eventually is the stability of the foliation around a compact leaf, and its relation with the dynamics of the foliation. Namely to generalize the *local Reeb stability theorem* (see for instance [MoMr03]) to the singular case in a way that keeps track of the choice of module \mathcal{F} . At the end of this article we outline the relation of the smoothness of the holonomy cover with stability around a compact leaf in the previous sense, as well as work of ours on the latter, which is presented in a separate paper [AZ2].

²This definition arises naturally from the work of Stefan [St74] and Sussmann [Su73].

Other topological invariants we would like to eventually understand for singular foliations, and their relation to the smoothness are: 1) Characteristic classes; in the regular case the crucial ingredient for the formulation of characteristic classes is the action of the holonomy groupoid on the normal bundle of the foliation (see [CrMo04]). In the singular case this action cannot even be written down, as neither the holonomy groupoid nor the normal bundle are smooth. The development of characteristic classes will allow calculations for the longitudinal analytic index. 2) In Noncommutative Geometry, having the smoothness of the holonomy covers simplifies a great deal the pseudodifferential calculus and index theory developed in [AnSk09] and [AnSk10], as it allows to define the left-regular representations (although, as it was shown in these papers, the existence of left-regular representations is not necessary for the theory to work).

Results.

The article at hand is the first part of our effort to understand the previous questions on the topology of singular foliations. We determine conditions for the smoothness of the holonomy cover of the leaf L_x at a point x , which is just the source-fiber H_x of the holonomy groupoid constructed in [AnSk06], and of the restriction H_{L_x} of the holonomy groupoid to L_x . This issue was discussed briefly already in [AnSk06, Rem. 3.13], where it was suggested that the smoothness is related with the integrability obstruction of a certain Lie algebroid attached to the leaf. More precisely, we show:

- The smoothness of H_{L_x} and H_x is controlled by a certain “essential isotropy” group attached to the leaf L_x . Precisely, H_{L_x} and H_x are smooth if and only if the essential isotropy group of L_x is closed (Thm. 3.16 and Prop. 3.17). This is always satisfied for regular leaves.
- There is a canonical transitive Lie algebroid A_{L_x} attached to the leaf L_x . It integrates to H_{L_x} if and only if the essential isotropy group is discrete (Thm. 4.1).

Recall that in [AnSk06, §3.3] a question was posed regarding the relation between the smoothness of H_x and the Crainic-Fernandes obstruction to the integrability of A_{L_x} [CrF]. This relation should be codified in some relation between our essential isotropy group and the monodromy group of A_{L_x} . The full clarification of such a relation is an issue we hope to clarify in a different paper. In Remark 4.3 we examine the case when the essential isotropy is discrete (then it is automatically central): It turns out that then the monodromy group lies inside our essential isotropy group.

Although it seems, from studying examples, that the essential isotropy groups may well always be discrete, we could not provide a concrete proof for this. However, we are able to present a condition which implies the discreteness of the essential isotropy groups. Choose a point $x \in M$, a transversal S_x to the leaf L_x , and consider the infinite-dimensional Lie algebra $I_x \mathcal{F}_{S_x}$, where \mathcal{F}_{S_x} is the restriction of \mathcal{F} to the transversal and I_x is the collection of functions on S_x which vanish at x . Then (Thm. 4.9):

- Assume that for any smooth time-dependent vector field $\{X_t\}_{t \in [0,1]}$ in $I_x \mathcal{F}_{S_x}$, there exists a vector field $Z \in I_x \mathcal{F}_{S_x}$ and a neighborhood S' of x in S_x such that $\exp(Z)|_{S'} = \phi|_{S'}$, where ϕ denotes the time-1 flow of $\{X_t\}_{t \in [0,1]}$. Then the essential isotropy group of L_x is discrete.

(This condition can be interpreted in terms of local surjectivity of the exponential map of $I_x \mathcal{F}_{S_x}$.)

The Lie algebra $I_x \mathcal{F}_{S_x}$ turns out to be a crucial ingredient in the understanding of the transversal action of the holonomy groupoid. This action, together with the smoothness results achieved in this note, provides the model for a generalization of the local Reeb stability theorem to singular foliations. This paper ends with an outline of these results, which will appear in [AZ2].

As a side-result of our investigations in this paper, we provide a negative answer to the old question of whether every singular foliation arises from a Lie algebroid. Indeed in Prop. 1.3 we show:

- Not all singular foliations arise from Lie algebroids.

Organization of the paper. In §1, §2 we recall the construction of the holonomy groupoid from [AnSk06] and prove certain technical results that we will need later in the sequel. We also discuss bi-submersions somewhat more deeply, showing that they provide an equivalent definition of a foliation. §3 introduces the essential isotropy groups and proves that H_x is smooth iff the associated essential isotropy group is closed. In §4 we study the transitive Lie algebroid A_L associated to a leaf L and prove that the discreteness of the essential isotropy group attached to L is the integrability obstruction of A_L . In the same section we discuss the condition on $I_x\mathcal{F}_{S_x}$ mentioned above. In §5 we outline the results of the forthcoming paper [AZ2].

Notation. Given a manifold M , we use $\mathfrak{X}(M)$ to denote its vector fields, and $\mathfrak{X}_c(M)$ its vector fields with compact support. For a vector field X and $x \in M$, we use $\exp_x(X) \in M$ to denote the time-one flow of X applied to x . By \mathcal{F} we will always denote a singular foliation on M and L a leaf. If $X \in \mathcal{F}$, we use $[X]$ to denote the class $X \bmod I_x\mathcal{F}$ (here $x \in M$). The notation $\langle X \rangle$ is used to denote either classes under several other equivalence relations or the foliation generated by X . Given a vector bundle $E \rightarrow M$, we denote by $C^\infty(M; E)$ its space of smooth sections. The holonomy groupoid of a singular foliation is denoted by H . Further, $H_x = \mathbf{s}^{-1}(x)$ is the source fiber and $H_x^x = \mathbf{s}^{-1}(x) \cap \mathbf{t}^{-1}(x)$ the isotropy group at x ; the same notation applies to bi-submersions U, W, \dots

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1 Foliations and the transitive Lie algebroid over a leaf

In this section we review singular foliations from [AnSk06] and construct explicitly a transitive Lie algebroid A_L over a leaf L . Namely, in §1.1 we recall the definition of a singular foliation adding some clarifications and several examples. Then, §1.2 provides an alternative version of [AnSk06, Prop. 1.12] which will be useful to us in this sequel, and §1.3 gives the construction of A_L .

1.1 Foliations

Let M be a smooth manifold. Given a vector bundle $E \rightarrow M$, we denote by $C_c^\infty(M; E)$ the $C^\infty(M)$ -module of compactly supported smooth sections of E .

- a) A (*singular*) *foliation* on M is a locally finitely generated submodule \mathcal{F} of the $C^\infty(M)$ -module $\mathfrak{X}_c(M) = C_c^\infty(M; TM)$, stable by the Lie bracket.

Recall from [AnSk06, §1.1, item 4] that the restriction of \mathcal{F} to an open $U \subseteq M$ is the submodule of $C^\infty(U; T_U M)$ generated by $f \cdot \xi|_U$, where $f \in C_c^\infty(U)$ and $\xi|_U$ is the restriction of a vector field $\xi \in \mathcal{F}$ to U . In this sequel we will denote the restriction of \mathcal{F} to an open U by \mathcal{F}_U .

Given this, the module \mathcal{F} being locally finitely generated (*cf.* [AnSk06, §1.1, items 3,5]) means that for every point of M there exists an open neighborhood U and a finite number of vector fields $X_1, \dots, X_n \in \mathfrak{X}(M)$ such that for every $k = 1, \dots, n$ we have $\mathcal{F}_U = C_c^\infty(U)X_1|_U + \dots + C_c^\infty(U)X_n|_U$.

- b) Stefan [St74] and Sussmann [Su73] showed that such modules induce a partition of M to injectively immersed submanifolds (leaves). Throughout this sequel we'll call a leaf L *regular* if there exists an open neighborhood W of L in M , where the dimension of L is equal to the dimension of any other leaf intersecting W . Otherwise L will be a *singular* leaf.
- c) A foliation \mathcal{F} provides an alternative topology for the manifold M ("longitudinal smooth structure", see [AnSk06, §1.3]). Denoting $M_{\mathcal{F}}$ the manifold M endowed with this topology, we have:

- $M_{\mathcal{F}}$ is a totally disconnected manifold, with connected components the leaves.
- The identity map $Id : M_{\mathcal{F}} \rightarrow M$ is continuous but not open.

In the sequel, when we refer to the smooth structure (and corresponding topology) on a leaf L , we mean the above longitudinal smooth structure. For embedded leaves, this topology coincides with relative topology to the usual topology of M . When L is immersed but not embedded though (*e.g.* a (regular) dense leaf of the irrational rotation foliation of the torus), the two topologies differ.

- d) Let (M, \mathcal{F}) be a foliation and W an open subset of M . We discussed in item (a) the restriction of \mathcal{F} to W . On the other hand, let $\{W_i\}_{i \in I}$ be an open cover of M and \mathcal{F}_i a foliation on every W_i . If \mathcal{F}_i and \mathcal{F}_j agree on $W_i \cap W_j$ for every $i, j \in I$, then using a partition of unity we can construct a unique foliation on M which restricts to \mathcal{F}_i for every $i \in I$.
- e) We can pull a foliation back over a smooth map: If $p : N \rightarrow M$ is smooth then $p^{-1}(\mathcal{F})$ is the submodule of $\mathfrak{X}_c(N)$ consisting of vector fields Y such that $y \mapsto dp(Y_y)$ lies in $p^*(\mathcal{F})$. The latter is defined as the submodule of $C_c^\infty(N; p^*(TM))$ generated by $f \cdot (X \circ p)$ for $f \in C_c^\infty(N)$ and $X \in \mathcal{F}$. In other words, $p^{-1}(\mathcal{F})$ consists of $C_c^\infty(N)$ -linear combinations of vector fields on N which are projectable and project to elements of \mathcal{F} .
- f) For $x \in M$ the space $\mathcal{F}(x) = \{X \in \mathcal{F} : X_x = 0\}$ is Lie subalgebra of \mathcal{F} . Let $I_x = \{f \in C^\infty(M) : f(x) = 0\}$. Then $I_x \mathcal{F} \subset \mathcal{F}(x)$ is a Lie ideal. Hence $\mathfrak{g}_x = \mathcal{F}(x)/I_x \mathcal{F}$ is a Lie algebra, called the *infinitesimal isotropy* of \mathcal{F} at x .

Lemma 1.1. *Let L be a leaf of \mathcal{F} . The Lie algebras \mathfrak{g}_x are isomorphic for all $x \in L$. Further, for any $x \in L$, $\mathfrak{g}_x = \{0\}$ iff L is a regular leaf.*

Proof. The first statement follows from the fact that the group $\exp(\mathcal{F})$, generated by $\exp(X)$ with $X \in \mathcal{F}$, acts transitively on the leaf L . We now prove the second statement. If L is a regular leaf, then nearby the point x we can find generators of \mathcal{F} which are linearly independent at x , implying that $\mathcal{F}(x) = I_x \mathcal{F}$. If L is a singular leaf, pick a neighborhood W

in M of some $x \in L$, and pick a set of generators X_1, \dots, X_n of \mathcal{F} defined on W . We may assume that this is a minimal set of generators, i.e. we may assume that none of the X_i can be written as a $C^\infty(W)$ linear combination of the others. Further, as $\{X_i(x)\}$ spans $T_x L$ and W contains leaves of dimension $> \dim(L)$, we may assume that $X_1(x)$ is a linear combination of the remaining $X_i(x)$, in other words $X_1 \in \mathcal{F}(x)$. However $X_1 \notin I_x \mathcal{F}$ because if we could write $X_1 = \sum_{i=1}^n f_i X_i$ with $f_i \in I_x$ then we would have $X_1 = \sum_{i \neq 1} f_i / (1 - f_1) X_i$, which contradicts the minimality assumption. \square

- g) The *fiber* of \mathcal{F} at x is the quotient $\mathcal{F}_x = \mathcal{F} / I_x \mathcal{F}$. The *tangent space of the leaf* is the image F_x of the evaluation map $ev_x : \mathcal{F} \rightarrow T_x M$. They are both finite-dimensional linear spaces. We have the extension

$$0 \rightarrow \mathfrak{g}_x \rightarrow \mathcal{F}_x \xrightarrow{ev_x} F_x \rightarrow 0. \quad (1.1)$$

Notice that the Lie bracket on \mathfrak{g}_x does not extend to \mathcal{F}_x .

- h) The dimension of F_x is lower semi-continuous while the dimension of \mathcal{F}_x is upper semi-continuous. The union of regular leaves is a dense open subset of M and the two fibres coincide in this set. The *co-dimension* of \mathcal{F} is the upper semi-continuous function $codim : M \rightarrow \mathbb{Z}$ which maps every $x \in M$ to the co-dimension of the leaf L_x at x .

Examples 1.2. (i) Let X be a vector field on M and $\mathcal{F} := \langle X \rangle$ the foliation generated by X , that is, $\mathcal{F} = C_c^\infty(M)X$. Denote by $\{X = 0\}$ the vanishing set of X . Then Lemma 1.1 shows that $\mathfrak{g}_x = \{0\}$ everywhere except if $x \in \partial\{X = 0\}$ (these are exactly the singular leaves), in which case $\mathfrak{g}_x \cong \mathbb{R}$ due to the fact that $\dim(\mathcal{F}_x) \leq 1$ since \mathcal{F} has just one generator. The short exact sequence (1.1) implies that $\mathcal{F}_x \cong \mathbb{R}$ except if x lies in the interior of $\{X = 0\}$, where \mathcal{F}_x vanishes.

- (a) Consider the foliation \mathcal{F} on \mathbb{R}^2 induced from the action of S^1 by rotations. Its regular leaves are circles concentric at the origin, and the origin is a singular leaf. The module \mathcal{F} is generated by the vector field $X = x\partial_y - y\partial_x$. We have $\mathcal{F}_q = \mathbb{R}$ at every point q , $\mathfrak{g}_0 = \mathbb{R}$, whereas \mathfrak{g}_q vanishes at every $q \neq 0$.
- (b) Consider the action of $SO(2) \subset SO(3)$ on S^2 by rotations about the z -axis. The module \mathcal{F} is generated by the image of the infinitesimal action. We have $\mathfrak{g}_N = \mathfrak{g}_S = Lie(SO(2)) \cong \mathbb{R}$, where N, S are the two poles.
- (c) The foliation \mathcal{F} on \mathbb{R} generated by $x\partial_x$ has $\mathcal{F}_x = \mathbb{R}$ at every point, whereas \mathfrak{g}_x vanishes at every point except for $\mathfrak{g}_0 \cong \mathbb{R}$.
- (d) Consider the foliation $\mathcal{F} = \langle f\partial_x \rangle$ on \mathbb{R} , where f is some smooth function that vanishes exactly on $\{x \leq 0\}$. Its leaves are $L^+ = \mathbb{R}^+$ and $L_x = \{x\}$ for every $x \leq 0$. All leaves are regular except for $L_0 = \{0\}$. Then \mathcal{F}_x vanishes if $x < 0$ and is one-dimensional otherwise. The infinitesimal isotropy \mathfrak{g}_x vanishes at every $x \neq 0$ except at 0, where $\mathfrak{g}_0 \cong \mathbb{R}$. Notice that \mathcal{F} vanishes in every order on every $x \leq 0$.
- (ii) In [AnSk06, Prop. 1.4] it was shown that if V is a (finite-dimensional) vector space and \mathcal{F} is the foliation defined by a linear action $G \subseteq GL(V)$ then $\mathcal{F}_0 = \mathfrak{g}$, where \mathfrak{g} is the Lie algebra of G . At points $x \neq 0$ there is a natural map from the isotropy Lie algebra of the action at x to \mathfrak{g}_x , which is always surjective but might fail to be injective: $\mathbb{R}^2 - \{0\}$ is a regular leaf of the action of $GL(\mathbb{R}^2)$ on \mathbb{R}^2 , hence for any $x \in \mathbb{R}^2 - \{0\}$ we have $\mathfrak{g}_x = \{0\}$.

- (iii) Let \mathcal{F}^k the module of vector fields in \mathbb{R}^2 vanishing “to order k ” at $(0,0)$. It is generated by $x^i y^j \partial_x, x^i y^j \partial_y$ for all $i, j \geq 0$ such that $i + j = k$. Then $\mathcal{F}_{(0,0)}^k = \mathfrak{g}_{(0,0)} = \mathbb{R}^{2k+2}$.
- (iv) Consider the foliation \mathcal{F} on \mathbb{R} generated by $f_1 \partial_x, f_2 \partial_x, \dots, f_k \partial_x$, where the f_i vanish at 0 to order at least 1, that is, $f_i(0) = f_i'(0) = 0$. Then

$$[f_i \partial_x, f_j \partial_x] = (f_i f_j' - f_j f_i') \partial_x \in I_x \mathcal{F}$$

hence the Lie algebra \mathfrak{g}_x is abelian.

- (v) Any Lie algebroid $A \rightarrow M$ gives rise to a singular foliation $\mathcal{F} := \{\rho(a) : a \in C_c^\infty(A; M)\}$, where $\rho: A \rightarrow TM$ is the anchor map.

Proposition 1.3. *Not all singular foliations arise from Lie algebroids.*

Proof. If \mathcal{F} is a foliation arising from a Lie algebroid A , then $\dim(\mathfrak{g}_x)$ is bounded above by the rank of A for all $x \in M$ (here \mathfrak{g}_x denotes the infinitesimal isotropy of \mathcal{F}). Indeed consider $\ker(\rho_x)$, the isotropy of the Lie algebroid A at x . There is a well-defined linear map $\ker(\rho_x) \rightarrow \mathfrak{g}_x$ mapping \bar{a} to $\langle \rho(a) \rangle$, where $a \in \Gamma(A)$ is any extension of \bar{a} . This map is surjective since any element of \mathfrak{g}_x is represented by some vector field $X \in \mathcal{F}$ vanishing at x , so that $X = \rho(a)$ for some $a \in C_c^\infty(A; M)$ with $\rho_x(a_x) = 0$.

For any $k \geq 1$, consider the foliation \mathcal{F}^k on \mathbb{R}^2 generated by $(x-k)^i y^j \partial_x, (x-k)^i y^j \partial_y$ for all $i, j \geq 0$ such that $i + j = k$ (a trivial variation of item (iii)). We have $\mathcal{F}_{(k,0)}^k = \mathbb{R}^{2k+2}$. Now take the foliation \mathcal{F} generated by $\cup_{k \geq 1} \varphi_k \mathcal{F}^k$, where φ_k is a fixed choice of bump function on \mathbb{R}^2 with small support concentrated around the point $(k, 0)$. Then the infinitesimal isotropy of \mathcal{F} satisfies $\mathfrak{g}_{(k,0)} = \mathbb{R}^{2k+2}$. In particular, the dimensions of the infinitesimal isotropies of \mathcal{F} are not bounded above, so \mathcal{F} can not arise from a Lie algebroid. \square

It is not clear whether the analog of Prop. 1.3 holds if M is compact, or even if it holds locally on M .

1.2 The splitting theorem for foliations

We show that locally every foliation is the product of a foliation vanishing at a point and a one-leaf foliation, in analogy with the splitting theorem for Poisson structures [We83]. This is done modifying slightly the proof of [AnSk06, Prop. 1.12].

Given singular foliations (M_1, \mathcal{F}_1) and (M_2, \mathcal{F}_2) , their product is defined as $(M_1 \times M_2, \mathcal{F}_1 \times \mathcal{F}_2)$, where $\mathcal{F}_1 \times \mathcal{F}_2$ is given by $C_c^\infty(M_1 \times M_2)(X_1 + X_2)$ with X_1 and X_2 are trivial extensions to $M_1 \times M_2$ of vector fields lying in \mathcal{F}_1 and \mathcal{F}_2 respectively.

Proposition 1.4. (Splitting theorem) *Let (M, \mathcal{F}) be a manifold with a foliation and $x \in M$, and set $k := \dim(F_x)$. Let \hat{S} be a slice at x , that is, an embedded submanifold such that $T_x \hat{S} \oplus F_x = T_x M$. Then there exists an open neighborhood W of x in M and a diffeomorphism of foliated manifolds*

$$(W, \mathcal{F}_W) \cong (I^k, TI^k) \times (S, \mathcal{F}_S). \quad (1.2)$$

Here \mathcal{F}_W is the restriction of \mathcal{F} to W , $I := (-1, 1)$, $S := \hat{S} \cap W$ and \mathcal{F}_S consists of the restriction to S of vector fields in W tangent to S (so $\mathcal{F}_S = \iota^{-1} \mathcal{F}$ for $\iota: S \hookrightarrow W$).

In particular, if we denote by s_1, \dots, s_k the canonical coordinates on I^k and X_1, \dots, X_l are generators of \mathcal{F}_S , then \mathcal{F}_W is generated by $\partial_{s_1}, \dots, \partial_{s_k}$ and the (trivial extensions of) X_1, \dots, X_l .

Proof. We proceed by induction. For $k = 0$ the result is clear, so assume $k \geq 1$ and take $X \in \mathcal{F}$ to be a vector field with $X(x) \neq 0$. Let $\hat{V} \subset M$ be a submanifold with $\hat{S} \subset \hat{V}$ and $T_x \hat{V} \oplus \mathbb{R}X(x) = T_x M$. The proof of [AnSk06, Prop. 1.12] shows that there is a neighborhood W of x in M and a diffeomorphism

$$W \cong I \times V \tag{1.3}$$

mapping the vector field X to ∂_s , where s is the canonical coordinate on I and $V = \hat{V} \cap W$. Under this identification, every compactly supported vector field on W is of the form $fX + Z$, where Z is tangent $\{s\} \times V$ for all s . Further it shows that $fX + Z \in \mathcal{F}$ iff $Z|_{\{s\} \times V} \in \mathcal{F}|_{\{s\} \times V}$ for all s , where the latter consists of the restrictions to $\{s\} \times V$ of vector fields in \mathcal{F} tangent to it.

We infer that $\mathcal{F} = TI \times \mathcal{F}_V$, where $\mathcal{F}_V := \mathcal{F}|_{\{0\} \times V}$. That is, eq. (1.3) is a diffeomorphism of manifolds with foliations

$$(W, \mathcal{F}_W) \cong (I, TI) \times (V, \mathcal{F}_V) \tag{1.4}$$

Now $(\mathcal{F}_V)_x = F_x \cap T_x V$ has dimension $k - 1$ and S is a slice to (V, \mathcal{F}_V) at x , so by the induction hypothesis there is a diffeomorphism of manifolds with foliations

$$(V, \mathcal{F}_V) \cong (I^{k-1}, TI^{k-1}) \times (S, \mathcal{F}_S). \tag{1.5}$$

(Notice that the foliation on S induced by \mathcal{F}_V agrees with the one induced by \mathcal{F} , which is \mathcal{F}_S .) Inserting eq. (1.5) into eq. (1.4) we obtain the sought eq. (1.2). \square

Examples 1.5. (i) Let m be a positive integer and consider the foliation $\mathcal{F}_m = \langle x^m \partial_x \rangle$ on \mathbb{R} . Its leaves are $(-\infty, 0)$, $\{0\}$ and $(0, +\infty)$. If x is non-zero, a transversal at x is just a point with the obvious (zero) foliation. At zero a transversal is $(\mathbb{R}, \mathcal{F}_m)$.

(ii) Consider the action of S^1 on \mathbb{R}^2 by rotations. The leaves of $(\mathbb{R}^2, \mathcal{F})$ are concentric circles with a singularity at zero. A transversal at zero is $(S_x, \mathcal{F}_{S_x}) = (\mathbb{R}^2, \mathcal{F})$. At an $x \neq 0$, any (affine) line $S_x \cong \mathbb{R}$ passing through x is a transversal and \mathcal{F}_V is the foliation by points (generated by the zero vector field in the identification $S_x \cong \mathbb{R}$). In this case there exists a canonical choice of S_x , namely the line connecting x with the origin.

Remark 1.6. Fix a slice S at x transverse to the foliation \mathcal{F} . By definition $(F_S)_x = F_x \cap T_x S = \{0\}$, so the exact sequence (1.1) applied for the foliation (S, \mathcal{F}_S) gives $(\mathcal{F}_S)_x = (\mathfrak{g}_S)_x$. Further, the latter is canonically isomorphic to \mathfrak{g}_x .

Indeed, by the splitting theorem, in every class of $\mathfrak{g}_x = \mathcal{F}(x)/I_x \mathcal{F}$ there is a representative Y tangent to S . Mapping the class of such a Y to the class of $Y|_S$ gives a well-defined map $\mathcal{F}(x)/I_x \mathcal{F} \rightarrow \mathcal{F}_S/I_x \mathcal{F}_S$. It is surjective, as we can extend an element of \mathcal{F}_S to one of $\mathcal{F}(x)$. Further both vector spaces have the same dimension, as a consequence of the splitting theorem (notice that $\mathcal{F}_I(x)/I_x \mathcal{F}_I = \{0\}$ since the foliation \mathcal{F}_I on I is regular).

1.3 The transitive Lie algebroid A_L over a leaf

Let L be a leaf of the foliation (M, \mathcal{F}) . There exists a transitive Lie algebroid A_L over L , described in [AnSk06, Remark 1.16]. Its integrability is important because if A_L is integrable then the construction of pseudodifferential calculus in [AnSk09] is simplified a great deal. This is discussed thoroughly in the beginning of §4.1.

In this subsection we assume that the leaf L is *embedded* (and not just immersed), and we make explicit the Lie algebroid A_L . Consider $\mathcal{F}/I_L \mathcal{F}$, where I_L is the space of functions in $C^\infty(M)$ which vanish on the leaf L .

Proposition 1.7. *Let L be an embedded leaf.*

- a) $A_L := \cup_{x \in L} \mathcal{F}_x$ is endowed with the structure of a smooth vector bundle over L .
- b) $\mathcal{F}/I_L\mathcal{F}$ is a $C^\infty(L)$ -module, and $\mathcal{F}/I_L\mathcal{F} = C_c^\infty(A_L; L)$, where the latter denotes the compactly supported sections.

Proof. a) Cover L by open subsets $\{W_\alpha\}_{\alpha \in A}$ of M as in the splitting theorem (Prop. 1.4), i.e. so that there exists an isomorphism of foliated manifolds

$$\psi_\alpha : (W_\alpha, \mathcal{F}_{W_\alpha}) \cong (I^k, TI^k) \times (S_\alpha, \mathcal{F}_{S_\alpha}). \quad (1.6)$$

Here S_α is a slice transverse to L at some fixed point x_α and $I = (-1, 1)$. We denote $L_\alpha := \psi_\alpha^{-1}(I^k \times \{x_\alpha\})$.

Consider the disjoint union of trivial vector bundles

$$\bigcup_{\alpha \in A} (\mathcal{F}_{x_\alpha} \times L_\alpha) \quad (1.7)$$

over $\cup_{\alpha \in A} L_\alpha$. Notice that ψ_α and the affine structure of I^k provides identifications $\mathcal{F}_{x_\alpha} \cong \mathcal{F}_x$ for all $x \in L_\alpha$. If $x \in L_\alpha \cap L_\beta$, we therefore obtain an isomorphism

$$(\mathcal{F}_{x_\alpha}, \{x\}) \cong \mathcal{F}_x \cong (\mathcal{F}_{x_\beta}, \{x\})$$

providing an equivalence relation on the vector bundle (1.7). The quotient is a vector bundle $A_L \rightarrow L$. Notice that, as a set, A_L is just the union $\cup_{x \in L} \mathcal{F}_x$.

b) To show that $\mathcal{F}/I_L\mathcal{F}$ is a $C^\infty(L)$ -module, put $f[X] = [\tilde{f} \cdot X]$ for every $f \in C^\infty(L)$ and $X \in \mathcal{F}$. Here \tilde{f} is any smooth extension of f from L to M , which exists since L is an embedded submanifold. Since any two extensions of f differ by an element of I_L , the above definition does not depend choices, and makes $\mathcal{F}/I_L\mathcal{F}$ a $C^\infty(L)$ -module.

Given $X \in \mathcal{F}$, there is an embedding τ that associates to $[X] \in \mathcal{F}/I_L\mathcal{F}$ the element of $C_c^\infty(A_L; L)$ given by

$$L \rightarrow A_L, x \mapsto X \bmod I_x\mathcal{F}.$$

(This section of A_L has compact support because X is a compactly supported vector field). In order to show that τ is surjective, we cover L by open subsets $\{W_\alpha\}_{\alpha \in A}$ of M as in the splitting theorem and show:

Claim: For every $\alpha \in A$, the map $\mathcal{F}_{W_\alpha}/I_{L_\alpha}\mathcal{F}_{W_\alpha} \rightarrow C^\infty(L_\alpha, A_L|_{L_\alpha})$ obtained restring τ is surjective.

Since both sides are $C^\infty(L)$ -modules, it is sufficient to show that for any constant section c of the trivial vector bundle $A_L|_{L_\alpha} \cong \mathcal{F}_{x_\alpha} \times L_\alpha$ there exists $Y \in \mathcal{F}_{W_\alpha}$ so that $\tau([Y]) = c$. Since c is a constant section, we can view it as an element $c \in \mathcal{F}_{x_\alpha} = T_0I^k \oplus (\mathcal{F}_{S_\alpha})_{x_\alpha}$ using eq. (1.6). The first component of c and any lift of the second component of c to \mathcal{F}_{S_α} give rise to an element $Y \in \mathcal{F}_{W_\alpha}$, invariant in the I^k direction, with the required property. This proves the claim.

The surjectivity of τ now follows: given $s \in C_c^\infty(A_L; L)$, there is a finite subset B of the index set A such that the support of s is contained in $\cup_{\alpha \in B} L_\alpha$. Using an associated partition of unity we can write $s = \sum_{\alpha \in B} s_\alpha$ where the elements s_α have support in L_α . By the claim we have $s_\alpha = \tau[Y_\alpha]$ some

$Y_\alpha \in \mathcal{F}$. So we conclude that $s = \tau[\sum_{\alpha \in B} Y_\alpha]$. □

From Prop.1.7 it follows that the Lie bracket of \mathcal{F} and the surjective map $\rho : C^\infty(L; A_L) \rightarrow \mathfrak{X}(L)$, $\rho[X] = X|_L$ make the triple $(A_L, [\cdot, \cdot], \rho)$ into a transitive Lie algebroid.

In fact, the evaluation map of eq. (1.1) is nothing else than the restriction of ρ to a fiber $(A_L)_x$. Whence the kernel of the anchor map ρ is the field of Lie algebras $\mathfrak{g}_L = \cup_{x \in L} \mathfrak{g}_x$. This is actually a locally trivial Lie algebra bundle from the general theory of transitive Lie algebroids (*cf.* [Ma05]).

Remark 1.8. The construction of A_L for an immersed leaf L (endowed with the longitudinal smooth structure) is exactly as in part a) of the above Prop. 1.7. However in that case $\mathcal{F}/I_L\mathcal{F}$ – which is not even a $C^\infty(L)$ module – is not contained in the space of sections of A_L . On the other hand, on open sets L_α of L , we have $C^\infty(L_\alpha, A_L|_{L_\alpha}) = \mathcal{F}_{W_\alpha}/I_{L_\alpha}\mathcal{F}_{W_\alpha}$. Notice that using a partition of unity for L we can glue elements $\xi_\alpha \in \mathcal{F}_{W_\alpha}/I_{L_\alpha}\mathcal{F}_{W_\alpha}$ to obtain elements of $C^\infty(L; A_L)$.

2 The holonomy groupoid

In §2.1 we review the notion of a bi-submersion introduced in [AnSk06] and clarify its role in the study of foliations in §2.1.1. Then, §2.2 gives explicit formulas of morphisms of bi-submersions that will be used in §3 to study the smoothness of the holonomy covers. For the convenience of the reader, we provide in §2.3 a brief review of the construction of the holonomy groupoid given in [AnSk06] as a quotient of an atlas of bi-submersions, together with explicit calculations of this groupoid for some foliations.

2.1 Bi-submersions

Let (M, \mathcal{F}) be a (singular) foliation. Here we recall the notion of bi-submersion from [AnSk06] and discuss how it provides an equivalent definition of a foliation.

- a) A *bi-submersion* of (M, \mathcal{F}) is a smooth manifold U endowed with two submersions $\mathbf{t}, \mathbf{s} : U \rightarrow M$ satisfying:

- (i) $\mathbf{s}^{-1}(\mathcal{F}) = \mathbf{t}^{-1}(\mathcal{F})$,
- (ii) $\mathbf{s}^{-1}(\mathcal{F}) = C_c^\infty(U; \ker d\mathbf{s}) + C_c^\infty(U; \ker d\mathbf{t})$.

Recall that $\mathbf{s}^{-1}(\mathcal{F})$ was defined in item e) of §1.1.

We say $(U, \mathbf{t}, \mathbf{s})$ is *minimal* at u if $\dim(U) = \dim(M) + \dim(\mathcal{F}_{\mathbf{s}(u)})$.

- b) Let $x \in M$, and $X_1, \dots, X_n \in \mathcal{F}$ inducing a basis of \mathcal{F}_x . In [AnSk06, Prop. 2.10 a)] it was shown that there is an open neighborhood U of $(x, 0)$ in $M \times \mathbb{R}^n$ such that $(U, \mathbf{t}_U, \mathbf{s}_U)$ is a bi-submersion minimal at $(x, 0)$, where $\mathbf{s}_U(y, \xi) = y$ and $\mathbf{t}_U(y, \xi) = \exp_y(\sum_{i=1}^n \xi_i X_i)$. (Recall

that the latter is the image of y under the time-1 flow of $\sum_{i=1}^n \xi_i X_i$.) Bi-submersions arising this way are called *path holonomy bi-submersions*.

- c) Let $(U_i, \mathbf{t}_i, \mathbf{s}_i)$ be bi-submersions, $i = 1, 2$. Then $(U_i, \mathbf{s}_i, \mathbf{t}_i)$ are bi-submersions, as well as $(U_1 \circ U_2, \mathbf{t}, \mathbf{s})$ where $U_1 \circ U_2 = U_1 \times_{\mathbf{s}_1, \mathbf{t}_2} U_2$, $\mathbf{t}(u_1, u_2) = \mathbf{t}(u_1)$ and $\mathbf{s}(u_1, u_2) = \mathbf{s}(u_2)$. They are called the *inverse* and *composite* bi-submersions respectively.

Examples 2.1. (i) In example 1.2(i)(a) the path holonomy bi-submersion, minimal at $x = 0$, associated to $X \in \mathcal{F}$ is given by $U \subset \mathbb{R}^2 \times \mathbb{R}$, $\mathbf{s} : U \rightarrow \mathbb{R}^2$ the projection and $\mathbf{t} : U \rightarrow \mathbb{R}^2$ the map

$$\left(\begin{pmatrix} x \\ y \end{pmatrix}, \varepsilon \right) \mapsto \begin{pmatrix} \cos(\varepsilon) & -\sin(\varepsilon) \\ \sin(\varepsilon) & \cos(\varepsilon) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(ii) In example 1.2(i)(c) the path holonomy bi-submersion, minimal at $x = 0$, associated to $x\partial_x$ is $(U, \mathbf{t}, \mathbf{s})$ where $U \subset \mathbb{R} \times \mathbb{R}$, \mathbf{s} is the first projection and $\mathbf{t}(x, \varepsilon) = xe^\varepsilon$.

2.1.1 Abstract bi-submersions

Note that bi-submersions determine the foliation \mathcal{F} quite directly. Indeed, if $f : U \rightarrow M$ is a bi-submersion, the foliation $f^{-1}(\mathcal{F})$ on U obviously determines the foliation \mathcal{F} on $f(U)$. This way a bi-submersion $(U, \mathbf{t}, \mathbf{s})$ determines \mathcal{F} on $\mathbf{s}(U)$ and $\mathbf{t}(U)$.

In fact, this observation leads to an abstraction of the notion of bi-submersion. Although we will not use this abstraction in the sequel, we present it here, as it allows the notion of bi-submersion to be better understood. In particular, it exhibits that bi-submersions are actually an equivalent definition of a foliation. This material was communicated to us by G. Skandalis. Let us start with a simple lemma:

Lemma 2.2. *Let $f : N \rightarrow M$ be a surjective submersion with connected fibres. Let \mathcal{F}_N be a foliation on N such that $C^\infty(N; \ker df) \subseteq \mathcal{F}_N$. Then there exists a unique foliation \mathcal{F} on M such that $f^{-1}(\mathcal{F}) = \mathcal{F}_N$.*

Proof. Assume first that $N = M \times \mathbb{R}^k$ and f is the first projection. In this case \mathcal{F}_N is the direct sum of its vertical part generated by $\partial/\partial x_i$ (where (x_1, \dots, x_k) are coordinates of \mathbb{R}^k) and a horizontal part \mathcal{F}_{hor} . Now \mathcal{F}_N is invariant by translations in the \mathbb{R}^k direction (cf. e.g. [AnSk06, Prop 1.6]), whence its horizontal part is constant along \mathbb{R}^k . It is the set of smooth functions with compact support from \mathbb{R}^k to a foliation \mathcal{F}_M of M . Now let $\iota : M \rightarrow M \times \mathbb{R}^k, x \mapsto (x, 0)$. It is transverse to \mathcal{F}_N . Since \mathcal{F}_N is of the form $f^{-1}(\mathcal{F}_M)$ we have $\iota^{-1}(\mathcal{F}_N) = \mathcal{F}_M$ and uniqueness follows.

In general, N is the union of open sets $W_i \cong U_i \times \mathbb{R}^k$, where the U_i are open sets in M and $f|_{W_i}$ corresponds to the projection $U_i \times \mathbb{R}^k \rightarrow U_i$. The restriction of \mathcal{F}_N to W_i is of the form $f^{-1}(\mathcal{F}_i)$, where \mathcal{F}_i is a foliation on U_i . By uniqueness, \mathcal{F}_i and \mathcal{F}_j agree on $f(W_i \cap W_j)$. By connectedness of the fibres, for every $x \in M$ and every $i, j \in I$ such that $x \in U_i \cap U_j$, there exists a finite sequence $i = i_1, \dots, i_r = j$ such that $x \in f(W_{i_\ell} \cap W_{i_{\ell+1}})$ for every $1 \leq \ell \leq r$. It follows that \mathcal{F}_i and \mathcal{F}_j coincide in a neighborhood of x . Using partitions of unity we deduce that \mathcal{F}_i and \mathcal{F}_j coincide in $U_i \cap U_j$, whence there exists a foliation \mathcal{F}_M whose restriction to U_i is \mathcal{F}_i . The foliation \mathcal{F}_M is unique in each U_i by the first case, therefore it is unique. \square

Definition 2.3. a) Let M, N be manifolds. An *abstract bi-submersion* between M and N is a triple $(U, \mathbf{t}, \mathbf{s})$, where U is a manifold and $\mathbf{s} : U \rightarrow M$, $\mathbf{t} : U \rightarrow N$ are submersions with connected fibres such that $C^\infty(U; \ker ds) + C^\infty(U; \ker dt)$ are foliations, i.e. stable under Lie brackets.

b) Let M, N_1, N_2 be manifolds. Let $(U_1, \mathbf{t}_1, \mathbf{s}_1)$ be a bi-submersion between M and N_1 , and $(U_2, \mathbf{t}_2, \mathbf{s}_2)$ a bi-submersion between M and N_2 . For every $i = 1, 2$ denote \mathcal{F}_i the foliation $C^\infty(U_i; \ker ds_i) + C^\infty(U_i; \ker dt_i)$ of U_i . Put $U = U_1 \times_M U_2$ and let $p_i : U \rightarrow U_i$ be the projection. The bi-submersions $(U_1, \mathbf{t}_1, \mathbf{s}_1)$ and $(U_2, \mathbf{t}_2, \mathbf{s}_2)$ are said to be *compatible* if $p_1^{-1}(\mathcal{F}_1) = p_2^{-1}(\mathcal{F}_2)$.

By Lemma 2.2, an abstract bi-submersion defines a foliation on $\mathbf{s}(U)$ and on $\mathbf{t}(U)$. Also, by uniqueness in Lemma 2.2, two bi-submersions $(U_1, \mathbf{t}_1, \mathbf{s}_1)$ and $(U_2, \mathbf{t}_2, \mathbf{s}_2)$ are compatible if and only if the foliations they define agree on $\mathbf{s}_1(U_1) \cap \mathbf{s}_2(U_2)$. The following proposition follows thanks to item (d) in §1.1.

Proposition 2.4. *Let M and $\{N_i\}_{i \in I}$ be manifolds. For every $i \in I$ let $(U_i, \mathbf{t}_i, \mathbf{s}_i)$ be an abstract bi-submersion between M and N_i . Suppose that:*

- a) $\bigcup_{i \in I} \mathbf{s}_i(U_i) = M$, and
- b) for every $i, j \in I$ the bi-submersions $(U_i, \mathbf{t}_i, \mathbf{s}_i)$ and $(U_j, \mathbf{t}_j, \mathbf{s}_j)$ are compatible.

Then there exists a unique foliation \mathcal{F} on M such that for all $i \in I$,

$$\mathbf{s}_i^{-1}(\mathcal{F}) = C^\infty(U_i; \ker d\mathbf{s}_i) + C^\infty(U_i; \ker d\mathbf{t}_i).$$

Proposition 2.4 shows that a foliation can be defined from abstract bi-submersions. Notice that our definition of an abstract bi-submersion (Def. 2.3) actually uses the notion of foliation. It is possible to abstract further the definition of a bi-submersion and completely do away with the notion of foliation in the definition, but this is far beyond the scopes of this sequel.

2.2 Morphisms of bi-submersions and bisections

Let us now recall from [AnSk06] the notion of a morphism of bi-submersions, as well as the notion of a bisection. We will be using both of them throughout this paper.

- a) Let $(U, \mathbf{t}_U, \mathbf{s}_U)$ and $(V, \mathbf{t}_V, \mathbf{s}_V)$ be two bi-submersions. A *morphism of bi-submersions* is a smooth map $f : U \rightarrow V$ such that $\mathbf{s}_V \circ f = \mathbf{s}_U$ and $\mathbf{t}_V \circ f = \mathbf{t}_U$.
- b) A *bisection* of $(U, \mathbf{t}, \mathbf{s})$ is a locally closed submanifold V of U on which the restrictions of \mathbf{s} and \mathbf{t} are diffeomorphisms to open subsets of M . We say that V is an *identity bisection* if $\mathbf{s}|_V = \mathbf{t}|_V$. A bisection is necessarily the image of a smooth map $\psi : M_0 \rightarrow U$ such that $\mathbf{s} \circ \psi = Id_{M_0}$, where M_0 is an open subset of M . In case $(U, \mathbf{t}, \mathbf{s})$ is a path holonomy bi-submersion $U \subseteq M \times \mathbb{R}^n$, a bisection is necessarily the graph of a smooth map $\phi : M_0 \rightarrow \mathbb{R}^n$. We will often switch freely between a bisection and the corresponding maps.
- c) We say that $u \in U$ carries the foliation-preserving local diffeomorphism ψ if there is a bisection V such that $u \in V$ and $\psi = \mathbf{t}|_V \circ (\mathbf{s}|_V)^{-1}$. For instance, if $U \subseteq M \times \mathbb{R}^n$ is a path holonomy bi-submersion which is minimal at $(x, 0)$ and $\phi : V \rightarrow \mathbb{R}^n$ a smooth map which vanishes at x then $(x, 0)$ carries the local diffeomorphism $y \mapsto \exp_y(\sum_{i=1}^n \phi_i(y) X_i)$. (Notice that in general this local diffeomorphism is not the time-1 flow of a vector field in any obvious way, due to the dependence of ϕ_i on y .) Putting ϕ the constant zero map, we see that $(x, 0)$ carries the identity as well.
- d) It was shown in [AnSk06, Cor. 2.11(b)] that if $\{(U_i, \mathbf{t}_i, \mathbf{s}_i)\}_{i \in I}$ are bi-submersions, $i = 1, 2$ then $u_1 \in U_1$ and $u_2 \in U_2$ carry the same local diffeomorphism iff there exists a morphism of bi-submersions g defined in an open neighborhood of $u_1 \in U_1$ such that $g(u_1) = u_2$. Such a morphism maps every bisection V of U_1 at u_1 to a bisection $g(V)$ of U_2 at u_2 .

Example 2.5. For the circle action on \mathbb{R}^2 we saw that a path holonomy bi-submersion is given by $U \subseteq \mathbb{R}^2 \times \mathbb{R}$ as in example 2.1(i). The action groupoid $S^1 \ltimes \mathbb{R}^2$ is also a bi-submersion. The familiar exponential map $\mathbb{R} \rightarrow S^1$ provides a (locally defined) map $U \rightarrow S^1 \ltimes \mathbb{R}^2$ by $(\vec{\lambda}, \theta) \mapsto (e^{i\theta}, \vec{\lambda})$. This is a morphism of bi-submersions.

The following Lemma is a special case of [AnSk06, Prop. 2.10 b)]. We include a proof, along the lines of that of [AnSk06, Prop. 2.10 b)] but more explicit. Recall that the latter states that if U, U' are bi-submersions with $x \in \sharp U$ and U' is a path holonomy bi-submersion minimal at x , then – shrinking U if necessary – there exists a morphism of bi-submersions $\Phi: U \rightarrow U'$ such that $(x, 0)$ lies in its image, and furthermore Φ is a submersion.

Lemma 2.6. *Fix $x \in M$ and consider vector fields X_1, \dots, X_n and X'_1, \dots, X'_n in \mathcal{F} whose images $\{[X_1], \dots, [X_n]\}$ and $\{[X'_1], \dots, [X'_n]\}$ form bases of \mathcal{F}_x . Let $(U, \mathbf{t}, \mathbf{s})$ and $(U', \mathbf{t}', \mathbf{s}')$ be the corresponding path holonomy bi-submersions. There exists a morphism $\Phi: U \rightarrow U'$ defined in a neighborhood of $(x, 0)$ and fixing $(x, 0)$, which is a diffeomorphism onto its image.*

Proof. Let $\{Y_i\}$ be vertical lifts of the $\{X_i\}$ w.r.t. the target map \mathbf{t} of the bi-submersion U , and $\{Y'_i\}$ vertical lifts of the $\{X'_i\}$ w.r.t. the target map \mathbf{t}' of the bi-submersion U' . Since $\{X'_i\}$ is a generating set for \mathcal{F} in a neighborhood of x (which we will still denote by M abusing notation) [AnSk06, Prop 1.5(a)], there exists smooth functions $\{c_{il}\}_{i,l \leq n}$ satisfying $X_i = \sum_l c_{il} X'_l$ for all i . In

particular, $\sum_i k_i [X_i] = \sum_{i,l} k_i c_{il}(x) [X'_l] \in \mathcal{F}_x$. We define Φ by

$$\exp_{(y,0)}\left(\sum_i \lambda_i Y_i\right) \mapsto \exp_{(y,0)}\left(\sum_{i,l} \lambda_i \mathbf{t}'^*(c_{il}) Y'_l\right). \quad (2.1)$$

Since Y_i and Y'_i are vertical vector fields, they lie in the kernels of the respective source maps, which are just the projections onto M . Hence we have $\mathbf{s}' \circ \Phi = \mathbf{s}$. On the other hand, notice that $\sum_l \mathbf{t}'^*(c_{il}) Y'_l$ maps under \mathbf{t}' to $\sum_l c_{il} X'_l = X_i$, so $\exp_{(y,0)}\left(\sum_{i,l} \lambda_i \mathbf{t}'^*(c_{il}) Y'_l\right)$ maps under \mathbf{t}' to $\exp_y\left(\sum_i \lambda_i X_i\right)$, which is the image under \mathbf{t} of $\exp_{(y,0)}\left(\sum_i \lambda_i Y_i\right)$. This shows that $\mathbf{t}' \circ \Phi = \mathbf{t}$ and hence that Φ is a morphism of bi-submersions.

To show that it is a diffeomorphism consider its derivative $(\Phi_*)_{(x,0)}$. It maps $Y_i|_{(x,0)}$ to $\sum_l c_{il}(x) Y'_l|_{(x,0)}$.

Since $\{[X_i]\}_{i \leq n} = \left\{\sum_l c_{il}(x) [X'_l]\right\}_{i \leq n}$ and $\{[X'_i]\}_{i \leq n}$ are both bases of \mathcal{F}_x , the matrix $[c_{il}(x)]_{il}$ is invertible.

We claim that the $\{Y'_i|_{(x,0)}\}_{i \leq n}$ are linearly independent (this immediately implies that Φ is a local diffeomorphism near x and concludes the proof). Indeed, assume that the constant linear combination $Z := \sum_{i \leq n} a_i Y'_i$ vanishes at $(x, 0) \in U'$. Choose coordinates along the fibers of $\mathbf{s}: U' \rightarrow$

M , giving rise to vertical coordinate vector fields $\{\partial_i\}_{i \leq n}$. Then $Z = \sum_{i \leq n} b_i \partial_i$ for unique $b_i \in C^\infty(U')$,

which satisfy $b_i(x, 0) = 0$. We have $\mathbf{t}_*(Z|_M) = \sum_{i \leq n} b_i|_M \mathbf{t}_*(\partial_i|_M) \in I_x \mathcal{F}$. (Here we use $\mathbf{t}_*(\partial_i|_M) \in \mathcal{F}$,

a direct consequence of the fact that U' is a bi-submersion.) So we have $[\mathbf{t}_*(Z|_M)] = 0 \in \mathcal{F}_x =$

$\mathcal{F}/I_x\mathcal{F}$. At the same time we have $\mathbf{t}_*(Z|_M) = \sum_{i \leq n} a_i X'_i$, which implies $[\mathbf{t}_*(Z|_M)] = \sum_{i \leq n} a_i [X'_i]$. The linear independence of $\{[X'_i]\}_{i \leq n}$ implies that $a_i = 0$ for all i , proving the claim. \square

2.3 The holonomy groupoid

We end this section with a recollection of the construction of the holonomy groupoid given in [AnSk06] and give a few examples.

Let $\mathcal{U} = ((U_i, \mathbf{t}_i, \mathbf{s}_i))_{i \in I}$ be a family of bi-submersions. Recall from [AnSk06] that a bi-submersion $(U, \mathbf{t}, \mathbf{s})$ is *adapted* to \mathcal{U} if for all $u \in U$ there exists an open subset $U' \subset U$ containing u , an $i \in I$, and a morphism of bi-submersions $U' \rightarrow U_i$. We say that \mathcal{U} is an *atlas* if

- a) $\bigcup_{i \in I} \mathbf{s}_i(U_i) = M$.
- b) The inverse of every element in \mathcal{U} is adapted to \mathcal{U} .
- c) The composition $U \circ V$ of any two elements in \mathcal{U} is adapted to \mathcal{U} .

An atlas $\mathcal{U}' = \{(U'_j, \mathbf{t}_j, \mathbf{s}_j)\}_{j \in J}$ is adapted to \mathcal{U} if every element of \mathcal{U}' is adapted to \mathcal{U} . We say \mathcal{U} and \mathcal{U}' are *equivalent* if they are adapted to each other. Given a foliation (M, \mathcal{F}) the *path holonomy atlas* is the one generated by all the path holonomy bi-submersions. (The latter were defined in §2.1 b.) In other words, the path holonomy atlas consists of finite compositions of path holonomy bi-submersions and their inverses.

The groupoid of an atlas $\mathcal{U} = ((U_i, \mathbf{t}_i, \mathbf{s}_i))_{i \in I}$ is the quotient

$$G(\mathcal{U}) := \coprod_{i \in I} U_i / \sim$$

by the equivalence relation for which $u \in U_i$ is equivalent to $u' \in U_j$ if there is a morphism of bi-submersions $f : W \rightarrow U_j$ defined in a neighborhood $W \subset U_i$ of u such that $f(u) = u'$. We denote the canonical quotient map by $\sharp : \coprod_{i \in I} U_i \rightarrow G(\mathcal{U})$.

The groupoid of the path holonomy atlas associated to \mathcal{F} is the *holonomy groupoid* of the foliation \mathcal{F} . It depends on the choice of module \mathcal{F} rather than the partition to leaves (see Ex. 2.7 below). We denote the holonomy groupoid of \mathcal{F} by $H(\mathcal{F})$ or by H (when the foliation \mathcal{F} is understood).

Examples 2.7. (i) Let (M, \mathcal{F}) be a foliation and S a transversal to a leaf. Consider the foliation (S, \mathcal{F}_S) introduced in §1.4. Its holonomy groupoid is the restriction of the holonomy groupoid of (M, \mathcal{F}) on S , namely $H(S, \mathcal{F}_S) = H(M, \mathcal{F})_S^S$. Notice that if L is a singular leaf then the isotropy groups $H(M, \mathcal{F})_x^x$ may not be discrete. Due to Remark 1.6, this constitutes a difference from the regular case, where the restriction of the holonomy groupoid to a transversal is always étale³.

³A very common technique in the study of regular foliations is to exploit the fact that the holonomy groupoid is always Morita equivalent to an étale one (its restriction to a transversal). For instance, in [CrMo04] it plays the crucial role in the extraction of explicit formulas for characteristic classes of regular foliations. Our observation in this example shows that this device cannot be used for singular foliations.

- (ii) Let X be a complete vector field on M and $\mathcal{F} = \langle X \rangle$. Assume that, for all $x \in \partial\{X = 0\}$, every neighborhood of x contains at least one point the integral curve through which is not periodic, where $\{X = 0\}$ denotes the vanishing set of X . Then

$$H(X) = H(X)|_{\{X \neq 0\}} \cup \text{Int}\{X = 0\} \cup (\mathbb{R} \times \partial\{X = 0\}),$$

where $H(X)|_{\{X \neq 0\}}$ is the (smooth) holonomy groupoid of the regular 1-dimensional foliation $\mathcal{F} = \langle X \rangle$ on $\{X \neq 0\}$, and the groupoid structure on $\mathbb{R} \times \partial\{X = 0\}$ is given by addition.

Indeed, the flow of X induces an action of the abelian group \mathbb{R} on M . The transformation groupoid $G := \mathbb{R} \ltimes M \rightrightarrows M$ is a bi-submersion and an atlas for (M, \mathcal{F}) , hence H is a quotient for the transformation groupoid (see [AnSk06, Ex. 3.4(4)]). If $x \in \text{Int}\{X = 0\}$ is an interior point of $\{X = 0\}$, then any two any points (λ, x) and (η, x) of G are related by a morphism of bi-submersions defined locally, namely the fibre translation by $\eta - \lambda$. If x is a boundary point of $\{X = 0\}$, then no morphism of bi-submersions relating distinct points (λ, x) and (η, x) can exist. If it existed, it would relate distinct points of $\mathbb{R} \times \{z\}$ for all z lying in some neighborhood U of x . This would contradict our assumption that there exist a $y \in U$ such that the target map of G is injective on $\mathbb{R} \times \{y\}$.

- (iii) Consider the foliation \mathcal{F} on \mathbb{R}^2 induced from the action of S^1 by rotations, i.e. $\mathcal{F} = \langle x\partial_y - y\partial_x \rangle$. (Notice that this vector field does not satisfies the assumption made in ii) above.) Then H coincides with the transformation groupoid $S^1 \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$. Indeed the latter is a bi-submersion and an atlas, so H is a quotient of it. We know that the quotient map restricted to $S^1 \times \{0\}$ is injective from [AnSk06, Prop. 1.4], and further on $\mathbb{R}^2/\{0\}$ the foliation is regular with trivial holonomy, so that $H|_{\mathbb{R}^2/\{0\}} = S^1 \times \mathbb{R}^2/\{0\}$.

2.4 Smoothness

We put here a few clarifications about the notion of smoothness for the holonomy groupoid, following [AnSk06, Remark 3.13, Def. 3.14].

First, let us recall the topology of the holonomy groupoid: H is endowed with the quotient topology, namely the finest topology which makes each quotient map $\sharp: U \rightarrow H$ continuous for each bi-submersion in the path holonomy atlas. Note that with this topology $\sharp U$ is open in H : The pre-image $\sharp^{-1}(\sharp U)$ consists of points y lying in bi-submersions U_α of the path holonomy atlas such that there exists a morphism of bi-submersions $V_\alpha \rightarrow U$ for some open neighborhood V_α of x in U_α . Hence $\sharp^{-1}(\sharp U)$ is open in $\cup_\alpha U_\alpha$, and by the definition of quotient topology it follows that $\sharp U$ is open in H .

In this sequel we will be concerned with the restriction $H_L = \mathbf{s}^{-1}(L) = \mathbf{t}^{-1}(L)$ of H to a leaf L , as well as the \mathbf{s} -fiber $H_x = \mathbf{s}^{-1}(x)$ and the isotropy group $H_x^x = \mathbf{s}^{-1}(x) \cap \mathbf{t}^{-1}(x)$, where $x \in L$.

Definition 2.8. Let \mathcal{F} be a singular foliation and L a leaf. We say that H_L is *smooth* if there exists a differentiable structure on it such that $\sharp: U_L \rightarrow H_L$ a submersion for all bi-submersions $(U, \mathbf{t}, \mathbf{s})$ in the path holonomy atlas, where $U_L := \mathbf{s}^{-1}(L)$. (If such a differentiable structure exists, it is unique.)

Similarly, when we talk about the smooth structure of H_x , we refer to the differential structure (unique if it exists) such that $\sharp: U_x := \mathbf{s}^{-1}(x) \rightarrow H_x$ a submersion for all bi-submersions U in the path holonomy atlas. The same applies to H_x^x .

Lemma 2.9. *If H_L is smooth (in the sense of Def. 2.8), then it is a Lie groupoid.*

Proof. First notice that the topology underlying the differentiable structure of H_L is exactly the quotient topology discussed above (this is a consequence of the fact that submersions are open maps).

If U_1 and U_2 are bi-submersions in the path holonomy atlas, then the diagram

$$\begin{array}{ccc} (U_1 \circ U_2)_L & & \\ \downarrow \# \times \# & \searrow \# & \\ (H_L)_s \times_{\mathbf{t}} (H_L) & \xrightarrow{\text{mult.}} & H_L \end{array}$$

commutes by the definition of the multiplication on H_L . Since $\# \times \#$ is a submersion, through every point of $(U_1 \circ U_2)_L$ we can find a submanifold S such that $(\# \times \#)|_S$ is a diffeomorphism onto its image. Hence the bottom map mult. is smooth, as locally it can be written as $\# \circ ((\# \times \#)|_S)^{-1}$. Similarly one shows that the inversion, the embedding of the identity section, the source map and the target map are smooth.

To show that the target map of H_L is a submersion, take a bi-submersion U in the path holonomy atlas and consider submanifolds $T \subset U_L$ transverse to the fibers of the submersion $\#: U_L \rightarrow H_L$ (so $\#|_T$ is a diffeomorphism onto its image). Since the $\#$ -fibers are contained in the fibers of $\mathbf{t}: U_L \rightarrow L$ (by definition of the equivalence relation \sim) and \mathbf{t} is a submersion, it follows that $\mathbf{t}|_T: T \rightarrow L$ is a submersion too. Hence the target map of H_L is a submersion. For the source map one proceeds similarly. \square

Remark 2.10. H_L is smooth for all leaves L iff the path holonomy atlas of (M, \mathcal{F}) , together with H , is a *holonomy pair* in the sense of [AnSk06, Def. 3.14]. This follows using Lemma 2.9.

To ensure that H_L is smooth, it is sufficient to consider bi-submersions that cover a neighborhood of the identity section L :

Proposition 2.11. *Assume that for every $x \in L$ there exists a path-holonomy bi-submersion U , minimal at x , such that $\#: U_L \rightarrow H_L$ is a submersion. Then H_L is smooth (in the sense of Def. 2.8).*

Proof. First notice that a path holonomy bi-submersion minimal at x , say $(U \subset M \times \mathbb{R}^n, \mathbf{t}, \mathbf{s})$, is isomorphic to its inverse U^{-1} , by the map

$$U \rightarrow U^{-1}, (y, \xi) \mapsto (\mathbf{t}(y, \xi), -\xi).$$

Hence in the following we will consider only products of path holonomy bi-submersions (and not of their inverses). It suffices to prove the following claim:

Claim: *Let U^1, \dots, U^m minimal path-holonomy bi-submersions. Then there exists a differential structure on the image of*

$$\#: U_L^1 \circ \dots \circ U_L^m \rightarrow H_L$$

such that the above map $\#$ is a submersion.

We prove the claim by induction. The case $m = 1$ holds by assumption together with Lemma 2.6. Assume the claim holds for $m - 1$, and use the short-hand notation $U_L^{(m-1)} := U_L^1 \circ \dots \circ U_L^{m-1}$. By

the induction assumption, there exists a smooth structure on $\sharp U_L^{(m-1)}$ such that $\sharp: U_L^{(m-1)} \rightarrow H_L$ is a submersion, and similarly for $\sharp U_L^m$.

Consider the following restriction of the multiplication on H_L :

$$mult: \sharp U_L^{(m-1)} \times_{\mathbf{s}} \times_{\mathbf{t}} \sharp U_L^m \rightarrow H_L.$$

For any q in the image of $mult$ we have

$$mult^{-1}(q) = \{(q \cdot r^{-1}, r) : r \in \mathbf{s}^{-1}(\mathbf{s}(q)) \cap \sharp U_L^m\}.$$

Notice that $\mathbf{s}^{-1}(\mathbf{s}(q)) \cap \sharp U_L^m$ is an embedded submanifold of $\sharp U_L^m$, as $\mathbf{s}: \sharp U_L^m \rightarrow L$ is a submersion by the proof of Lemma 2.9. Let σ be a smooth local section of \mathbf{s} . Then

$$\{(q \cdot \sigma(\mathbf{s}(q))^{-1}, \sigma(\mathbf{s}(q))) : q \in \text{image}(mult)\}$$

is submanifold transverse to the fibers of $mult$, and induces a differentiable structure on $\text{image}(mult) \subset H_L$ such that $mult$ is a submersion. The commuting diagram

$$\begin{array}{ccc} U_L^{(m-1)} \circ U_L^m & & \\ \downarrow \sharp \times \sharp & \searrow \sharp & \\ \sharp U_L^{(m-1)} \times_{\mathbf{s}} \times_{\mathbf{t}} \sharp U_L^m & \xrightarrow{mult} & H_L \end{array}$$

and the induction assumption imply that the diagonal map \sharp is a submersion, proving the claim.

The differentiable structure on an open subset of H_L defined in the above claim is independent of the choice of bi-submersion in the path holonomy atlas. Indeed if U, U' are two such bi-submersions and $u \in U, u' \in U'$ map to the same point of H_L under the quotient map, then there exists a (smooth) morphism a bi-submersions Φ defined near u' such that $\sharp_{U'} = \sharp_U \circ \Phi$, showing that $\sharp_{U'}$ is a smooth map for the differentiable structure defined by U . \square

3 Smoothness of the holonomy groupoid H restricted to a leaf

In this section we discuss conditions for the smoothness of the isotropy groups of the holonomy groupoid, of the fibres of its source map, and of its restriction to a leaf. Namely, after some preparation in §3.1-3.2, we construct a morphism $\varepsilon: G_x \rightarrow H_x^x$ from the simply connected Lie group integrating the isotropy Lie algebra \mathfrak{g}_x into the isotropy group at x of the holonomy groupoid. We use this morphism in §3.3 and §3.4 to show that when $\ker(\varepsilon)$ is closed then the isotropy group H_x^x and the restriction H_L of the holonomy groupoid to a leaf are both smooth.

We are going to consider the *path holonomy atlas*

3.1 A local group structure attached to bi-submersions

Let (M, \mathcal{F}) be a singular foliation. This section proves the following result: If $x \in M$ and $(U, \mathbf{t}, \mathbf{s})$ is a path holonomy bi-submersion, minimal at $(x, 0)$, then $U_x^x = \mathbf{s}^{-1}(x) \cap \mathbf{t}^{-1}(x)$ is a local Lie group.

Lemma 3.1. *Let $(U, \mathbf{t}, \mathbf{s})$ be a path holonomy bi-submersion, minimal at $(x, 0)$. Let $X \in \mathcal{F}$ and $Y, Y' \in C^\infty(U; \ker d\mathbf{s})$ such that $\mathbf{t}_*(Y) = \mathbf{t}_*(Y') = X$. Then the restrictions of Y, Y' to U_x^x are equal.*

Proof. We have to show that if $Z \in C^\infty(U; \ker ds) \cap C^\infty(U; \ker dt)$ then $Z|_{U_x^x} = 0$. Choose $X_1, \dots, X_n \in \mathcal{F}$ whose images in \mathcal{F}_x form a basis, and choose lifts $Y_i \in C^\infty(U; \ker ds)$, that is, $\mathbf{t}_* Y_i = X_i$. Since the Y_i are linearly independent (cf. proof of Lemma 2.6 or [De1, Prop. 1]) and $\dim(\mathcal{F}_x) = \dim(\ker ds)$ there exist (unique) $g_i \in C^\infty(U)$ for which $\sum_{i=1}^n g_i Y_i = Z$. Let $u \in U_x^x$, let D be a small submanifold of U through u transverse to the \mathbf{t} -fibers, and define $\tilde{g}_i \in C^\infty(M)$ by $(g_i)|_D = \tilde{g}_i \circ \mathbf{t}|_D$. We have

$$0 = \mathbf{t}_*(Z) = \sum_i \tilde{g}_i X_i,$$

so

$$[0] = [\sum_i \tilde{g}_i X_i] = \sum_i \tilde{g}_i(x) [X_i] \in \mathcal{F}_x.$$

As the $[X_i]$ form a basis of \mathcal{F}_x , we obtain $\tilde{g}_i(x) = 0$ for all i , so $g_i(u) = 0$ for all i , showing that $Z_u = 0$. As $u \in U_x^x$ was arbitrary, we conclude that $Z|_{U_x^x} = 0$. \square

Lemma 3.2. *Let $(U, \mathbf{t}, \mathbf{s})$ be a path holonomy bi-submersion, minimal at $(x, 0)$. There exists a well-defined Lie algebra isomorphism*

$$\delta: \mathfrak{g}_x \rightarrow \{Y \in \mathfrak{X}(U_x^x) : Y \text{ admits a } \mathbf{t}\text{-projectable extension } \hat{Y} \in C^\infty(U, \ker ds)\}, \quad [X] \mapsto \hat{Y}|_{U_x^x}$$

where $X \in \mathcal{F}(x)$ and \hat{Y} is any lift of X by \mathbf{t} lying in $C^\infty(U; \ker ds)$.

Remark 3.3. Let $\{X_i\}_{i \leq n} \in \mathcal{F}$ be vector fields whose images in \mathcal{F}_x form a basis of \mathcal{F}_x and such that the images of $\{X_i\}_{i \leq \ell}$ form a basis of \mathfrak{g}_x . Let U be the corresponding path holonomy bi-submersion. Choose lifts $Y_i \in C^\infty(U; \ker ds)$ of the X_i . Then the map δ is given by $[X_i] \mapsto Y_i|_{U_x^x}$ for all $i \leq \ell$.

Proof. We show that δ is well-defined. Fix an element of \mathfrak{g}_x and choose a representative X ; then $\hat{Y}|_{U_x^x}$ is uniquely defined by lemma 3.1. The independence from the choice of representative X is as follows: a lift of fZ , where $f \in \mathcal{I}_x$ and $Z \in \mathcal{F}$, is given by $\mathbf{t}^*(f) \cdot \hat{Z}$, where \hat{Z} is a lift of Z , hence on U_x^x it vanishes.

The injectivity of the map is clear from Rem. 3.3 together with the fact that the Y_i are linearly independent at every point of U_x^x . The surjectivity holds because $Y = \delta[\mathbf{t}_* \hat{Y}]$ for any \mathbf{t} -projectable extension \hat{Y} of Y . Hence δ is a linear isomorphism.

To see that δ is a Lie algebra morphism, take $X, X' \in \mathcal{F}$ and lifts $\hat{Y}, \hat{Y}' \in C^\infty(U; \ker ds)$ and use that $[\hat{Y}, \hat{Y}']$ is a lift of $[X, X']$, so that $\delta[[X, X']] = [\hat{Y}, \hat{Y}']|_{U_x^x} = [Y, Y'] = [\delta X, \delta X']$. \square

Proposition 3.4. *Let $(U, \mathbf{t}, \mathbf{s})$ be a path holonomy bi-submersion, minimal at x . Let G_x be the simply connected Lie group integrating \mathfrak{g}_x and denote by \tilde{G}_x a neighborhood of the unit. Shrinking \tilde{G}_x and U if necessary, there is a canonical diffeomorphism $\Delta: \tilde{G}_x \rightarrow U_x^x$ making U_x^x into a local Lie group with identity element $(x, 0)$.*

Proof. First notice that U_x^x is a submanifold of $U_x^x \subset \{x\} \times \mathbb{R}^n$ of dimension $\dim(\mathfrak{g}_x)$ containing $(x, 0)$. This is immediate if the vector fields $\{X_i\} \subset \mathcal{F}$ used to define the bi-submersion arise from a local splitting of (M, \mathcal{F}) as $(I^k, TI^k) \times (S, \mathcal{F}_S)$ as in Proposition 1.4. The general case holds because any path holonomy bi-submersion minimal at x is isomorphic to one arising from a local splitting of (M, \mathcal{F}) , by Lemma 2.6.

By Lemma 3.2, δ is an infinitesimal action of the Lie algebra \mathfrak{g}_x on the manifold U_x^x . Evaluating every $\delta([X])$ at x we obtain a map $\mathfrak{g}_x \rightarrow T_x(U_x^x)$ which is injective, as a consequence of the fact

that the $Y_i|_x$ are linearly independent (by the proof of Lemma 2.6). By dimension count, it follows that the previous map is an isomorphism, whence the infinitesimal action δ is locally free at x . By standard arguments, δ can be integrated to a local right group action of \tilde{G}_x on U_x^x . Evaluation at x gives a map

$$\Delta: \tilde{G}_x \rightarrow U_x^x, \quad g \mapsto x \cdot g, \quad (3.1)$$

which (shrinking \tilde{G}_x and U if necessary) is a diffeomorphism. \square

It's easy to see that for any $v \in \mathfrak{g}_x$, under the identification Δ , the infinitesimal generator $\delta(v) \in \mathfrak{X}(U_x^x)$ corresponds to $\overleftarrow{v} \in \mathfrak{X}(\tilde{G}_x)$, the left-invariant vector field whose value at the unit is v . Since $\exp_{\mathfrak{g}}(tv) \subset G$ is the integral curve of \overleftarrow{v} starting at the identity element, we obtain an alternative formula for Δ (here X_i and Y_i are as in Remark 3.3):

$$\Delta: \tilde{G}_x \rightarrow U_x^x, \quad \exp_{\mathfrak{g}}\left(\sum_{i=1}^l k_i [X_i]\right) \mapsto \exp_{(x,0)}\left(\sum_{i=1}^l k_i Y_i\right). \quad (3.2)$$

3.2 The canonical morphism into the isotropy groups of H

Here we introduce a canonical morphism of local topological groups from U_x^x to the isotropy group H_x^x of the holonomy groupoid.

Lemma 3.5. *Denote by \sharp the quotient map onto the holonomy groupoid as in §2.3. The composition*

$$\tilde{\varepsilon}: \tilde{G}_x \xrightarrow{\Delta} U_x^x \xrightarrow{\sharp} H_x^x \quad (3.3)$$

is independent of the choice of path holonomy bi-submersion U .

Proof. Let U and U' be bi-submersions, minimal at x . Then, from Lemma 2.6 there exists an *isomorphism* of bisubmersions $\Upsilon: U \rightarrow U'$ defined near x and fixing the point x . Since the maps δ of Lemma 3.2 are given in terms of the source and target maps, Υ intertwines the maps δ and δ' of Lemma 3.2, and therefore intertwines also the diffeomorphisms $\Phi: \tilde{G}_x \rightarrow U_x^x$ and $\Phi': \tilde{G}_x \rightarrow U'^x_x$. We conclude by noticing that Υ , as every morphism of bi-submersions, intertwines the quotient maps to H . \square

We prove two properties of the map $\tilde{\varepsilon}$ of eq. (3.3).

Lemma 3.6. *The image of the map $\tilde{\varepsilon}$ is open in H_x^x .*

Proof. Recall that H is endowed with the quotient topology given by the map $\sharp: \cup_{\alpha} U_{\alpha} \rightarrow H$. One has $\sharp U_x^x = \sharp U \cap H_x^x$, so it suffices to show that $\sharp U$ is open in H , which is true by the beginning of §2.4. \square

Proposition 3.7. *The map $\tilde{\varepsilon}$ in eq. (3.3) is a morphism of local topological groups.*

Proof. The map $\tilde{\varepsilon}$ is continuous since Δ and \sharp are. We check explicitly that formula (3.3) respects products, using eq. (3.2). Let $\tilde{\mathfrak{g}}_x$ be a neighborhood of the origin in \mathfrak{g}_x so that the exponential map $\exp_{\mathfrak{g}}: \tilde{\mathfrak{g}}_x \rightarrow \tilde{G}_x$ is a diffeomorphism. The (partial) product on \tilde{G}_x corresponds to the Baker-Campbell-Hausdorff formula (BCH) in $\tilde{\mathfrak{g}}_x$.

Let $\{X_i\}_{i \leq n} \in \mathcal{F}$ be vector fields whose images in \mathcal{F}_x form a basis of \mathcal{F}_x and such that the images of $\{X_i\}_{i \leq \ell}$ form a basis of \mathfrak{g}_x . Let $v^\alpha = \sum_{i \leq \ell} k_i^\alpha [X_i] \in \tilde{\mathfrak{g}}_x$ ($\alpha = 1, 2$) so that

$$v := BCH(v^1, v^2) = v^1 + v^2 + \frac{1}{2}[v^1, v^2] + \frac{1}{12}[[v^1, [v^1, v^2]] + \dots$$

also lies in $\tilde{\mathfrak{g}}_x$. Write $v = \sum_{i \leq \ell} k_i [X_i]$ for some coefficients $\vec{k} \in \mathbb{R}^\ell$. Consider the bi-submersion $U \subset M \times \mathbb{R}^n$ associated to X_1, \dots, X_n , and let $\{Y_i\}$ be a choice of vertical vector fields near $(x, 0)$ in $U \subset M \times \mathbb{R}^n$ which lifts X_i w.r.t. the target map. Put $u_\alpha := \Delta(\exp_{\mathfrak{g}} v^\alpha) = \exp_{(x,0)}(\sum_{i \leq \ell} k_i^\alpha Y_i)$ and $u = \exp_{(x,0)}(\sum_{i \leq \ell} k_i Y_i)$. Notice that we have $(u_1, u_2) \in U \circ U$, that is $\mathbf{s}(u_1) = x = \mathbf{t}(u_2)$, since the vector field $\sum_{i \leq \ell} k_i^2 Y_i$ maps under \mathbf{t}_* to $\sum_{i \leq \ell} k_i^2 X_i$, which vanishes at x . Since

$$\begin{aligned} \tilde{\varepsilon}(\exp_{\mathfrak{g}} v_1 \cdot \exp_{\mathfrak{g}} v_2) &= \tilde{\varepsilon}(\exp_{\mathfrak{g}} v) = \sharp u \\ \tilde{\varepsilon}(\exp_{\mathfrak{g}} v_1) \cdot \tilde{\varepsilon}(\exp_{\mathfrak{g}} v_2) &= \sharp(u_1, u_2), \end{aligned}$$

we have to show that there exists a morphism of bi-submersions $U \circ U \rightarrow U$ defined in a neighborhood of (u_1, u_2) such that $(u_1, u_2) \mapsto u$.

Consider the map

$$\varphi: \mathcal{F}_x \rightarrow \mathcal{F}, \quad [X_i] \mapsto X_i.$$

The map φ is not a Lie algebra homomorphism; rather, $\varphi[v^1, v^2]$ and $[\varphi v^1, \varphi v^2]$ differ by an element of $I_x \mathcal{F}$. Using the fact that φv^1 and φv^2 are vector fields vanishing on x one shows that $BCH(\varphi v^1, \varphi v^2)$ and $\varphi(BCH(v^1, v^2)) = \sum_{i \leq \ell} k_i X_i$ differ by an element $Z \in I_x \mathcal{F}$, which hence is of the form $\sum_{i \leq n} f_i X_i$ where $f_i \in I_x$. Therefore

$$BCH(\varphi v^1, \varphi v^2) = \varphi(BCH(v^1, v^2)) + Z = \sum_{i \leq n} \phi_i X_i \quad (3.4)$$

where $\phi_i = k_i + f_i \in C^\infty(M)$ (setting $k_i = 0$ for $\ell < i \leq n$). Notice that in particular $\phi_i(x) = k_i$.

Consider the following local diffeomorphism of M :

$$\exp\left(\sum_{i \leq \ell} k_i^1 X_i\right) \circ \exp\left(\sum_{i \leq \ell} k_i^2 X_i\right) = \exp(BCH(\varphi v^1, \varphi v^2)) = \exp\left(\sum_{i \leq n} \phi_i X_i\right), \quad (3.5)$$

where we used eq. (3.4) to write down the last term. The local diffeomorphism on the l.h.s. of (3.5) is carried in $U \circ U \subset M \times \mathbb{R}^n \times \mathbb{R}^n$ by the bisection

$$\{(\mathbf{t}(y, \vec{\lambda}_y), \exp_{(\mathbf{t}(y, \vec{\lambda}_y), 0)}(\sum_{i \leq \ell} k_i^1 Y_i)) \ , \ (y, \vec{\lambda}_y) : y \in M\}$$

where we use the short-hand notation $\vec{\lambda}_y := \exp_{(y,0)}(\sum_{i \leq \ell} k_i^2 Y_i)$ for all $y \in M$. This bisection passes through the point (u_1, u_2) since $\mathbf{t}(u_2) = x$. The local diffeomorphism on the r.h.s. of (3.5) is carried by the following bisection of U :

$$\{(y, \exp_{(y,0)}(\mathbf{t}^* \phi_i \cdot Y_i)) : y \in M\}.$$

This bisection passes through the point $u = \exp_{(x,0)}\left(\sum_{i \leq \ell} k_i Y_i\right)$. Indeed the curve

$$\gamma: \epsilon \mapsto \exp_{(x,0)}\left(\epsilon \sum_{i \leq \ell} \mathbf{t}^* \phi_i \cdot Y_i\right)$$

maps under \mathbf{t} to the constant curve at x (since the vector field $\sum_{i \leq \ell} \phi_i X_i$ vanishes at x), so along γ we have $\mathbf{t}^* \phi_i = \phi_i(x) = k_i$.

In conclusion, we found bisections of $U \circ U$ and U , passing respectively through (u_1, u_2) and u , which carry the same diffeomorphism. It follows from item d) in §2.2 that there exists a morphism of bi-submersions $U \circ U \rightarrow U$ such that $(u_1, u_2) \mapsto u$. \square

Last, let us discuss the kernel of $\tilde{\varepsilon}$.

Lemma 3.8. *Let $(U, \mathbf{t}, \mathbf{s})$ a path holonomy bi-submersion minimal at $x \in M$ and consider the map $\tilde{\varepsilon}: \tilde{G}_x \rightarrow H_x^x$. Then $g \in \ker \tilde{\varepsilon}$ iff there exists a bisection of $(U, \mathbf{t}, \mathbf{s})$ through $\Delta(g)$ carrying the identity, where U and Δ are as in Prop. 3.4.*

Proof. Let $g \in \ker \tilde{\varepsilon}$, i.e. $\Delta(g) \in \ker \sharp$. Since $(x, 0) \in U \subset M \times \mathbb{R}^n$ also belongs to $\ker \sharp$, there exists a morphism of bi-submersions $U \rightarrow U$ such that $(x, 0) \mapsto \Delta(g)$, and the image of $U \cap (M \times \{0\})$ is a bisection through $\Delta(g)$ carrying the identity. The converse follows by reversing the previous argument, which is allowed due to item d) in §2.2. \square

3.3 Smoothness of the isotropy groups of H

In this section we show that, under suitable hypotheses the isotropy groups of the holonomy groupoid are smooth.

Proposition 3.9. *Let $x \in (M, \mathcal{F})$ and let G_x be the connected and simply connected Lie group integrating the Lie algebra \mathfrak{g}_x . There exists a morphism of topological groups*

$$\varepsilon: G_x \rightarrow H_x^x$$

extending the map $\tilde{\varepsilon}$ in eq. (3.3). Further, ε maps surjectively onto \hat{H}_x^x , the connected component of the identity in H_x^x .

Proof. The map $\tilde{\varepsilon}$ is a morphism of local topological groups by Prop. 3.7 and G_x is simply connected, so $\tilde{\varepsilon}$ can be extended uniquely to a topological group morphism $\varepsilon: G_x \rightarrow H_x^x$ (cf. [Bo, III, Lemma 1, p. 304]).

By Lemma 3.6 $\tilde{\varepsilon}(\tilde{G}_x)$ is open in the topological group H_x^x , hence any point h in the connected component of the identity of H_x^x can be written as a product of elements of $\tilde{\varepsilon}(\tilde{G}_x) \cap (\tilde{\varepsilon}(\tilde{G}_x))^{-1}$. The point h is then the image under ε of a product of elements of \tilde{G}_x , so ε is surjective onto \hat{H}_x^x .

As a Lie group, G_x is second countable, so to prove the continuity of ε it suffices [Ja84, §VI.3] to show that if $(g_n)_{n \in \mathbb{N}}$ is a sequence in G_x which converges to $g \in G_x$ then $(\varepsilon(g_n))_{n \in \mathbb{N}}$ converges to $\varepsilon(g)$. The sequence $(g_n g^{-1})_{n \in \mathbb{N}}$ converges to the identity and we may assume that all of its terms except finitely many lie in a neighborhood of the identity in the domain of $\tilde{\varepsilon}$. We have:

$$\varepsilon(g_n) = \varepsilon(g_n g^{-1} g) = \tilde{\varepsilon}(g_n g^{-1}) \varepsilon(g)$$

We conclude passing to the limit and using the continuity of $\tilde{\varepsilon}$. \square

From Prop. 3.9 we have an exact sequence of topological groups

$$1 \rightarrow \ker \varepsilon \rightarrow G_x \xrightarrow{\varepsilon} \mathring{H}_x^x \rightarrow 1. \quad (3.6)$$

The group $\ker \varepsilon$ plays an important role: In §4.1 we will see that the discreteness of the group $\ker \varepsilon$ is the integrability obstruction for the Lie algebroid A_L . For these reasons it is worth giving $\ker \varepsilon$ a special name. To this end, notice that the isotropy group G_x is the same for every x in the leaf L , and so is the group $\ker \varepsilon$.

Definition 3.10. Let L be a leaf of the foliation (M, \mathcal{F}) . The group $\ker \varepsilon$ is called the *essential isotropy group* of L .

Notice that the group \mathring{H}_x^x is endowed with the quotient topology, which might be very coarse. In particular its identity element might *not* form a closed subset of \mathring{H}_x^x (an example of group with this property is \mathbb{R}/\mathbb{Q}). Hence \mathring{H}_x^x might not be a Lie group, and from the continuity of ε one can not conclude that the essential isotropy group is closed. Of course, this issue arises only for singular leaves, since on a regular leaf the infinitesimal isotropy vanishes, whence the group G_x is trivial.

Theorem 3.11. *The identity component \mathring{H}_x^x is smooth and is a Lie group if and only if the essential isotropy group is closed in G_x . In this case, $\varepsilon: G_x \rightarrow H_x^x$ is a submersion.*

Proof. Assume that the essential isotropy is closed. Then it is an embedded normal Lie subgroup of G_x , whence it follows from Cartan's theorem [Hel78, Thm 4.2, p. 123] that the quotient $G_x/\ker \varepsilon$ is a Lie group and the projection $G_x \rightarrow G_x/\ker \varepsilon$ is a submersion. Since the sequence (3.6) is exact, we conclude that \mathring{H}_x^x is a Lie group and ε a submersion.

Notice that the induced differentiable structure on \mathring{H}_x^x is the one defined in §2.4. Indeed, for any path holonomy bi-submersion U minimal at x , we have that \tilde{G}_x is diffeomorphic to a neighborhood of x in U_x^x , so $\sharp: U_x^x \rightarrow H_x^x$ is a submersion (see Lemma 3.5). [AnSk06, Prop. 2.10 b)] implies that the same holds for any bi-submersion U in the path holonomy atlas covering x .

Conversely, if \mathring{H}_x^x is a Lie group, then its identity element forms a closed subset, and since ε is continuous it follows that the essential isotropy group is closed. \square

Remark 3.12. Going through examples, we couldn't find any foliation (M, \mathcal{F}) whose groups \mathring{H}_x^x are not Lie groups for every $x \in M$. It may well be the case that the essential isotropy groups are always closed, however we couldn't provide a proof for this statement.

Here is an example of the map ε :

Corollary 3.13. *Let V be a vector space and $G \subset GL(V)$ a closed connected subgroup, and consider the foliation on V induced by the action. At $x = 0$ we have that $\varepsilon: G_x \rightarrow H_x^x = G$ is a covering map.*

Proof. Let $x = 0$. Recall that we have $\mathfrak{g}_x = \mathfrak{g} := Lie(G)$, so in particular G_x is the simply connected Lie group integrating \mathfrak{g} . Further we have $H_x^x \cong G$, and the identification is given by (the restriction to $\{x\} \times G$ of) a natural quotient map ψ from the transformation groupoid $V \rtimes G$ to the holonomy groupoid H (see [AnSk06, Prop. 1.4] and [AnSk06, Ex. 3.7(2)]).

Consider the natural minimal bi-submersion, namely $U \subseteq V \rtimes \mathfrak{g}$, with target $\mathfrak{t}(y, u) = exp_y(u_V)$, where u_V denotes the linear vector field on V induced by the infinitesimal action of $u \in \mathfrak{g}$. It is the path holonomy bi-submersion associated to $\{(u_i)_V\}$ for $\{u_i\}$ any basis of \mathfrak{g} . Notice that

U_x^x , for $x = 0$, is a neighborhood of the origin in \mathfrak{g} . One computes using eq. (3.1) that the map $\Delta: \tilde{G}_x \rightarrow U_x^x$ is given by the inverse of the Lie group exponential map $\mathfrak{g} \rightarrow G_x$. Further the quotient map $\sharp: U_x^x \rightarrow H_x^x = G$ is the Lie group exponential map, as a consequence of the fact that the morphism of bi-submersions $\phi: U \rightarrow V \times G, (y, u) \mapsto (y, \exp_{\mathfrak{g}}(u))$ intertwines the natural quotient maps $\sharp: U \rightarrow H$ and $\psi: V \times G \rightarrow H$ (see the text just before [AnSk06, Def. 3.5]). Hence

$$\tilde{\varepsilon}: \tilde{G}_x \xrightarrow{\Delta} U_x^x \xrightarrow{\sharp} H_x^x = G$$

is the local Lie group morphism that integrates $id_{\mathfrak{g}}$. \square

3.4 Smoothness of H_L

Here we show that, under suitable assumptions, the restriction of H to a leaf is a Lie groupoid. We start with a technical lemma:

Lemma 3.14. *Let (M, \mathcal{F}) be a foliated manifold and x a point lying in a leaf L of dimension k . As in Prop. 1.4, in a neighborhood W of x , choose a splitting $\psi: (W, \mathcal{F}_W) \cong (I^k, TI^k) \times (S, \mathcal{F}_S)$ and elements $\{X_i\}_{i \leq n}$ of \mathcal{F} inducing a basis of \mathcal{F}_x and compatible with the splitting⁴. Let $U \subset W \times \mathbb{R}^n$ be the associated path holonomy bi-submersion. Assume for simplicity that $U = W \times B^k \times B^\ell$ where B^k, B^ℓ are open neighborhoods of the origin in $\mathbb{R}^k, \mathbb{R}^\ell$ respectively.*

Then for any $y_0 \in L_W := \psi^{-1}(I^k \times \{x\})$, for any $\vec{\eta}_0, \hat{\eta}_0 \in B^k$ and $\vec{\lambda}_0, \hat{\lambda}_0 \in B^\ell$ we have:

$$(y_0, \vec{\eta}_0, \vec{\lambda}_0) \sim (y_0, \hat{\eta}_0, \hat{\lambda}_0) \Leftrightarrow \begin{cases} (x, 0, \vec{\lambda}_0) \sim (x, 0, \hat{\lambda}_0) \\ \vec{\eta}_0 = \hat{\eta}_0 \end{cases}$$

Remark 3.15. Lemma 3.14 is saying the the equivalence relation \sim introduced in §2.3, restricted to $U_{L_W} := L_W \times B^k \times B^\ell$, is trivial in the L_W and B^k components, while in the B^ℓ component it is the restriction of \sim to $U_x^x = \{x\} \times \{0\} \times B^\ell$.

Proof. Notice first that, since any of the $\{X_i\}_{i \leq k}$ commutes with any of the $\{X_i\}_{i > k}$, the target map sends $(z, \vec{\eta}, \vec{\lambda}) \in U$ to the element

$$(\vec{z}_I + \vec{\eta}, \exp_z(\sum_{i > k} \lambda_i X_i)) \in I^k \times S = W, \quad (3.7)$$

where \vec{z}_I is the first component of $z \in W = I^k \times S$.

Consider now an arbitrary morphism of bi-submersions $\Phi: U \rightarrow U$. From the first component of eq. (3.7) we deduce that Φ is of the form

$$(z, \vec{\eta}, \vec{\lambda}) \mapsto (z, \vec{\eta}, \vec{f}(z, \vec{\eta}, \vec{\lambda})), \quad (3.8)$$

for some smooth $\vec{f}: U \rightarrow B^\ell$, and from the second component of eq. (3.7) we deduce further that

$$\exp_z(\sum_{i > k} \lambda_i X_i) = \exp_z(\sum_{i > k} f_i(z, \vec{\eta}, \vec{\lambda}) X_i) \quad (3.9)$$

⁴In particular the first elements X_1, \dots, X_k are coordinate vector fields on L commuting with all X_i 's, and the remaining $\ell := n - k$ elements X_{k+1}, \dots, X_n vanish on L .

for all $z \in W$ and $\vec{\lambda} \in B^\ell$ (in particular the r.h.s. is independent of $\vec{\eta}$). Conversely, any map $U \rightarrow U$ of the form (3.8) satisfying eq. (3.9) is a morphism of bi-submersions.

To prove the implication “ \Leftarrow ”, assume that $(x, 0, \vec{\lambda}_0) \sim (x, 0, \vec{\lambda}_0)$. Then there exists a morphism Φ defined near $(x, 0, \vec{\lambda}_0)$ mapping one point to the other, that is, the function \vec{f} satisfies $\vec{f}(x, 0, \vec{\lambda}_0) = \vec{\lambda}_0$. Given $y_0 \in L_W = \psi^{-1}(I^k \times \{x\})$ and $\vec{\eta}_0 \in B^k$, define Φ' in a neighborhood of $(y_0, \vec{\eta}_0, \vec{\lambda}_0)$ in U by

$$(z, \vec{\eta}, \vec{\lambda}) \xrightarrow{\Phi'} (z, \vec{\eta}, \vec{f}(z - y_0, \vec{\eta} - \vec{\eta}_0, \vec{\lambda}))$$

where $z - y_0 = (z_I - y_0, z_S) \in I^k \times S = W$. The map Φ' is a morphism of bi-submersions for it satisfies eq. (3.9), as a consequence of the fact that Φ satisfies eq. (3.9) and of the invariance of the vector fields X_i in the I^k -direction. Further Φ' maps $(y_0, \vec{\eta}_0, \vec{\lambda}_0)$ to $(y_0, \vec{\eta}_0, \vec{\lambda}_0)$, showing that these two elements are equivalent. The converse implication is proven similarly. \square

Theorem 3.16. *Let L be a leaf and $H_L \rightrightarrows L$ the transitive groupoid $H_L := \mathfrak{s}^{-1}(L) = \mathfrak{t}^{-1}(L)$. The essential isotropy group (Def. 3.10) of L is closed if and only if H_L is smooth and is a Lie groupoid.*

Proof. If H_L is a (transitive) Lie groupoid then H_x^x is a Lie group, so Thm. 3.11 implies that the essential isotropy group of L is closed.

For the converse, we show that the assumptions of Prop. 2.11 are satisfied, and then apply Lemma 2.9. Let $x \in L$, let U be a bi-submersion as in Lemma 3.14, and set $U_{L_W} := L_W \times B^k \times B^\ell$.

Consider first U_x^x . Thm. 3.11 implies that $\sharp|_{U_x^x}: U_x^x \rightarrow H_x^x$ is a submersion between smooth manifolds. Let $S^x \subset U_x^x$ be a smooth submanifold through x transverse to the fibres of $\sharp|_{U_x^x}$, and without loss of generality assume that $\sharp S^x = \sharp U_x^x$ (an open subset of H_x^x , by Lemma 3.6). Clearly $\sharp|_{S^x}: S^x \rightarrow \sharp U_x^x$ is a diffeomorphism.

Notice that $\sharp U_{L_W}$ is an open neighborhood of x in H_L , since $\sharp U$ is open in H (see the beginning of §2.4). Since by definition the fibres of $\sharp: U_{L_W} \rightarrow H_L$ are given by the equivalence classes of \sim in U_{L_W} , Lemma 3.14 (see also Remark 3.15) implies that at every point $(y, \vec{\eta}, \vec{\lambda})$ of U_{L_W} we have

$$(\text{Fibre of } \sharp: U_{L_W} \rightarrow H_L \text{ through } (y, \vec{\eta}, \vec{\lambda})) = \{y\} \times \{\vec{\eta}\} \times (\text{Fibre of } \sharp|_{U_x^x} \text{ through } (x, 0, \vec{\lambda})).$$

From this we conclude that $T^x := L_W \times B^k \times S^x$ is a smooth submanifold of U_{L_W} through $(x, 0, 0)$ transverse to the \sharp -fibres. Therefore $\sharp: U_{L_W} \rightarrow H_L$ restricts to a homeomorphism

$$\sharp|_{T^x}: T^x \rightarrow \sharp U_{L_W},$$

which endows $\sharp U_{L_W}$ with a differentiable structure such that that $\sharp: U_{L_W} \rightarrow H_L$ is a submersion. As $x \in L$ is arbitrary, we showed that the assumptions of Prop. 2.11 are satisfied. \square

Proposition 3.17. *Let $x \in M$ and L the leaf of \mathcal{F} at x . Its essential isotropy group is closed if and only if the source fiber H_x of the holonomy groupoid is smooth (in the sense of §2.4). In this case, $\mathfrak{t}: H_x \rightarrow L$ is a principal H_x^x -bundle.*

Proof. As we showed in Thm. 3.16, if the essential isotropy is closed then H_L is smooth, whence H_x is smooth as well. For the converse, it suffices to show that H_x^x is smooth, because it follows then from Thm. 3.11 that the essential isotropy group is closed.

Let $(U, \mathfrak{t}, \mathfrak{s})$ be a bi-submersion in the path holonomy atlas with $x \in \mathfrak{s}(U)$. Put $U_x = \mathfrak{s}^{-1}(x)$ and $\mathfrak{t}_x = \mathfrak{t}|_{U_x}: U_x \rightarrow L$. The latter is a submersion, as a consequence of the definition of bi-submersion.

Hence $U_x^x = \mathbf{t}_x^{-1}(x)$ is an embedded submanifold of U_x (if it is not empty). By assumption, H_x is endowed with a smooth structure such that $\sharp: U_x \rightarrow H_x$ is a submersion. U_x^x is a union of fibers of \sharp , by the definition of the equivalence relation \sim . This and the local form theorem for submersions imply that $\sharp|_{U_x^x}: U_x^x \rightarrow \sharp U_x^x$ is a submersion. As the bi-submersion U was arbitrary, it follows that H_x^x is smooth in the sense of §2.4.

To prove the second statement, use Thm. 3.16 and recall that any source fibre of a Lie groupoid is a principal bundle for the corresponding isotropy group [MoMr03, Thm. 5.4]. \square

Proposition 3.17 and Thm. 4.1 are crucial for the generalization of Heitsch's [He73, Thm. 3] to singular foliations. We will explain this in a subsequent article [AZ2].

4 Essential isotropy and integrability

We prove that the Lie algebroid A_L is integrable provided the essential isotropy group of the leaf L is discrete (§4.1). Then, in §4.2 we provide a sufficient condition for the discreteness of the essential isotropy group.

4.1 Integrability of A_L

Fix a leaf L and consider the associated transitive Lie algebroid, $A_L = \cup_{x \in L} \mathcal{F}_x$, see §1.3. Here we discuss the integrability of this Lie algebroid, and show that the integrability of A_L implies the smoothness of the \mathfrak{s} -fibers of the holonomy groupoid. When the \mathfrak{s} -fibers are smooth, then the construction of the pseudodifferential calculus in [AnSk09] is simplified a great deal because then the left-regular representation is defined.

Theorem 4.1. *Let (M, \mathcal{F}) be a foliation and L a leaf. The transitive groupoid H_L is smooth and integrates the Lie algebroid $A_L = \cup_{x \in L} \mathcal{F}_x$ if and only if the essential isotropy group of L (Def. 3.10) is discrete.*

Remark 4.2. Notice that the Lie algebroid A_L integrates to H_L iff the Lie algebra \mathfrak{g}_x integrates to \mathring{H}_x^x (for any $x \in L$), for both conditions are equivalent to the discreteness of the essential isotropy group of L .

Proof. It is known that every Lie algebroid integrates to a *local* Lie groupoid. We start by describing explicitly a local Lie groupoid that integrates A_L . Recall from §1.1 that the leaf L is endowed with the longitudinal smooth structure. Cover L by open subsets W_α of M as in the splitting theorem (Prop. 1.4), i.e. so that there exists an isomorphism of foliated manifolds $\psi_\alpha: (W_\alpha, \mathcal{F}_{W_\alpha}) \cong (I^k, TI^k) \times (S_\alpha, \mathcal{F}_{S_\alpha})$ where $I = (-1, 1)$ and S_α is a slice transverse to L at some point x_α . We fix a choice of generators $\{X_i^\alpha\}_{1 \leq i \leq n}$ of \mathcal{F}_{W_α} as in Lemma 3.14. Denote $L_\alpha := \psi_\alpha^{-1}(I^k \times \{x_\alpha\})$.

The vector fields $\{X_i^\alpha\}_{1 \leq i \leq n} \subset \mathcal{F}$ induce a frame $\{\xi_i^\alpha\}_{1 \leq i \leq n}$ of $A_L|_{L_\alpha}$ via the projection $\mathcal{F}_{W_\alpha} \rightarrow C^\infty(L_\alpha, A_L|_{L_\alpha}) = \mathcal{F}_{W_\alpha}/I_{L_\alpha}\mathcal{F}_{W_\alpha}$. This frame induces an isomorphism of Lie algebroids over L_α

$$\nu_\alpha: A_L|_{L_\alpha} \cong TL_\alpha \oplus \mathfrak{g}_x$$

which maps $\xi_i^\alpha \mapsto X_i^\alpha|_{L_\alpha}$ for $i \leq k$ and $\xi_i^\alpha \mapsto [X_i^\alpha] \in \mathfrak{g}_x$ for $k < i \leq n$. A Lie groupoid integrating $TL_\alpha \oplus \mathfrak{g}_x \rightarrow L_\alpha$ is $(L_\alpha \times L_\alpha) \times G_x$, the product of the pair groupoid over L_α and the simply

connected Lie group integrating the Lie algebra \mathfrak{g}_x . We will now relate this Lie groupoid the path holonomy bi-submersion U_α defined by the $\{X_i^{\alpha}\}_{1 \leq i \leq n} \subset \mathcal{F}$.

As in Lemma 3.14, assume that U_α is of the form $W_\alpha \times B^k \times B^\ell$ where B^k, B^ℓ are open neighborhoods of the origin in $\mathbb{R}^k, \mathbb{R}^\ell$ respectively. We make use of $\mathbb{R}^\ell \cong \mathfrak{g}_x$ given by the basis $\{[X_i^{\alpha}]\}_{k < i \leq n}$ of \mathfrak{g}_x , and we use the splitting theorem to identify L_α with I^k . The identification

$$\begin{aligned} L_\alpha \times B^k \times B^\ell &\rightarrow \text{neighborhood of the identity section of } (L_\alpha \times L_\alpha) \times G_x, \\ (z, \vec{\eta}, \vec{\lambda}) &\mapsto (z + \vec{\eta}, z, \exp_{\mathfrak{g}_x} \vec{\lambda}) \end{aligned} \quad (4.1)$$

makes $L_\alpha \times B^k \times B^\ell$ into a local Lie groupoid, which we denote by Ω_α . It has the following properties: Ω_α integrates $A_L|_{L_\alpha}$ (because $(L_\alpha \times L_\alpha) \times G_x$ does), and its source and target map of Ω_α agree with those of the path holonomy bi-submersion U_α by eq. (3.7).

Consider the restriction $\Omega_\alpha|_{L_{\alpha\beta}}$ of Ω_α to $L_{\alpha\beta} := L_\alpha \cap L_\beta$. The Lie algebroid isomorphisms

$$\nu_\beta \circ \nu_\alpha^{-1}: (TL_\alpha \oplus \mathfrak{g}_x)|_{L_{\alpha\beta}} \rightarrow (TL_\beta \oplus \mathfrak{g}_x)|_{L_{\alpha\beta}}$$

integrate to isomorphisms of local Lie groupoids $\mu_{\beta\alpha}: \Omega_\alpha|_{L_{\alpha\beta}} \rightarrow \Omega_\beta|_{L_{\alpha\beta}}$ which satisfy

$$\mu_{\alpha\alpha} = id, \quad \mu_{\alpha\beta}^{-1} = \mu_{\beta\alpha}, \quad \mu_{\alpha\beta} \circ \mu_{\beta\gamma} = \mu_{\alpha\gamma}.$$

Now define the equivalence relation

$$\Omega_\alpha \ni \omega_\alpha \sim \omega_\beta \in \Omega_\beta \iff \omega_\alpha \in \Omega_\alpha|_{L_{\alpha\beta}}, \quad \omega_\beta \in \Omega_\beta|_{L_{\alpha\beta}} \quad \text{and} \quad \mu_{\beta\alpha}(\omega_\alpha) = \omega_\beta$$

Then the local Lie groupoid $\Omega := \coprod_{\alpha} \Omega_\alpha / \sim$ is a local Lie groupoid integrating A_L .

For every α , consider the quotient map $\sharp: \Omega_\alpha \rightarrow H_L$, and recall that Ω_α is a local Lie groupoid whose underlying bi-submersion is U_α . Since the $\mu_{\beta\alpha}$ are in particular morphisms of bi-submersions, by the definition of holonomy groupoid these quotient maps assemble to a map $\phi: \Omega \rightarrow H_L$.

We claim: $\phi: \Omega \rightarrow H_L$ is a morphism of local groupoids.

To prove the claim it suffices to show that the quotient map $\sharp: \Omega_\alpha \rightarrow H_L$ is a morphism of local groupoids, for every α . In turn, this is a consequence of the fact that the multiplication and inversion of any local Lie groupoid are morphisms of bi-submersions. More explicitly: $\sharp: \Omega_\alpha \rightarrow H_L$ preserves multiplications because in the diagram

$$\begin{array}{ccc} \Omega_\alpha \circ \Omega_\alpha & \xrightarrow{\text{mult.}} & \Omega_\alpha \\ \downarrow \sharp \times \sharp & \searrow \sharp|_{\Omega_\alpha \circ \Omega_\alpha} & \downarrow \sharp \\ (H_L)_s \times_{\mathfrak{t}} (H_L) & \xrightarrow{\text{mult.}} & H_L \end{array}$$

the lower triangle commutes by the definition of multiplication on H and the upper triangle commutes since the multiplication on Ω_α is a morphism of bi-submersions. This proves the claim.

Notice that ϕ maps onto an open neighborhood of L in H_L , as a consequence of the fact that $\sharp U$ is open in H for any path holonomy bi-submersion U (see the beginning of §2.4). At every point $x \in L$, by eq. (3.3), in a neighborhood of the unit in G_x , the essential isotropy group is identified with $\phi^{-1}(x)$. Hence the essential isotropy group of the leaf L is discrete iff the fibers of $\ker(\phi) = \cup_{x \in L} \phi^{-1}(x)$ are discrete.

Assume now that the essential isotropy group is discrete. Restricting the extension

$$0 \rightarrow \ker \phi \rightarrow \Omega \rightarrow H_L \rightarrow 0$$

to a neighborhood of the set of units L we deduce that a neighborhood of L in H_L is a local Lie groupoid integrating A_L . Since Ω is a union of path holonomy bi-submersions, from Lemma 2.9 and Prop. 2.11 we deduce that the H_L is smooth and is a Lie groupoid. Conversely, if we assume that H_L is a Lie groupoid integrating A_L , then the above extension shows that $\ker(\phi)$ is discrete. \square

Remark 4.3. Crainic and Fernandes [CrF] determined the exact obstruction for the integrability of the Lie algebroid A_L to a *source simply connected* Lie groupoid Γ : it is the discreteness (or equivalently, the closeness) of a certain subgroup $\tilde{\mathcal{N}}_x(A_L)$ of the center of G_x , called *monodromy group*.

When the essential isotropy group $\ker \varepsilon$ is discrete then the Lie algebroid A_L is integrable by Thm. 4.1, and we have

$$\tilde{\mathcal{N}}_x(A_L) \subset \ker \varepsilon.$$

Indeed, $\tilde{\mathcal{N}}_x(A_L)$ is defined as the kernel of the morphism $G_x \rightarrow \Gamma_x^x$ induced by the inclusion $\mathfrak{g}_x \hookrightarrow A_L$, there is a morphism $\Gamma_x^x \rightarrow H_x^x$ induced by (the restriction of) Id_{A_L} , and ε factors as $G_x \rightarrow \Gamma_x^x \rightarrow H_x^x$ (since ε is induced by $Id_{\mathfrak{g}_x}$). At this stage we do not know how $\ker \varepsilon$ relates to the monodromy group in general. For instance, it is not clear if, in the general case, the discreteness of $\tilde{\mathcal{N}}_x(A_L)$ implies the discreteness of $\ker \varepsilon$, for the former controls the integration of A_L to a source simply connected Lie groupoid, whereas the latter controls the integration of A_L to H_L . The general comparison of the two integrability obstructions is beyond the scopes of this work; this might be discussed in a different paper.

Notice that, whenever $\ker \varepsilon$ is totally disconnected (for instance, discrete) it automatically contained in the center of G_x . Indeed if $k \in \ker \varepsilon$ and $g \in G_x$, consider a continuous path $t \mapsto g_t$ in G_x from the identity to g . The path $t \mapsto g_t k g_t^{-1}$ is contained in $\ker \varepsilon$ and sits at k at time zero, so the fact that $\ker \varepsilon$ is totally disconnected implies that $g k g^{-1} = k$, i.e. k commutes with g .

Remark 4.4. The structure of the local Lie groupoid $\Omega_\alpha = L_\alpha \times B^k \times B^\ell \rightrightarrows W_\alpha$ induced by the identification (4.1) is the following⁵:

- $\mathbf{s}(z, \vec{\eta}, \vec{\lambda}) = z$ and $\mathbf{t}(z, \vec{\eta}, \vec{\lambda}) = z + \vec{\eta}$;
- The unit map is the embedding of W_α ;
- The local product is given by

$$(z_1, \vec{\eta}_1, \vec{\lambda}_1) \cdot (z_2, \vec{\eta}_2, \vec{\lambda}_2) = (z_2, \vec{\eta}_1 + \vec{\eta}_2, BCH(\vec{\lambda}_1, \vec{\lambda}_2))$$

where $BCH(\vec{\lambda}_1, \vec{\lambda}_2)$ stands for the Baker-Campbell-Hausdorff product of $\vec{\lambda}_i \in B^\ell \subset \mathfrak{g}_x$;

- The inversion map is

$$\iota : g = (z, \vec{\eta}, \vec{\lambda}) \mapsto (z + \vec{\eta}, -\vec{\eta}, -\vec{\lambda}).$$

Example 4.5. Let X be a complete vector field on M and $\mathcal{F} = \langle X \rangle$. As in Ex. 2.7 (ii) assume that, for all $x \in \partial\{X = 0\}$, every neighborhood of x contains at least one point the integral curve through which is not periodic. Fix such a point x and denote $L = \{x\}$. Then $\varepsilon : G_x \rightarrow H_x^x$ is a surjective group morphism $(\mathbb{R}, +) \rightarrow (\mathbb{R}, +)$, i.e. a non-zero multiple of $Id_{(\mathbb{R}, +)}$, hence $\ker(\varepsilon) = \{0\}$. Clearly $H_L = (\mathbb{R}, +)$ integrates the Lie algebroid $A_L = \mathfrak{g}_x = \mathbb{R}$, as predicted by Thm. 4.1.

⁵Recall that we use the splitting theorem to identify W_α with I^k

Example 4.6. Let V be a finite dimensional vector space and $G \subset G(V)$ a closed connected subgroup. Consider the singular foliation given by the action of G on V . Let $L = \{x\}$ for $x = 0$. Then $\varepsilon: G_x \rightarrow H_x^x = G$ is a covering map by Cor. 3.13, where G_x denotes the simply connected cover of G . Hence $\ker(\varepsilon)$ is discrete. Clearly $H_L = H_x^x = G$ integrates the Lie algebroid $A_L = \mathfrak{g}_x = \mathfrak{g}$, as predicted by Thm. 4.1.

Example 4.7. Consider $M = \mathbb{T}^2 \times \mathbb{R}^2$, with coordinates $(\theta_1, \theta_2, t_1, t_2)$, endowed with the singular foliation \mathcal{F} spanned by

$$\begin{aligned} v_1 &= \partial_{\theta_1} + t_1 t_2 \partial_{t_1} & w_1 &= t_1^2 t_2 \partial_{t_1} \\ v_2 &= \partial_{\theta_2} + t_1 t_2 \partial_{t_2} & w_2 &= t_1 t_2^2 \partial_{t_2}. \end{aligned}$$

\mathcal{F} is involutive, as

$$[v_1, v_2] = w_2 - w_1, \quad [v_1, w_1] = t_2 w_1, \quad [v_1, w_2] = t_2(w_2 - w_1), \quad [w_1, w_2] = t_1 t_2(w_2 - w_1).$$

(The remaining brackets are deduced easily from the symmetry that relates v_1 to v_2 and w_1 to w_2 .)

For $L = \mathbb{T}^2 \times \{0\}$ the Lie algebroid A_L , as a vector bundle, is the trivial rank 4 vector bundle. The images of the above vector fields under the quotient map $\mathcal{F} \rightarrow \mathcal{F}/I_L \mathcal{F} = C^\infty(L; A_L)$ form a trivializing frame of sections $\underline{v}_1, \underline{v}_2, \underline{w}_1, \underline{w}_2$ of A_L . The Lie algebroid structure of A_L in terms of this frame is as follows: the anchor maps $\underline{v}_i \mapsto \partial_{\theta_i}$ and $\underline{w}_i \mapsto 0$, while the only non-zero bracket is $[\underline{v}_1, \underline{v}_2]_{A_L} = \underline{w}_2 - \underline{w}_1$.

The Lie algebroid A_L is integrable, as every transitive Lie algebroid over a base with trivial second homotopy group π_2 [CrF, Cor. 5.5]. Explicitly, denote by \mathfrak{k} the 4-dimensional Lie algebra whose basis we denote (abusing slightly notation) by $\underline{v}_1, \underline{v}_2, \underline{w}_1, \underline{w}_2$, satisfying the above bracket relations. A_L is the transformation algebroid for the action of \mathfrak{k} on L given by the above anchor map, so a Lie groupoid integrating A_L is the transformation groupoid $K \times L \rightrightarrows L$ where K is the simply connected Lie group integrating \mathfrak{k} . Notice that K is the product of $\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$ with $(\mathbb{R}, +)$, since \mathfrak{k} is the direct sum of the Heisenberg Lie algebra with \mathbb{R} .

Example 4.8. Let L be a manifold and $\alpha \in \Omega^1(L)$ a closed 1-form. On $M := L \times \mathbb{R}$ consider the singular foliation spanned by

$$w := t^2 \partial_t \quad \text{and} \quad X^\alpha := X + \alpha(X) t \partial_t,$$

as X ranges over all vector fields of L (extended trivially to vector fields on $L \times \mathbb{R}$). Here t denotes the coordinate on \mathbb{R} . The singular foliation is involutive: $[X^\alpha, w] = \alpha(X)w$, and $[X^\alpha, Y^\alpha] = [X, Y]^\alpha$ since α is closed.

The Lie algebroid A_L is isomorphic as a vector bundle to $TL \times \mathbb{R} \rightarrow L$ with the obvious anchor and bracket given by $[(X, 0), (Y, 0)]_{A_L} = ([X, Y], 0)$ and $[(X, 0), (0, 1)]_{A_L} = (0, \alpha(X))$. This Lie algebroid is integrable. Indeed, the formula $\nabla_X 1 = \alpha(X)$ determines a flat connection on the trivial \mathbb{R} -bundle over L , that is, a representation of TL on the trivial \mathbb{R} -bundle. The above Lie algebroid is precisely the transformation algebroid of this representation, and as such it is integrable (by the transformation groupoid induced by the holonomy of ∇).

4.2 A criterion for the discreteness of the essential isotropy group

This section provides a condition for the discreteness of $\ker(\varepsilon)$.

Theorem 4.9. *Let S_x be a slice to the leaf L_x at x . Assume that the following condition is satisfied: for any smooth time-dependent vector field $\{X_t\}_{t \in [0,1]}$ in $I_x \mathcal{F}_{S_x}$, there exists a vector field $Z \in I_x \mathcal{F}_{S_x}$ and a neighborhood S' of x in S_x such that $\exp(Z)|_{S'} = \phi|_{S'}$, where ϕ denotes the time-1 flow of $\{X_t\}_{t \in [0,1]}$.*

Then the essential isotropy group of L_x (Def. 3.10) is discrete.

Remark 4.10. $I_x \mathcal{F}_{S_x}$ is a Lie subalgebra of the infinite dimensional Lie algebra $\mathfrak{X}_c(S_x)$, and it is well-known that the exponential map (the time-1 flow) $\mathfrak{X}_c(S_x) \rightarrow \text{Diff}(S_x)$ is *not* locally surjective [Mi83, Ex.6.6] [KrMi97, Ex. 43.2]. The above condition amounts to a local surjectivity statement for the restriction of the exponential map to $I_x \mathcal{F}_{S_x}$ and to a subgroup of $\text{Diff}(M)$, where one is allowed to shrink S_x to a small neighborhood of x . We do now know how restrictive the above condition is.

Proof. Let $\{X_i\}_{i \leq n}$ be generators of \mathcal{F} in a neighborhood of x , chosen compatibly with a splitting $W \cong I^k \times S_x$ as in Lemma 3.14 (so X_{k+1}, \dots, X_n are tangent to S_x). Denote by $U \subset W \times \mathbb{R}^n$ the corresponding path holonomy bi-submersion. We have $U_x^x = \{x\} \times \{0\} \times \mathbb{R}^\ell$ by eq. (3.7). Take $D \subset \mathbb{R}^\ell$ to be a neighborhood of the origin such that the matrix exponential $\mathfrak{gl}(T_x S_x) \rightarrow GL(T_x S_x)$ is injective on $\{\sum_{i>k} \gamma_i X_i^{lin} : \gamma \in D\}$, where $X_i^{lin} \in \mathfrak{X}_{lin}(T_x S_x) \cong \mathfrak{gl}(T_x S_x)$ denotes the linearization

of X_i at x . We show that $\ker(\varepsilon) \cap \Delta^{-1}(\{x\} \times \{0\} \times D)$ consists of just one point, namely the identity element of G_x , see eq. (3.3).

Let $u := (x, 0, \lambda^0) \in U_x^x$ with $\lambda^0 \in D$. Assume that $\Delta^{-1}(u) \in \ker(\varepsilon)$. By Lemma 3.8 this is equivalent to the existence of a bisection s of U through u (say given by $z \mapsto (z, \eta(z), \lambda(z))$) carrying the identity. Notice that in particular $\lambda(x) = \lambda^0$. We want to show that $\lambda^0 = 0$.

The diffeomorphism carried by s is (essentially) given by eq. (3.7), so that we necessarily have $\eta(z) = 0$ for all z . From now on we consider only $U|_{S_x}$. Restricting the identity diffeomorphism to the slice S_x we obtain that

$$S_x \rightarrow S_x, \quad z \mapsto \exp_z\left(\sum_{i>k} \lambda_i(z) X_i\right)$$

is Id_{S_x} . Now consider the vector field

$$Y := \sum_{i>k} \lambda_i^0 X_i|_{S_x}$$

on S_x . Denote by $\phi = \exp(Y)$ its flow, which is also the diffeomorphism carried by the *constant bisection* $z \mapsto (z, 0, \lambda^0)$ of $U|_{S_x}$. We apply a theorem of [AZ2] to the foliation (S_x, \mathcal{F}_{S_x}) (of which $\{x\}$ is a singular leaf) at $u \in U_x^x$ in order to compare the diffeomorphisms carried by the bisection $s|_{S_x}$ (which is Id_{S_x}) and the constant bisection λ^0 , and we deduce that ϕ is the time-1 flow of a *time-dependent* vector field $\{X_t\}_{t \in [0,1]}$ in $I_x \mathcal{F}_{S_x}$. The condition we assumed implies that there is $Z \in I_x \mathcal{F}_{S_x}$ such that $\exp(Z) = \phi = \exp(Y)$, shrinking S_x if necessary. Hence

$$Id_{S_x} = \exp(Y)\exp(-Z) = \exp(BCH(Y, Z)). \quad (4.2)$$

Now consider the vector field $BCH(Y, Z)$. Notice that $BCH(Y, Z) = Y + Z'$ for some $Z' \in I_x \mathcal{F}_{S_x}$. The flow $\varphi^t := \exp(t(Y + Z'))$ fixes x . The flow of the linearization $Y^{lin} = (Y + Z')^{lin}$ is the linearization of the flow of $Y + Z'$, which is $d_x \varphi^1 = d_x Id_M = Id_{T_x M}$. Since the flow of linear vector fields on $T_x S_x$ is given by the matrix exponential $\mathfrak{gl}(T_x S_x) \rightarrow GL(T_x S_x)$ and $\lambda^0 \in D$, the injectivity condition on D implies that $(Y + Z')^{lin} = 0$. Hence its flow $d_x \varphi^t$ is the identity for any t . Bochner's linearization theorem [DuKo00, Thm. 2.2.1] implies that the circle⁶ actions on S_x (by $t \rightarrow \varphi^t$) and

⁶We have an S^1 action because of eq. (4.2).

on $T_x S_x$ (by $t \mapsto d_x \varphi^t = Id$) are equivalent, hence we conclude that $\varphi^t = Id_M$ for all t , that is, that $Y + Z' = 0$. Hence $Y = -Z' \in I_x \mathcal{F}_{S_x}$. Taking the image of Y under $\mathcal{F}(x) \rightarrow \mathfrak{g}_x$ we obtain $0 = [-Z'] = [Y] = \sum_{i>k} \lambda_i^0 [X_i]$, and since the $\{[X_i]\}_{i>k}$ for a basis of \mathfrak{g}_x we conclude $\lambda^0 = 0$. \square

5 Outlook: Holonomy transformations and the normal form of a singular foliation

In the present note we considered smoothness statements about the holonomy groupoid H . This, together with an analysis of the notion of holonomy for singular foliations, allows to construct a model for the singular foliation near a leaf, leading to the question of when the singular foliation is isomorphic to the local model. We provide a brief outline of this thoughts, which will be exposed thoroughly in a subsequent paper [AZ2].

Let us start by recalling the following well-known facts about regular foliations:

- Geometrically, the holonomy of a co-dimension q (regular) foliation (M, F) at a point $x \in M$ is realized by a map $h: \pi_1(L_x) \rightarrow GermDiff(S)$, where L_x is the leaf at x , S is a transversal at x , and $GermDiff(S)$ is the space of germs of local diffeomorphisms of S . Its linearisation $Lin(h): \pi_1(L_x) \rightarrow GL(q)$ gives rise to a representation of the holonomy groupoid on the normal bundle $(TM/F)|_{L_x}$.
- The (local) Reeb stability theorem (see [MoMr03, Thm 2.9]) states that around a compact leaf L with finite holonomy, the manifold is diffeomorphic to the quotient $\frac{\tilde{L} \times \mathbb{R}^q}{\pi_1(L)}$, where \tilde{L} is the universal cover of the leaf, and $\pi_1(L)$ acts on $\tilde{L} \times \mathbb{R}^q$ diagonally by deck transformations and linearized holonomy.

One encounters serious problems trying to define holonomy for singular foliations naively in terms of paths. Rather than using paths, we threat the notion of holonomy as follows. At the end we discuss the analog of the Reeb stability theorem.

- For x, y in the same leaf consider transversals S_x, S_y of the leaf at x and y respectively. There is a map

$$\Phi_x^y: H(M, \mathcal{F})_x^y \rightarrow \frac{GermAut_{\mathcal{F}}(S_x; S_y)}{exp(I_x \mathcal{F})|_{S_x}},$$

whose target is the space of germs of foliation-preserving local diffeomorphisms between S_x and S_y , quotiented by the group generated by exponentials of elements in the maximal ideal $I_x \mathcal{F}$. The maps Φ_x^y assemble to a morphism of groupoids. We refer to elements of the target as “*holonomy transformations from x to y* ”.

- The map Φ linearises to a morphism of groupoids $Lin(\Phi): H(M, \mathcal{F}) \rightarrow Iso(N)$, whose target is the groupoid of isomorphisms between the fibres of the (singular) normal bundle N to the leaves (linear holonomy). This map can be interpreted as the adjoint representation.
- We can make sense of the quotient

$$\frac{H_x \times N_x L}{H_x^x}. \tag{5.1}$$

Namely, H_x^x acts on N_xL by $Lin(\Phi)$. Our Thm. 3.16 ensures that H_x and H_x^x are both smooth, provided the essential isotropy group of L is closed, so that the quotient (5.1) is a smooth manifold if H_x^x is compact. Endowing H_x with the foliation having just one leaf and N_xL with the (H_x^x -invariant) singular foliation obtained linearizing \mathcal{F} at x , the above space is further endowed with a singular foliation.

The generalization of Reeb's stability theorem to singular foliations answers the question: When L is compact and H_x^x is a compact Lie group, under which other conditions (if any) is there a diffeomorphism of foliated manifolds between a neighborhood of L and the local model (5.1)?

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