THE CONVOLUTION ALGEBRA OF SCHWARTZ KERNELS ON A SINGULAR FOLIATION

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Abstract. Motivated by the study of Hörmander's sums-of-squares operators and their generalizations, we define the convolution algebra of proper distributions associated to a singular foliation. We prove that this algebra is represented as continuous linear operators on the spaces of smooth functions and generalized functions on the underlying manifold. This generalizes Schwartz kernel operators to singular foliations. We also define the algebra of smoothing operators in this context and prove that it is a two-sided ideal.

Contents

Key words and phrases. foliation, singular foliation, distributions, Schwartz kernel, convolution, pseudodifferential operators, subriemannian geometry.

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1. Introduction

The goal of this article is to lay the analytical foundations for a study of an extremely broad class of pseudodifferential operators, including as special cases:

- the Heisenberg calculus and its generalizations [BG88, Tay, CGGP92, Mel82],
- Hörmander's sums-of-square operators [H67̈],
- pseudodifferential operators on a singular foliation [AS11].

In these situations, which are very far from the usual elliptic calculus, it is typically more convenient to use the singular integral approach (*i.e.*, Schwartz kernels) than the Fourier analysis approach (*i.e.*, symbols). The primary goal of this paper is to construct the appropriate algebra of Schwartz kernel operators, as well as the analogue of the ideal of smoothing operators. These algebras are crucial to the groupoid approach to pseudodifferential operators introduced in [vEY] (see also [DS14]), and will used for this purpose in a future paper.

Let M be a smooth closed manifold and a a distribution on $M\times M$. Recall that the Schwartz kernel operator $\mathrm{Op}(a)$ defined by

$$(\mathrm{Op}(a)f)(x) := \int_y a(x,y)f(y)\,dy \tag{1.1}$$

gives a continuous linear operator $\mathrm{Op}(a) : C^\infty(M) \to C^\infty(M)$ if and only if the kernel a is smooth (*i.e.*, semiregular) in the first variable. In groupoid language, this condition corresponds to requiring that a be a smooth family of distributions along the range fibres of the pair groupoid $M \times M$. This point of view was introduced by Skandalis and the first author in [AS11] under the name of "distributions transversal to the range map," and studied extensively in [LMV17]. Here, we will use the name "r-fibred distributions" as in [vEY17].

Lescure, Manchon and Vassout show in [LMV17] that the space of r-fibred distributions on a Lie groupoid $G \rightrightarrows M$ forms an algebra under convolution, with a natural representation as Schwartz kernel operators on $C^\infty(M)$. This algebra plays an essential role in [vEY], where a simple characterization of pseudodifferential operators on a filtered manifold (see [BG88, Tay, Mel82]) is given in terms of an associated tangent groupoid (see also [DS14]).

The purpose of the present article is to construct and study the convolution algebra of r-fibred distributions on the holonomy groupoid of a singular foliation. This groupoid was introduced in [AS09] as a device to carry the analysis of differential operators along the leaves of the foliation. Its topology is typically highly singular; this is the main difficulty in building and understanding our convolution algebra. We are encouraged to proceed by the observation in [AS11] that an appropriate class of smooth submersions to this groupoid, called "bisubmersions", enables quite sophisticated analysis. In particular, the authors of [AS11] produce a calculus of pseudodifferential operators adapted to the geometry of a singular foliation.

Let us explain the guiding philosophy here, which is due to the first author and Skandalis. We begin with the classical case of Schwartz kernel operators on $\mathbb{R}^n$. In order to define linear operators on $C^\infty(\mathbb{R}^n)$, we have the standard form (1.1) above. But nothing is stopping us from adding additional dimensions to the kernel a. If b is a distribution on $\mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n$, smooth in x, then we could define an operator $\mathrm{Op}(b) : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by

$$(\mathrm{Op}(b)f)(x) = \int_{(z,y)\in\mathbb{R}^k\times\mathbb{R}^n} b(x,z,y)f(y)\,dz\,dy. \tag{1.2}$$

(The support of b needs to be compact in the z-direction for this to converge, but let's defer discussion of support conditions until later.) For instance, if $\varphi \in C_c^\infty(\mathbb{R}^n)$ is a smooth function of total mass 1, then putting $b(x,z,y) = a(x,y)\varphi(z)$ we obtain exactly the same operator as (1.1).

Of course, adding these extraneous dimensions is completely unnecessary in this situation, since the Schwartz kernel theorem tells us that all continuous linear operators on $C^\infty(M)$ can be written uniquely in the form (1.1). But as pointed out in [AS11], these additional dimensions are crucial for making sense of pseudodifferential operators on singular foliations, where the dimensions of the leaves can vary from point to point.

Ultimately, we will use this framework to study pseudodifferential operators associated to Hörmander's sums of squares operators. We leave that for a separate article.

The paper is structured as follows. In Section 2 we recall the construction of the holonomy groupoid of a singular foliation $(M,\mathcal{F})$ from [AS09]. Its building blocks are bisubmersions. Consequently, all of our constructions throughout the paper are carried out at the level of bisubmersions rather than on the holonomy groupoid per se.

In Section 3 we define r-fibred distributions on bisubmersions and discuss various algebraic operations, including convolution and transposition. The algebra $\mathcal{E}'_r(\mathcal{F})$ of properly supported r-fibred distributions on the holonomy groupoid is constructed in Section 4. In §5 we discuss the action of $\mathcal{E}'_r(\mathcal{F})$ on $C^\infty(M)$. The right ideal of smooth r-fibred densities is discussed in §6. In Section 7 we define proper distributions on the holonomy groupoid. Roughly speaking, these are distributions which can be realized as smooth families of distributions on both the r-fibres and the s-fibres. We also give a condition in terms of wavefront sets which is sufficient to deduce that an r-fibred distribution is proper. Finally, in §8 we show that the proper distributions act on each of the spaces $C^\infty(M)$, $C_c^\infty(M)$, $\mathcal{E}'(M)$ and $\mathcal{D}'(M)$.

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2. The path holonomy groupoid of a singular foliation

Let M be a smooth finite dimentional manifold. Following [AS09], we define a singular foliation $(M,\mathcal{F})$ as a $C^\infty(M)$-submodule $\mathcal{F}$ of the module $\mathcal{X}_c(M)$ of compactly supported vector fields of M which is locally finitely generated and involutive.

2.1. **Bisubmersions.** To keep this paper self-contained, we recall here the notion of a bisubmersion from [AS09]. For the reader unfamiliar with bisubmersions, keep in mind that the archetypal example is an open subset U of a Lie groupoid G over M, equipped with the restrictions of the range and source maps r and s. In that example, the underlying foliation $\mathcal{F}$ is the foliation of M by the orbits of G. Specifically, the module $\mathcal{F}$ is the image by the anchor map $\rho : AG \to TM$ of the $C^\infty(M)$-module of compactly supported sections $\Gamma_c(AG)$.

General bisubmersions replace charts for more singular groupoids. They serve as a lifting of a region in the singular space to a nice locally euclidean manifold.

Let $(M, \mathcal{F})$ be a foliation. If $\varphi : N \to M$ is a smooth map between manifolds, we write

$$\varphi^{-1}\mathcal{F} = \{Y \in \mathcal{X}_c(N) : d\varphi \circ Y = \sum_{i=1}^{n} f_i(X_i \circ \varphi) \text{ for } f_i \in C_c^\infty(N), X_i \in \mathcal{F}\}.$$

If φ is a submersion, which will always be the case in what follows, this means that the elements of $\varphi^{-1}\mathcal{F}$ are vector fields on N which project under φ to vector fields in $\mathcal{F}$. Then $\phi^{-1}(\mathcal{F})$ is a also a singular foliation (locally finitely genrated and involutive).

a) A *bisubmersion* (U, r_U, s_U) of $(M, \mathcal{F})$ is a smooth, finite dimensional, Hausdorff manifold U equipped with two submersions $r_U, s_U : U \to M$, called *range* and *source*, such that

$$s_U^{-1}\mathcal{F} = r_U^{-1}\mathcal{F} = C_c^\infty(U; \ker ds_U) + C_c^\infty(U; \ker dr_U).$$

   We will often blur the distinction between a bisubmersion (U, r_U, s_U) and its underlying space U, and we write r and s instead of r_U and s_U when the bisubmersion U is evident from the context.

b) A *morphism* of bisubmersions from (U, r_U, s_U) to (V, r_V, s_V) is a smooth map $\varphi : U \to V$ such that $r_U = r_V \circ \varphi$ and $s_U = s_V \circ \varphi$. A *local morphism* at $u \in U$ is a morphism of bisubmersions from from (U', r_U, s_U) to (V, r_V, s_V) for some neighbourhood U' of u.

c) The *inverse* of a bisubmersion (U, r_U, s_U) is the bisubmersion (U, s_U, r_U). We will sometimes use U^{t} to denote the space U equipped with this inverse bisubmersion structure.

d) The *composition* $(U \circ V, r_{U \circ V}, s_{U \circ V})$ of two bisubmersions (U, r_U, s_U) and (V, r_V, s_V) is the bisubmersion with $U \circ V := U_{s_U} \times_{r_V} V$ and maps $r_{U \circ V}(u, v) = r_U(u)$ and $s_{U \circ V}(u, v) = s_V(v)$.

   More generally, if $A \subset U$ and $B \subset V$, we write $A \circ B = A\,{}_s\!\times_r B$.

e) A bisection of a bisubmersion U is a locally closed submanifold $S \subset U$ such that the restrictions of r_U and s_U to S are diffeomorphisms onto open subsets of M. Any bisection S induces a local diffeomorphism Φ_S on M

$$\Phi_S = r_U|_S \circ s_U|_S^{-1}.$$

   If Φ_S is the identity on its domain, S is called an *identity bisection.*

We will really only be interested in bisubersions of the following particular type. Let $\mathbf{X} = (X_1, \ldots, X_m)$ be a generating family of vector fields for $\mathcal{F}$ in an open subset $M_0 \subseteq M$. Consider the map

$$\operatorname{Exp}_{\mathbf{X}} : \mathbb{R}^m \times M_0 \to M \tag{2.1}$$

where $\operatorname{Exp}_{\mathbf{X}}(\xi, x)$ is the unit-time flow of x along the vector field $\sum_i \xi_i X_i$, when defined. This map is defined on some sufficiently small neighbourhood U of $\{0\} \times M_0$ in $\mathbb{R}^m \times M_0$.

Definition 2.1 ([AS09])**.** With the above notation, the set $U \subseteq \mathbb{R}^n \times M_0$ is a bisubmersion when equipped with the maps

$$s_U(\xi, x) = x, \qquad r_U(\xi, x) = \operatorname{Exp}_{\mathbf{X}}(\xi, x).$$

We call (U, r_U, s_U) a *path-holonomy bisubmersion* associated to the local generating family $\mathbf{X} = (X_1, \ldots, X_m)$. If $\mathbf{X}$ is a minimal generating family at $x \in M_0$ then (U, r_U, s_U) is called a *miminal path-holonomy bisubmersion at x*.

2.2. **Atlas of bisubmersions.** The holonomy groupoid of a singular foliation is defined in terms of an atlas of bisubmersions, as follows.

f) A bisubmersion (V, r_V, s_V) is *adapted* to a family $\mathcal{U}$ of bisubmersions if for every $v \in V$ there exists a local morphism at v from (V, r_V, s_V) to some bisubmersion in $\mathcal{U}$.
g) A family $\mathcal{U}$ of second countable bisubmersions of $(M, \mathcal{F})$ is called a *singular groupoid atlas*, or just *atlas*, if the source images $\{s(U) : U \in \mathcal{U}\}$ cover M and all inverses and compositions of elements of $\mathcal{U}$ are adapted to $\mathcal{U}$.
h) An atlas $\mathcal{U}$ is called *maximal* if every bisubmersion which is adapted to $\mathcal{U}$ is already in $\mathcal{U}$. Any atlas $\mathcal{U}$ can be completed to the maximal atlas $\tilde{\mathcal{U}}_{\max}$ of all bisubmersions adapted to $\mathcal{U}$.
i) Given any family $\mathcal{U}_0$ of bisubmersions of $(M, \mathcal{F})$ whose source images $\{s_U(U) : U \in \mathcal{U}_0\}$ cover M, the *minimal atlas generated by* $\mathcal{U}_0$ is the set of all iterated compositions of elements of $\mathcal{U}_0$ and their inverses, and the *maximal atlas generated by* $\mathcal{U}_0$ is the maximal completion of this atlas in the sense above.

Definition 2.2. Let $(M, \mathcal{F})$ be a singular foliation. We write $\mathcal{U}_{\mathrm{hol}}(\mathcal{F})$, or just $\mathcal{U}_{\mathrm{hol}}$, for the maximal atlas generated by all path-holonomy bisubmersions.

Remark 2.3. Let us make a couple of technical remarks about this definition. Firstly, a maximal atlas is too large to be a set. This doesn't actually matter in what follows, but if desired, the problem could averted by allowing only bisubmersions where U is an embedded submanifold of $\mathbb{R}^n$ for some n.

Secondly, in [AS11], the authors work mainly with the minimal path-holonomy atlas, and not its maximal completion. This doesn't change anything in practice. We are favouring the maximal atlas because it slightly simplifies the natural equivalence relation for distributional kernels, see Section 4.

In this article for simplicity we will only work with maximal atlases.

2.3. **Holonomy groupoid.** An atlas of bisubmersions has an associated singular groupoid (*cf.* [AS09]). We won't ever actually use this groupoid in what follows, since we'll define the convolution algebra of fibred distributions directly on the atlas of bisubmersions. But for the sake of completeness, let us finish this section with the construction.

Definition 2.4 ([AS09]). Let $\mathcal{U}$ be an atlas of bisubmersions for the foliation $(M, \mathcal{F})$. The *groupoid associated with* $\mathcal{U}$ is $\mathcal{G}(\mathcal{U}) = \coprod_{U \in \mathcal{U}} U/\sim$ where the equivalence relation is defined as follows: $U \ni u \sim v \in V$ if there exists a local morphism from (U, r_U, s_U) to (V, r_V, s_V) sending u to v.

This is indeed a topological groupoid with base M, see [AS09, Proposition 3.2]. It is proved in [Deb13] that it is longitudinally smooth.

3. Distributions on a singular groupoid

3.1. **Fibred distributions on a submersion.** We recall some basic definitions concerning fibred distributions. This material is adapted from [LMV17] and [vEY]. The basic ideas appear already in [AS11].

Let $q : U \to M$ be a submersion. Then $C^\infty(U)$ becomes a left $C^\infty(M)$-module with

$$f \cdot \phi = (q^* f)\phi$$

for all $\phi \in C^\infty(U)$ and $f \in C^\infty(M)$.

a) A *properly supported q-fibred distribution* is a continuous linear map

$$a : C^\infty(U) \to C^\infty(M); \qquad \phi \mapsto (a, \phi)$$

which is $C^\infty(M)$-linear with respect to the above $C^\infty(M)$-structure on $C^\infty(U)$.

b) The space of properly supported q-fibred distributions on U is denoted $\mathcal{E}'_q(U)$. We equip $\mathcal{E}'_q(U)$ with the topology inherited as a closed subspace $\mathbb{L}(C^\infty(U), C^\infty(M))$ with the topology of uniform convergence on bounded subsets.

c) Let $x \in M$. By $C^\infty(M)$-linearity, the value of (a, ϕ) at x depends only on the restriction of ϕ on the fibre $q^{-1}(x)$, and this defines a compactly supported distribution $a_x \in \mathcal{E}'(q^{-1}(x))$. The q-fibred distribution a is uniquely determined by the family of distributions $(a_x)_{x \in M}$.

d) Let $V \subseteq U$ be open. The q-fibred distribution a *vanishes on* V if $(a, \phi) = 0$ whenever $\mathrm{supp}(\phi) \subseteq V$. The *support* of a, denoted by $\mathrm{supp}(a)$, is the complement of the largest open subset of U on which a vanishes. For any $a \in \mathcal{E}'_q(U)$, the support $\mathrm{supp}(a)$ is a q-proper set, meaning that restriction $q : \mathrm{supp}(a) \to M$ is a proper map.

Example 3.1. Let $S \subset U$ be a local section of $q : U \to M$, meaning that S is a locally closed submanifold of U and $q|_S$ is a diffeomorphism of S onto an open subset of M. Fix also a smooth function $c \in C_c^\infty(M)$ with support in $q(S)$. The *q-fibred Dirac distribution on S (with coefficient c)*, denoted $c\Delta_S \in \mathcal{E}'_q(U)$, is the q-fibred distribution defined by the formula

$$(c\Delta_S, \phi)(x) = \begin{cases} c(x)(\phi \circ q|_S^{-1})(x), & \text{if } x \in q(S) \\ 0, & \text{otherwise,} \end{cases}$$

for $\phi \in C^\infty(U)$. In other words, upon identifying S with its image $q(S) \subset M$, $c\Delta_S$ is given by evaluating ϕ on S and then multiplying by c (to ensure the result is smooth). For any $x \in q(S)$, the restriction of $c\Delta_S$ to the fibre $q^{-1}(x)$ is the multiple $c(x)\delta_{\tilde{x}}$ of the Dirac distribution at the preimage $\tilde{x} = q|_S^{-1}(x)$.

We end this section with two operations on fibred distributions which will be heavily used in what follows.

3.1.1. *Pushforward, or integration along fibres.*

Definition 3.2. Let $q : U \to M$ and $q' : U' \to M$ be submersions and let $\pi : U' \to U$ be a smooth morphism of submersions, meaning that $q' = q \circ \pi$. There is an induced linear map $\pi_* : \mathcal{E}'_{q'}(U') \to \mathcal{E}'_q(U)$, called *pushforward* or *integration along the fibres*, defined by

$$(\pi_* a, \phi) := (a, \pi^* \phi)$$

for $a \in \mathcal{E}'_{q \circ \pi}(U')$ and $\phi \in C^\infty(U)$.

Example 3.3. Let U' be an open subset of a bisubmersion U. The inclusion map $\iota : U' \to U$ is a morphism of bisubmersions. The pushforward $\iota_* : \mathcal{E}'_q(U') \to \mathcal{E}'_q(U)$ corresponds to extension by zero. We can thus identify $\mathcal{E}'_q(U')$ as a subspace of $\mathcal{E}'_q(U)$.

Lemma 3.4. *If the morphism of submersions $\pi : U' \to U$ is a surjective submersion, then $\pi_* : \mathcal{E}'_{q'}(U') \to \mathcal{E}'_q(U)$ is surjective.*

Proof. We begin with the case where $U' = U \times \mathbb{R}^k$ for some $k \in \mathbb{N}$ and π is the projection onto the first variable. Fix a positive function $\omega \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \omega(\xi)\, d\xi = 1$. If $a \in \mathcal{E}'_q(U)$, then we define a $a' \in \mathcal{E}'_{q'}(U')$ by the pairing

$$(a', \phi) = (a, \textstyle\int_{\mathbb{R}^k} \phi(\,\cdot\,, \xi)\omega(\xi)\, d\xi) \qquad \phi \in C^\infty(U \times \mathbb{R}^k).$$

Then $\pi_* a' = a$, as desired.

The general case follows by from the above by using a partition of unity argument. □

3.1.2. *Pullback over a base map.* The following construction is described in [LMV17, Proposition 2.15].

Definition 3.5. Let $p : N \to M$ be any smooth map, and consider the pullback diagram

$$\begin{tikzcd} N\,{}_p\!\times_q U \arrow[r, "\mathrm{pr}_U"] \arrow[d, "\mathrm{pr}_N"'] & U \arrow[d, "q"] \\ N \arrow[r, "p"] & M. \end{tikzcd}$$

There is an associated linear map $p^* : \mathcal{E}'_q(U) \to \mathcal{E}'_{\mathrm{pr}_N}(N\,{}_p\!\times_q U)$, called the *pullback along p*, which is uniquely characterized by the property

$$(p^* a, \mathrm{pr}_U^* \phi) = p^*(a, \phi) \tag{3.1}$$

for all $a \in \mathcal{E}'_q(U)$, $\phi \in C^\infty(U)$. Explicitly, $p^* a$ is defined on the fibres of pr_U by $(p^* a)_y = a_{p(y)}$, where $y \in N$ and we are using the canonical identification $\mathrm{pr}_N^{-1}(y) = q^{-1}(p(y))$.

Lemma 3.6. *Let $q : U \to M$ be a submersion and $p : N \to M$ a smooth map, as above.*

a) *If $q' : U' \to M$ is a submersion and $\pi : U' \to U$ a morphism of submersions, then for any $a \in \mathcal{E}'_{q'}(U')$,*

$$p^*(\pi_* a) = (id \times \pi)_*(p^* a).$$

b) *Let $p' : N' \to N$ be a smooth map, then $(p \circ p')^* = p'^* \circ p^*$.*

Proof. a) Let $\phi \in C^\infty(U)$. One has

$$\begin{aligned}((\mathrm{id} \times \pi)_* p^* a, \mathrm{pr}_U^* \phi) &= (p^* a, (\mathrm{pr}_U \circ (\mathrm{id} \times \pi)))^* \phi) \\ &= (p^* a, (\pi \circ \mathrm{pr}_{U'})^* \phi) \\ &= p^*(a, \pi^* \phi) \\ &= p^*(\pi_* a, \phi).\end{aligned}$$

The result then follows from uniqueness in Equation (3.1).

b) Let $\mathrm{pr}'_U : N'{}_{p \circ p'}\!\times_q U \to U$ be the natural projection, $\phi \in C^\infty(U)$. One has

$$(p'^*(p^* a), (\mathrm{pr}'_U)^* \phi) = p'^*(p^* a, \mathrm{pr}_U^* \phi) = p'^* p^*(a, \phi) = (p \circ p')^*(a, \phi).$$

The result then follows from uniqueness in Equation (3.1). □

Finally, we remark that pushforward and pullback are continuous with respect to the topologies on fibred distributions.

3.2. **Convolution of fibred distributions on bisubmersions.** Now we consider fibred distributions on a bisumbersion U for a foliation $(M, \mathcal{F})$. Thus we have two submersions $r, s : U \to M$ and we can define the spaces $\mathcal{E}'_r(U)$ and $\mathcal{E}'_s(U)$ of linear maps from $C^\infty(U)$ to $C^\infty(M)$. For $a \in \mathcal{E}'_s(U)$, we will write a_x for the distribution on the s-fibre $s^{-1}(x)$ and for $b \in \mathcal{E}'_r(U)$ we write b^x for the distribution on the r-fibre $r^{-1}(x)$.

Recall that if U and V are bisubmersions over M, then their composition is $U \circ V = U_s \times_r V$ with range and source maps $r_{U \circ V}(u, v) = r(u)$ and $s_{U \circ V}(u, v) = s(v)$. Therefore, given $b \in \mathcal{E}'_r(V)$ we can define a pullback distribution $s_U^* b \in \mathcal{E}'_{\mathrm{pr}_U}(U \circ V)$ as in point d) above, and hence the following definition.

Definition 3.7. Let U, V be bisubmersions for $(M, \mathcal{F})$. We define the *convolution product* of $a \in \mathcal{E}'_r(U)$ and $b \in \mathcal{E}'_r(V)$ to be the r-fibred distribution

$$a * b := a \circ s_U^* b \in \mathcal{E}'_r(U \circ V).$$

Similarly, the *convolution product* of $a \in \mathcal{E}'_s(U)$ and $b \in \mathcal{E}'_s(V)$ is the s-fibred distribution

$$a * b := b \circ r_V^* a \in \mathcal{E}'_s(U \circ V).$$

The *transpose* $a^{\mathrm{t}} \in \mathcal{E}'_s(U^{\mathrm{t}})$ of an r_U-fibred distribution $a \in \mathcal{E}'_r(U)$ is $a^{\mathrm{t}} := a$ but viewed as an $s_{U^{\mathrm{t}}}$-fibred distribution on the inverse bisubmersion $U^{\mathrm{t}} = U$. Likewise, the transpose of an s-fibred distribution $b \in \mathcal{E}'_s(U)$ is $b^{\mathrm{t}} = b \in \mathcal{E}'_r(U^{\mathrm{t}})$.

Note that convolution $\mathcal{E}'_r(U) \times \mathcal{E}'_r(V) \to \mathcal{E}'_r(U \circ V)$ is separately continuous, since it is built from the continuous operations of pullback and composition. Likewise for convolution of s-fibred distributions and transposition.

Remark 3.8. In the definition of convolution, we should not exclude the possibility of the empty bisubmersion $(\emptyset, r_\emptyset, s_\emptyset)$ where $r_\emptyset$ and $s_\emptyset$ are the empty maps. Here, convention says that $C^\infty(\emptyset) = \{0\}$, so that $\mathcal{E}'_r(\emptyset)$ and $\mathcal{E}'_s(\emptyset)$ contain only the zero map. This comes into play when considering a convolution of r-fibred distributions $a \in \mathcal{E}'_r(U)$ and $b \in \mathcal{E}'_s(V)$ such that $s(U)$ and $r(V)$ are disjoint, since then $U \circ V = \emptyset$ and hence $a * b = 0$.

This is a particular case of the following lemma.

Lemma 3.9. *For any $a \in \mathcal{E}'_r(U)$ and $b \in \mathcal{E}'_r(V)$ we have*

$$\operatorname{supp}(a \circ b) \subseteq \operatorname{supp}(a) \circ \operatorname{supp}(b).$$

In particular, if $s(\operatorname{supp}(a)) \cap r(\operatorname{supp}(b)) = \emptyset$ then $a * b = 0$.

Analogous statements hold for $a \in \mathcal{E}'_s(U)$, $b \in \mathcal{E}'_s(V)$.

The convolution product is associative. It is also compatible with integration along fibres in the following sense.

Lemma 3.10. *If $\pi : U \to U'$ and $\rho : V \to V'$ are submersive morphisms of bisubmersions then $\pi \times \rho : U \circ V \to U' \circ V'$ is a submersive morphism of bisubmersions and we have $(\pi_* a) * (\rho_* b) = (\pi \times \rho)_*(a * b)$ for all $a \in \mathcal{E}'_r(U)$, $b \in \mathcal{E}'_r(V)$.*

Proof. Let $\phi \in C^\infty(U \, {}_s\!\times_r V)$. Using the functorial properties of Lemma 3.6 and Equation (3.1), we have

$$\begin{aligned}
((\pi_* a) * (\rho_* b), \phi) &= (\pi_* a, (s_U^*(\rho_* b), \phi)) \\
&= (\pi_* a, ((\mathrm{id} \times \rho)_*(s_U^* b), \phi)) \\
&= (a, \pi^*(s_U^* b, (\mathrm{id} \times \rho)^* \phi)) \\
&= (a, (s_{U'}^* b, (\pi \times \mathrm{id})^* (\mathrm{id} \times \rho)^* \phi)) \\
&= ((\pi \times \rho)_*(a * b), \phi)
\end{aligned}$$ ☐

Lemma 3.11. *For any $a \in \mathcal{E}'_r(U)$ and $b \in \mathcal{E}'_r(V)$ we have*

$$(a * b)^{\mathrm{t}} = b^{\mathrm{t}} * a^{\mathrm{t}}$$

as elements of $\mathcal{E}'_s(U \circ V)$.

Proof. We calculate $(a * b)^{\mathrm{t}} = (a \circ s_U^* b)^{\mathrm{t}} = a^{\mathrm{t}} \circ r_U^*(b^{\mathrm{t}}) = b^{\mathrm{t}} * a^{\mathrm{t}}$. ☐

4. The convolution algebra of fibred distributions on the holonomy groupoid

In this section, we will define the convolution algebra $\mathcal{E}'_r(\mathcal{F})$ of properly supported r-fibred distributions on the holonomy groupoid of a singular foliation. In §5, we will show that $\mathcal{E}'_r(\mathcal{F})$ acts by continuous linear operators on the spaces $C^\infty(M)$, via a representation which we call Op. These are what we refer to, informally, as the 'Schwartz kernel operators' associated to a singular foliation.

There is likewise a convolution algebra $\mathcal{E}'_s(\mathcal{F})$ of properly supported s-fibred distributions. This algebra admits a representation as operators on the distribution space $\mathcal{E}'(M)$. For an algebra which acts at once on all four spaces $C^\infty(M)$, $C_c^\infty(M)$, $\mathcal{D}'(M)$, $\mathcal{E}'(M)$, we will need to add further conditions, which we deal with in §7.

4.1. **The convolution algebra of r-fibred distributions.** Let $(M, \mathcal{F})$ be a foliation and $\mathcal{U}$ a maximal atlas of bisubmersions.

Definition 4.1. The space $\mathcal{E}'_r(\mathcal{U})$ denotes the vector space of all families $(a_U)_{U\in\mathcal{U}}$ that are *r-locally finite*. This means that for every compact $K \subseteq M$ there are only finitely many $U \in \mathcal{U}$ with

$$r(\mathrm{supp}(a_U)) \cap K \neq \emptyset.$$

If $U_0 \in \mathcal{U}$, then we will naturally identify $\mathcal{E}'_r(U_0)$ with the subspace of $\mathcal{E}'_r(\mathcal{U})$ consisting of $(a_U)_{U\in\mathcal{U}}$ such that $a_U = 0$ if $U \neq U_0$.

We will also represent an element $\mathbf{a} = (a_U)_{U\in\mathcal{U}} \in \mathcal{E}'_r(\mathcal{U})$ as a sum $\mathbf{a} = \sum_U a_U$.

Proposition 4.2. *The convolution defined by* $\sum a_U \circ \sum b_U = \sum_{U,V} a_U \circ b_V$ *is well defined.*

Proof. By this we mean that $\sum_{U,V} a_U \circ b_V$ is r-locally finite. Let $K \subseteq M$ be compact, and denote by V_i the bisubmersions such that $r(\mathrm{supp}(b_{V_i}))\cap K \neq \emptyset$. Since a is r-locally finite and $\mathrm{supp}(b_{V_i}) \cap r^{-1}(K)$ is compact, there exists a finite number of U_j such that there exists i with $r(\mathrm{supp}\, a_{U_j}) \cap s(\mathrm{supp}(b_{V_i}) \cap r^{-1}(K)) \neq \emptyset$. The set of bisubmersions $U_j \circ V_i$ includes all those $U \circ V$ for which $r(\mathrm{supp}(a_U \circ b_V)) \cap K \neq \emptyset$. The result follows. $\square$

We equip $\mathcal{E}'_r(\mathcal{U})$ with the following topology. A generalized sequence $\mathbf{a}_i = (a_{i,U})$ converges to $\mathbf{a} = (a_U)$ if the families $\mathbf{a}_i$ are uniformly r-locally finite—meaning that for every compact $K \subseteq M$ there are only finitely many $U \in \mathcal{U}$ for which $r(\mathrm{supp}(a_{i,U})) \cap K \neq \emptyset$ for some $i \in I$—and $a_{i,U} \to a_U$ for every U.

Now let us focus on the path-holonomy atlas $\mathcal{U}_{hol}$.

Definition 4.3. We define $\mathcal{N}_r \subseteq \mathcal{E}'_r(\mathcal{U}_{hol})$ to be the closure of the ideal generated by all elements of the form $a - \pi_* a$, where $a \in \mathcal{E}'_r(U_{hol})$ for some bisubmersion $U \in \mathcal{U}_{\mathrm{hol}}$ and $\pi : U \to V$ is a morphism of bisubmersions. For elements $a \in \mathcal{E}'_r(U)$, $b \in \mathcal{E}'_r(V)$ we will write $a \equiv b$ when $a - b \in \mathcal{N}_r$. We define

$$\mathcal{E}'_r(\mathcal{F}) = \mathcal{E}'_r(\mathcal{U}_{\mathrm{hol}})/\mathcal{N}_r.$$

We define spaces $\mathcal{N}_s$ and $\mathcal{E}'_s(\mathcal{F}) = \mathcal{E}'_s(\mathbf{U})/\mathcal{N}_s$ analogously.

Remark 4.4. It is necessary to take a closure in the definition of the ideal $\mathcal{N}_r$ in order to allow the equivalences in $\mathcal{E}'_r(\mathcal{U}_{hol})$ to take place on an infinite (but r-locally finite) family of bisubmersions.

The point of quotienting by the ideal $\mathcal{N}_r$ is that, as we will see in Section 5, the r-fibred distributions a and $\pi_* a$ will induce the same kernel operators on $C^\infty(M)$. For a simple example, if $\iota : U' \to U$ is the inclusion of an open set of a bisubmersion, then every kernel $a \in \mathcal{E}'_r(U')$ is identified in the quotient with its extension by zero $\iota_* a \in \mathcal{E}'_r(U)$.

Proposition 4.5.
a) *The convolution product on $\mathcal{E}'_r(\mathcal{U}_{hol})$ descends to a seperately continuous associative product on the quotient space $\mathcal{E}'_r(\mathcal{F})$. Similarly for $\mathcal{E}'_s(\mathcal{F})$.*
b) *Transposition descends to a bijective anti-algebra isomorphism $\mathcal{E}'_r(\mathcal{F}) \to \mathcal{E}'_s(\mathcal{F})$.*

Proof. a) It follows from Lemma 3.10 that $\mathcal{N}_r$ is a closed two-sided ideal in $\mathcal{E}'_r(\mathcal{U}_{hol})$, which proves the first statement. The statement for $\mathcal{E}'_s(\mathcal{F})$ is proven similarly.

b) It is clear that transposition defines a continuous linear isomorphism from $\mathcal{E}'_r(\mathcal{U}_{hol})$ to $\mathcal{E}'_s(\mathcal{U}_{hol})$ and that $\mathcal{N}_r$ maps to $\mathcal{N}_s$. The result then follows from Lemma 3.11.

□

5. Action on smooth functions

The most important feature of the convolution algebra $\mathcal{E}'_r(\mathcal{F})$ is that it acts by continuous linear operators on $C^\infty(M)$.

From this point all bisubmersions will be considered in the maximal path holonomy atlas $\mathcal{U}_{hol}$.

Proposition 5.1. *Let $U \rightrightarrows M$ be a bisubmersion and $a \in \mathcal{E}'_r(U)$.*

a) *The formula*
$$\mathrm{Op}(a)f = (a, s_U^* f)$$
defines a continuous linear operator $\mathrm{Op}(a)$ *on* $C^\infty(M)$.

b) *If $b \in \mathcal{E}'_r(V)$ for another bisubmersion $V \rightrightarrows M$, we have*
$$\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(a * b).$$

c) *If $\pi : U \to U'$ is a morphism of bisubmersions then* $\mathrm{Op}(a) = \mathrm{Op}(\pi_*(a))$.

Proof. The linear maps $s_U^* : C^\infty(M) \to C^\infty(U)$ and $a : C^\infty(U) \to C^\infty(M)$ are continuous, which proves a).

The statement in b) follows from the calculation
$$\begin{aligned}\mathrm{Op}(a * b)f &= (a \circ s_U^* b, s_{U\circ V}^* f)\\ &= (a, (s_U^* b, \mathrm{pr}_V^*\, s_V^* f))\\ &= (a, s_U^*(b, s_V^* f)) && \text{(by Eq. (3.1))}\\ &= \mathrm{Op}(a)\,\mathrm{Op}(b)f.\end{aligned}$$

Finally, by the definition of a morphism of bisubmersions we have
$$\mathrm{Op}(\pi_*(a))(f) = (a, \pi^* s_{U'}^* f) = (a, s_U^* f) = \mathrm{Op}(a)(f),$$
which proves c). □

If $\mathbf{a} = \sum_U a_U \in \mathcal{E}'_r(\mathcal{U}_{hol})$, then for any $f \in C^\infty(M)$ we define
$$\mathrm{Op}(\mathbf{a})f = \sum_{U \in \mathcal{U}_{\mathrm{hol}}} (a_U, s_U^* f).$$

The sum is well-defined by r-local finiteness. As an immediate consequence of Proposition 5.1, we have the main theorem of this section.

Theorem 5.2. *The map* $\mathrm{Op} : \mathcal{E}'_r(\mathcal{U}_{hol}) \to \mathbb{L}(C^\infty(M))$ *descends to a continuous representation of $\mathcal{E}'_r(\mathcal{F})$ on $C^\infty(M)$.* □

Example 5.3. Let S be a local bisection of a bisubmersion U, as in item (e) of §2.1, and let $\Phi_S = r|_S \circ s|_S^{-1}$ be the local diffeomorphism that it carries. Fix also a smooth function $c \in C^\infty(M)$ with support contained in $r(S)$. Recall from Example 3.1 that the r-fibred Dirac distribution supported on S with coefficient c is defined such that its pairing with $\phi \in C^\infty(M)$ is given by restriction to $S \cong r(S)$ and then multiplication by c. We will use the notation $c\Delta^S$ for this, with S in superscript to indicate it is an r-fibred Dirac distribution.

We claim that for all $x \in r(S)$ we have

$$(\mathrm{Op}(c\Delta^S)f)(x) = c(x)f(\Phi_S^{-1}(x)),$$

and $(\mathrm{Op}(c\Delta^S)f)(x) = 0$ when $x \notin r(S)$. Indeed, if $x \in r(S)$ then

$$(\mathrm{Op}(c\Delta^S)f)(x) = (c\Delta^S, s^*f)(x) = c(x)f(s(r|_S^{-1}(x)).$$

In particular, if $X \subseteq s(S)$ is a closed subset and if $c \in C^\infty(M)$ is a smooth bump function with $c|_{\Phi_S(X)} \equiv 1$ and $c|_{M\setminus r(S)} \equiv 0$, then for all f supported in X we have

$$(\mathrm{Op}(c\Delta^S)f) = f \circ \Phi_S^{-1}.$$

For instance, using a path-holonomy bisubmersion, we obtain flows along vector fields tangent to $\mathcal{F}$ in this way.

In particular, this example shows that the quotient algebra $\mathcal{E}'_r(\mathcal{F})$ is not trivial.

5.1. **Propagation of supports.** To describe propagation of supports under the action of $\mathrm{Op}(a)$, we introduce some further notation.

Definition 5.4. Let $U \rightrightarrows M$ be a bisubmersion. For subsets $N \subseteq M$ and $V \subseteq U$ we define the following subsets of M:

$$V \circ N = \{r(v) : v \in V \text{ with } s(v) \in N\},$$
$$N \circ V = \{s(v) : v \in V \text{ with } r(v) \in N\}.$$

Proposition 5.5. *Let $U \rightrightarrows M$ be a bisubmersion and $a \in \mathcal{E}'_r(U)$. For any $f \in C^\infty(M)$ we have*

$$\mathrm{supp}(\mathrm{Op}(a)f) \subseteq \mathrm{supp}(a) \circ \mathrm{supp}(f).$$

Proof. The support of $\mathrm{Op}(a)f$ lies in $r_U(\mathrm{supp}(a) \cap s_U^{-1}(\mathrm{supp}\, f)) = \mathrm{supp}(a) \circ \mathrm{supp}(f)$. □

Definition 5.6. Let U be a bisubmersion. A subset $X \subseteq U$ is called *proper* if it is both r- and s-proper.

Corollary 5.7. *If $a \in \mathcal{E}'_r(U)$ has proper support, then $\mathrm{Op}(a)$ maps $C_c^\infty(M)$ into itself.*

Proof. If $\mathrm{supp}(a)$ is s-proper and $\mathrm{supp}(f)$ is compact, then $\mathrm{supp}(a) \cap s_U^{-1}(\mathrm{supp}(f))$ is compact so $\mathrm{supp}(\mathrm{Op}(a)f)$ is compact. □

The algebra $\mathcal{E}'_s(\mathcal{F})$ of s-fibred distributions acts naturally on the distribution space $\mathcal{E}'(M)$ via the transpose: if $b \in \mathcal{E}'_s(U)$, we define $\widetilde{\mathrm{Op}}(b) \in \mathbb{L}(\mathcal{E}'(M))$ by

$$(\widetilde{\mathrm{Op}}(b)\omega, f) = (\omega, \mathrm{Op}(b^{\mathrm{t}})f),$$

for all $\omega \in \mathcal{E}'(M)$, $f \in C^\infty(M)$. It follows from Proposition 4.5 that this defines an algebra representation of $\mathcal{E}'_s(\mathcal{F})$ on $\mathcal{E}'(M)$. Moreover, if $b \in \mathcal{E}'_s(U)$ has r-proper support, then $\widetilde{\mathrm{Op}}$ extends to an action on $\mathcal{D}'(M)$.

Ultimately, we will want an algebra of distributions that acts on all four of the spaces $C^\infty(M)$, $C_c^\infty(M)$, $\mathcal{E}'(M)$ and $\mathcal{D}'(M)$. This requires the notion of proper distributions, which we treat in Section 7.

6. The ideal of smooth fibred densities

Inside the algebra of continuous operators on $C^\infty(M)$ is the right ideal of smoothing operators which map $\mathcal{D}'(M)$ into $C^\infty(M)$. This consists of Schwartz kernel operators with kernels in $C^\infty(M) \hat{\otimes} C^\infty_{\mathrm{c}}(M; |\Omega|)$. The generalization of this in our context is the right ideal of smooth r-fibred densities.

Let $q : U \to M$ be a submersion. Let $|\Omega_q|$ denote the bundle of 1-densities along the longitudinal tangent bundle of the q-fibration $\ker(dq) \subseteq TU$. Let us write $C^\infty_q(U; |\Omega_q|)$ for the space of smooth sections of $|\Omega_q|$ with q-proper support. Any such section a defines a q-fibred distribution on U by the formula

$$(a, \phi)(x) = \int_{u \in q^{-1}(x)} a(u)\phi(u).$$

We call these elements *smooth q-fibred densities.*

In particular, if (U, r, s) is a bisubmersion for $\mathcal{F}$ then we have the subspaces $C^\infty_r(U; |\Omega_r|) \subset \mathcal{E}'_r(U)$ and $C^\infty_s(U; |\Omega_s|) \subset \mathcal{E}'_s(U)$.

We note the following.

Lemma 6.1. *Let $\pi : U' \to U$ be a submersive morphism of bisubmersions. Integration along the fibres restricts to a map*

$$\pi_* : C^\infty_r(U'; |\Omega_r|) \to C^\infty_r(U; |\Omega_r|).$$

Moreover, if π is onto then this map is surjective.

Analogous statements hold with s in place of r throughout.

Proof. The proof that integration along the fibres preserves smoothness is essentially contained in [AS09], see the comments after Definition 4.2. For surjectivity, observe that the lifting process in the proof of Lemma 3.4 sends smooth r-fibred densities to smooth r-fibred densities. $\square$

Definition 6.2. We write

$$C^\infty_r(\mathcal{U}_{\mathrm{hol}}; |\Omega_r|) = \{\mathbf{a} = (a_U) \in \mathcal{E}'_r(\mathcal{U}_{\mathrm{hol}}) : a_U \in C^\infty_r(U; |\Omega_r|) \text{ for all } U \in \mathcal{U}_{\mathrm{hol}}\},$$

and define $C^\infty_r(\mathcal{F}; |\Omega_r|)$ to be the image of $C^\infty_r(\mathcal{U}_{\mathrm{hol}}; |\Omega_r|)$ in the quotient $\mathcal{E}'_r(\mathcal{F})$.

We define $C^\infty_s(\mathcal{F}) \subseteq \mathcal{E}'_s(\mathcal{F})$ similarly.

The main result of this section is the following.

Theorem 6.3. *The space $C^\infty_r(\mathcal{F}; |\Omega_r|)$ is a right ideal in the algebra $\mathcal{E}'_r(\mathcal{F})$. Likewise, $C^\infty_s(\mathcal{F}; |\Omega_s|)$ is a left ideal in $\mathcal{E}'_s(\mathcal{F})$.*

The rest of this section is dedicated to proving Theorem 6.3 for $C^\infty_s(\mathcal{F}; |\Omega_s|)$. The result for $C^\infty_r(\mathcal{F}; |\Omega_r|)$ follows from this by applying the transpose and using Proposition 4.5(b).

By linearity, it suffices to treat the case where $\mathbf{a}$ and $\mathbf{b}$ each live on a single bisubmersion, i.e., taking $a \in \mathcal{E}'_s(U)$ and $b \in \mathcal{E}'_s(V)$ for some bisubmersions U and V.

We begin with the special case where $U = V$ is a minimal path holonomy bisubmersion at some point $x_0 \in M$. Therefore, let $\mathbf{X} = (X_1, \ldots, X_m)$ be a minimal generating family at $x_0 \in M$ and let U be an associated path holonomy bisubmersion. This means we fix a small enough neighbourhood

$M_0 \subseteq M$ of x_0 and a small enough neighbourhood $U \subseteq \mathbb{R}^m \times M_0$ of $\{0\} \times M_0$ so that we can define source and range maps on U by

$$s(\xi, x) = x, \qquad r(\xi, x) = \mathrm{Exp}_{\mathbf{X}}(\xi, x),$$

see Definition 2.1 for notation.

By Proposition 2.7 of [AS11], there is a neighbourhood $U' \subseteq U$ of $(0, x_0)$ which admits a submersive morphism of bisubmersions

$$\pi : U' \circ U' \to U$$

with $\pi((0,x),(0,x)) = (0,x)$ for all $x \in M_0$. Recall that

$$U' \circ U' = \{(\eta, y), (\xi, x) \in U' \times U' : y = \mathrm{Exp}_{\mathbf{X}}(\xi, x)\}.$$

The coordinate y is superfluous, so let us henceforth make the identification

$$U' \circ U' = \{(\eta, \xi, x) \in \mathbb{R}^m \times \mathbb{R}^m \times M_0 : (\xi, x) \in U' \text{ and } (\eta, \mathrm{Exp}_{\mathbf{X}}(\xi, x)) \in U'\},$$

with range and source maps

$$s(\eta, \xi, x) = x, \qquad r(\eta, \xi, x) = \mathrm{Exp}_{\mathbf{X}}(\eta, \mathrm{Exp}_{\mathbf{X}}(\xi, x)).$$

Let $\pi_1 : U' \circ U' \to \mathbb{R}^m$ be the function determined by

$$\pi(\eta, \xi, x) = (\pi_1(\eta, \xi, x), x).$$

If we fix $\xi = 0$ then we have $\pi_1(\eta, 0, x) = \eta$ for all $(\eta, 0, x) \in U'$, so the derivative $D_\eta \pi_1$ of π_1 with respect to the η variables is the identity at all $(\eta, 0, x) \in U'$. Therefore, the map

$$\begin{aligned} \Pi : U' \circ U' &\to \mathbb{R}^m \times \mathbb{R}^m \times M_0 \\ \Pi(\eta, \xi, x) &= (\pi_1(\eta, \xi, x), \xi, x), \end{aligned}$$

has invertible derivative at every $(\eta, 0, x) \in U'$. By further restricting the neighbourhood U' of $(0, x)$ in U, the map Π is a diffeomorphism onto its image. We thus have a smooth function $\theta : \mathbb{R}^m \times \mathbb{R}^m \times M_0 \to \mathbb{R}^m$ such that

$$\Pi(\theta(\eta, \xi, x), \xi, x) = (\eta, \xi, x)$$

for all (η, ξ, x) in $\Pi(U' \circ U')$, or equivalently

$$\pi(\theta(\eta, \xi, x), \xi, x) = (\eta, x).$$

Lemma 6.4. *Let U be a minimal path holonomy bisubmersion at $x_0 \in M$. With the above notation, there exists a neighbourhood U' of $(0, x_0)$ in U such that the map Π is a diffeomorphism onto its image. Then for any $a \in C_s^\infty(U'; |\Omega_s|)$, and $b \in \mathcal{E}_s'(U')$ we have*

$$\pi_*(a * b) \in C_s^\infty(U; |\Omega_s|).$$

Proof. The existence of the neighbourhood U' was proven in the discussion preceding the lemma. Let $\phi \in C^\infty(U' \circ U')$. We have

$$(\pi_*(a * b), \phi) = (b \circ r^* a, \pi^* \phi).$$

Let us write $a = a_0(\xi, x) d\xi$ where a_0 is some s-properly supported smooth function on U' and $d\xi$ is Lebesgue measure on $\mathbb{R}^n$. For every $(\eta, \xi, x) \in U' \circ U'$

we have $\pi(\eta,\xi,x) = (\pi_1(\eta,\xi,x),x)$, so that

$$\begin{aligned}(r^*a_0,\pi^*\phi)(\xi,x) &= \int_{\eta\in\mathbb{R}^m} r^*a_0(\eta,\xi,x)\phi(\pi_1(\eta,\xi,x),x)\,d\eta \\ &= \int_{\eta\in\mathbb{R}^m} r^*a_0(\theta(\eta,\xi,x),\xi,x)\phi(\eta,x)|D_\eta\theta(\eta,\xi,x)|\,d\eta,\end{aligned}$$

where in the last line we have used the change of variables $\eta \to \theta(\eta,\xi,x)$. Let us write

$$\tilde{a}_0(\eta,\xi,x) = r^*a_0(\theta(\eta,\xi,x),\xi,x)|D_\eta\theta(\eta,\xi,x)|,$$

which is a smooth function on $\Pi(U'\circ U') \subseteq \mathbb{R}^m\times\mathbb{R}^m\times M_0$. Using the formal notation $\int_\xi b(\xi,x)\psi(\xi,x)\,d\xi$ to denote the value of $(b,\psi)(x)$, Fubini's Theorem for distributions gives

$$\begin{aligned}(\pi_*(a*b),\phi)(x) &= \int_{\xi\in\mathbb{R}^m} b(\xi,x)\left(\int_{\eta\in\mathbb{R}^m}\tilde{a}_0(\eta,\xi,x)\phi(\eta,x)\,d\eta\right)d\xi \\ &= \int_{\eta\in\mathbb{R}^m}\left(\int_{\xi\in\mathbb{R}^m} b(\xi,x)\tilde{a}_0(\eta,\xi,x)\,d\xi\right)\phi(\eta,x)\,d\eta.\end{aligned}$$

The integral in brackets in the last line is a smooth function of $(\eta,x)\in\mathbb{R}^m\times M_0$. Since $\pi_*(a*b)$ is automatically s-properly supported, the result follows. □

Lemma 6.5. *Let $V,W\in\mathcal{U}_{\mathrm{hol}}$ and suppose $S\subseteq V$ and $T\subset W$ are local identity bisections. For every $v\in S$ and $w\in T$ there exist open neighbourhoods $V_v\subseteq V$ of v (depending only on v) and $W_{v,w}\subseteq W$ of w (depending on both v and w) such that for any $a\in C^\infty_s(V_v;|\Omega_s|)$ and $b\in\mathcal{E}'_s(W_{v,w})$ we have $a*b\in C^\infty_s(\mathcal{F};|\Omega_s|)$.*

Proof. Let us put $x_0 = s_V(v) = r_V(v)$. Let U be a minimal path-holonomy bisubmersion at x_0 and let $U'\subseteq U$ be a neighbourhood of $(0,x_0)$ of the kind described in Lemma 6.4. Since S is an identity bisection at v, Proposition 2.10 of [AS09] shows that we can find a submersive morphism $\pi_V: V_v\to U'$ with $\pi_V(v) = (0;x_0)$ for some neighbourhood V_v of v.

Now put $y_0 = s_W(w) = r_W(w)$. To define $W_{v,w}$, we consider two cases:

- Suppose $y_0\notin s_V(V_v)$. Then we can find a neighbourhood $W_{v,w}$ of w such that $r_W(W_{v,w})\cap s_V(V_v) = \emptyset$. In this case, Lemma 3.9 shows that for all $a\in C^\infty_s(V_v;|\Omega_s|)$ and $b\in\mathcal{E}'_s(W_{v,w})$ we have $a*b = 0$, which proves the claim.
- Suppose $y_0\in s_V(V_v)$. Let $\widetilde{U}$ be a minimal path-holonomy bisubmersion at y_0. Given that we have identity bisubmersions $T\subset W$ passing through w and $\{0\}\times M_0\subset U$ passing through $(0,y_0)$, Proposition 2.10 of [AS09] shows that there exist submersive morphisms

  $$\begin{aligned}&\pi_W: W_{v,w}\to\widetilde{U} &&\text{with } \pi_W(w) = (0,y_0),\\ &\pi_U: U_{v,w}\to\widetilde{U} &&\text{with } \pi_U(0,y_0) = (0,y_0),\end{aligned}$$

  for some neighbourhoods $W_{v,w}\subseteq W$ of w and $U_{v,w}\subseteq U'$ of $(0;y_0)$. Moreover, by reducing $\widetilde{U}$ sufficiently, we may assume that π_U is surjective. Now, given $a\in C^\infty_s(V_v;|\Omega_s|)$ and $b\in\mathcal{E}'_s(W_{v,w})$, Lemma 3.4

shows that we can find $\widetilde{b} \in \mathcal{E}'_s(U_{v,w})$ such that $\pi_{U*}\widetilde{b} = \pi_{W*}b$. Then Lemma 6.4 shows that

$$a * b \equiv \pi_{V*}a * \widetilde{b} \in C_s^\infty(\mathcal{F}; |\Omega_s|),$$

which again proves the claim.

□

In order to extend this result to general bisubmersions, we use translations by bisections.

Definition 6.6. Let $U, V \in \mathcal{U}_{\mathrm{hol}}$ be bisubmersions. Let $S \subset V$ be a local bisection, and let $\Phi_S = r|_S \circ s|_S^{-1}$ be the local diffeomorphism induced by S. We define the *right-translate* of (U, r_U, s_U) by S to be (U_S, r_{U_S}, s_{U_S}) with

$$U_S = s_U^{-1}(r_V(S)) \subseteq U, \qquad r_{U_S} = r_U, \qquad s_{U_S} = \Phi_S^{-1} \circ s_U.$$

Likewise, the *left-translate* of (U, r_U, s_U) by S is (U^S, r_{U^S}, s_{U^S}) with

$$U^S = r_U^{-1}(s_V(S)) \subseteq U, \qquad r_{U^S} = \Phi_S \circ r_U, \qquad s_{U_S} = s_U.$$

Using the fact that the local diffeomorphism Φ_S preserves the foliation $\mathcal{F}$, we see that U_S and U^S satisfy the bisubermsion axioms. We want to further check that they belong to the maximal path-holonomy atlas.

For this, consider the subset $U \circ S = U\, {}_s\!\times_r S$ of $U \circ V$. The projection $\mathrm{pr}_U : U \circ S \to U$ is a diffeomorphism onto U_S which intertwines the range and source maps, and it follows that the embedding

$$\iota_S : U_S \xrightarrow{\mathrm{pr}_U^{-1}} U \circ S \hookrightarrow U \circ V \tag{6.1}$$

is a morphism of bisubmersions. This proves that U_S is adapted to $\mathcal{U}_{\mathrm{hol}}$. The proof for U^S is similar, using the morphism of bisubmersions

$$\iota^S : U^S \xrightarrow{\mathrm{pr}_U^{-1}} S \circ U \hookrightarrow V \circ U \tag{6.2}$$

defined in the analogous way.

Next, we want to introduce the left and right translates of an s-fibred distribution.

We note that on the open subset $U_S \subseteq U$ the fibres of the two source maps s_U and $s_{U_S} = \Phi_S^{-1} \circ s_U$ are the same, although the maps themselves are different. Therefore, if $a \in \mathcal{E}'_s(U)$ has $\mathrm{supp}(a) \subseteq U_S$, then a also defines an s-fibred distribution on U_S, which we denote by a_S and call the *right translate of a by S*. Explicitly, a_S is determined by

$$(a_S, \phi|_{U_S})(x) = \begin{cases} (a, \phi)(\Phi_S(x)), & \text{if } x \in s_V(S) \\ 0, & \text{otherwise,} \end{cases}$$

for all $\phi \in C^\infty(U)$.

A similar definition can be made for the left translate, and in fact is even easier since $s_{U^S} = s_U$ on U^S. In this case, for $b \in \mathcal{E}'_s(U)$ with $\mathrm{supp}(b) \subseteq U^S$ we define

$$(b^S, \phi|_{U^S}) = (b, \phi),$$

for all $\phi \in C^\infty(U)$.

We collect some basic facts about left and right translates.

Lemma 6.7. *Let U, V, W be bisubmersions and let $S \subset V$ be local bisections. If $a \in \mathcal{E}'_s(U)$ with $\operatorname{supp}(a) \subseteq U_S$ then:*

a) *We have $(W \circ U)_S = W \circ U_S$, and for any $b \in \mathcal{E}'_s(W)$ we have $(b*a)_S = b * a_S$.*
b) *The right translate a_S is a smooth s-fibred density if and only if a is.*

Similarly, if $a \in \mathcal{E}'_s(U)$ with $\operatorname{supp}(a) \subseteq U^S$ then:

c) *We have $(U \circ W)^S = U^S \cap W$, and for any $b \in \mathcal{E}'_s(W)$ we have $(a * b)^S = a^S * b$.*
d) *The left translate a^S is a smooth s-fibred density if and only if a is.*

Proof. For (a), note that

$$(W \circ U)_S = \{(w,u) \in W \,{}_s\!\times_r U : s_U(u) \in r_V(S)\} = W \circ U_S.$$

Thus, the right translate $(b * a)_S$ makes sense, and its restrictions to the s-fibres of $W \circ U_S$ are identical to those of $b * a$, and hence to those of $b * a_S$. Part (b) follows immediately from the fact that the restrictions of a and a_S to the fibres of U_S are identical.

The other two statements are analogous. □

Lemma 6.8. *Let $V, W \in \mathcal{U}_{\mathrm{hol}}$. For every $v \in V$ and $w \in W$ there exist open neighbourhoods $V_v \subseteq V$ of v (depending only on v) and $W_{v,w} \subseteq W$ of w (depending on both v and w) such that for any $a \in C^\infty_s(V_v; |\Omega_s|)$ and $b \in \mathcal{E}'_s(W_{v,w})$ we have $a * b \in C^\infty_s(\mathcal{F}; |\Omega_s|)$.*

Proof. Let $S \subset V$ be a local bisection passing through v. Note that S^{t} is a local bisection of V^{t}, so we may consider the left translate $V^{S^{\mathrm{t}}}$ of V by S^{t}. In this bisubmersion, the set S is an identity bisection, since

$$r_{U^{S^{\mathrm{t}}}}(z) = \Phi_{S^{\mathrm{t}}} \circ r_U(z) = s_U(z) = s_{U^{S^{\mathrm{t}}}}(z)$$

for all $z \in S$.

Likewise, if $T \subset W$ is a bisection passing through w, then T is an identity bisection in the right translate $W_{T^{\mathrm{t}}}$.

Therefore, by Lemma 6.5, we can find neighbourhoods V_v of v in $V^{S^{\mathrm{t}}}$ and $W_{v,w}$ of w in $W_{T^{\mathrm{t}}}$ verifying the conditions of Lemma 6.5.

Let $a \in C^\infty_s(V; |\Omega_s|)$ with $\operatorname{supp}(a) \subseteq V_v$ and $b \in \mathcal{E}'_s(W)$ with $\operatorname{supp}(b) \subseteq W_{v,w}$. By Lemmas 6.5 and 6.7 (a) and (c) we have

$$a^{S^{\mathrm{t}}} * b_{T^{\mathrm{t}}} = ((a * b)^{S^{\mathrm{t}}})_{T^{\mathrm{t}}} \in C^\infty_s(\mathcal{F}; |\Omega_s|).$$

By Lemma 6.7 (b) and (d), this implies $a*b \in C^\infty_s(\mathcal{F}; |\Omega_s|)$, as claimed. □

Proof of Theorem 6.3. Let $a \in C^\infty_s(V; |\Omega_s|)$ and $b \in \mathcal{E}'_s(W)$ for some $V, W \in \mathcal{U}_{\mathrm{hol}}$.

For each $v \in V$, pick an open neighbourhood V_v of v as in Lemma 6.8. Using a smooth locally finite partition of unity subordinate to the cover $(V_v)_{v \in V}$, we can reduce to the case where a is supported on V_v for some $v \in V$.

Next, for each $w \in W$, pick an open neighbourhood $W_{v,w}$ of w as in Lemma 6.8. Again, using a partition of unity we can reduce to the case where b is supported in $W_{v,w}$ for some $w \in W$. Then Lemma 6.8 completes the proof. □

7. Proper distributions

We now want to consider Schwartz kernels which are both r- and s-fibred. As defined above, the spaces $\mathcal{E}'_r(U)$ and $\mathcal{E}'_s(U)$ are not comparable, so we must put both of them into the usual space of distributions $\mathcal{D}'(U)$.

Throughout this section, we will fix a choice of nowhere-vanishing smooth 1-density on the base space, $\mu \in C^\infty(M; |\Omega|)$. As we will show, the particular choice of μ ultimately will not affect the results.

Definition 7.1. Let U be a bisubmersion and μ a nowhere-vanishing smooth density on M. We define maps

$$\mu_r : \mathcal{E}'_r(U) \hookrightarrow \mathcal{D}'(U); \qquad \mu_r(a) = \mu \circ a,$$
$$\mu_s : \mathcal{E}'_s(U) \hookrightarrow \mathcal{D}'(U); \qquad \mu_s(b) = \mu \circ b.$$

The maps μ_r and μ_s are continuous injective linear maps. The image $\mu_r(\mathcal{E}'_r(U)) \subseteq \mathcal{D}'(U)$ coincides with the space of distributions (with r-proper support) transversal to the submersion r, as defined in [AS11, §1.2.1]; see also [LMV17]. This shows that the image is independent of the choice of smooth density μ. Analogous statements hold for μ_s, of course.

Definition 7.2. Let U be a bisubmersion.

a) A distribution in $\mathcal{D}'(U)$ is called *proper* if it belongs to $\mu_r(\mathcal{E}'_r(U)) \cap \mu_s(\mathcal{E}'_s(U))$. The space of proper distributions on U is denoted $\mathcal{D}'_{\mathrm{p}}(U)$.
b) An r-fibred distribution $a \in \mathcal{E}'(U)$ is called *proper* if $\mu_r(a)$ is proper. The space of proper r-fibred distributions on U is denoted $\mathcal{E}'_{r,s}(U)$.
c) An s-fibred distribution $b \in \mathcal{E}'(U)$ is called *proper* if $\mu_s(b)$ is proper. The space of proper s-fibred distributions on U is denoted $\mathcal{E}'_{s,r}(U)$.

Since r-fibred distributions have r-proper support, and s-fibred distributions have s-proper support, we see that proper distributions of any kind have proper support in the sense of Definition 5.6.

Let us write $C^\infty_{\mathrm{p}}(U)$ for the space of properly supported functions on U and $C^\infty_{\mathrm{p}}(U; E)$ for the space of properly supported sections of any bundle E over U. Note that

$$\mu_r(C^\infty_{\mathrm{p}}(U; |\Omega_r|)) = C^\infty_{\mathrm{p}}(U; |\Omega|) = \mu_s(C^\infty_{\mathrm{p}}(U; |\Omega_s|)),$$

so that properly supported smooth r-fibred densities are automatically proper as r-fibred distributions, and likewise for properly supported smooth s-fibred densities.

Lemma 7.3. *Let U and V be bisubmersions and let $a \in \mathcal{E}'_{r,s}(U)$ and $b \in \mathcal{E}'_{r,s}(U)$ be proper r-fibred distributions, so that there exist $\tilde{a}, \tilde{b} \in \mathcal{E}'_s(U)$ with $\mu_r(a) = \mu_s(\tilde{a})$ and $\mu_r(b) = \mu_s(\tilde{b})$. Then*

$$\mu_r(a * b) = \mu_s(\tilde{a} * \tilde{b}).$$

In particular, the convolution product of two proper r-fibred distributions is again proper, and likewise for proper s-fibred distributions.

Proof. We first claim that for any $\tilde{a} \in \mathcal{E}'_s(U)$ and $b \in \mathcal{E}'_r(V)$ we have $\tilde{a} \circ s_U^* b = b \circ r_V^* \tilde{a}$ as maps $C^\infty(U \circ V) \to C^\infty(M)$. Note that both these maps are

$C^\infty(M)$-linear with respect to the 'middle' submersion,

$$q : U \circ V \to M; \qquad q(u,v) = s_U(u) = r_v(V),$$

that is, we have

$$(\tilde{a} \circ s_U^* b, (q^* f)\phi) = f(\tilde{a} \circ s_U^* b, \phi)$$

for all $\phi \in C^\infty(U \circ V)$ and $f \in C^\infty(M)$, and similarly for $b \circ r_V^* \tilde{a}$. The q-fibre at $x \in M$ is $q^{-1}(x) = U_x \times V^x$, and we calculate

$$(\tilde{a} \circ s_U^* b, \phi)(x) = (\tilde{a}_x \otimes b^x, \phi|_{U_x \times V^x}) = (b \circ r_V^* \tilde{a}, \phi)(x),$$

which proves the claim.

Finally, we obtain

$$\begin{aligned}\mu_r(a * b) &= \mu \circ a \circ s_U * b = \mu \circ \tilde{a} \circ s_U^* b \\ &= \mu \circ b \circ r_V^* \tilde{a} = \mu \circ \tilde{b} \circ r_V^* \tilde{a} = \mu_s(\tilde{a} \circ \tilde{b}).\end{aligned}$$

□

Definition 7.4. We define $\mathcal{E}'_{r,s}(\mathcal{F})$ to be the subspace of $\mathcal{E}'_r(\mathcal{F})$ consisting of classes of proper r-fibred distributions on $\mathbf{U}$, and $\mathcal{E}'_{s,r}(\mathcal{F})$ as the subspace of $\mathcal{E}'_s(\mathcal{F})$ consisting of classes of proper s-fibred distributions on $\mathbf{U}$.

Proposition 7.5. *The subspace $\mathcal{E}'_{r,s}(\mathcal{F})$ is a closed subalgebra of $\mathcal{E}'_r(\mathcal{F})$ and $C^\infty_{\mathrm{p}}(\mathcal{F}; |\Omega_r|)$ is a two-sided ideal.*

Similarly for $\mathcal{E}'_{s,r}(\mathcal{F})$ and $C^\infty_{\mathrm{p}}(\mathcal{F}; |\Omega_s|)$.

Proof. The fact that $\mathcal{E}'_{r,s}(\mathcal{F})$ is closed under convolution follows from Lemma 7.3. Theorem 6.3 shows immediately that $C^\infty_{\mathrm{p}}(\mathcal{F}; |\Omega_r|)$ is a right ideal, and also that it is a left ideal if we take advantage of Lemma 7.3. □

We note the relationship of the transpose map from Definition 3.7 with the maps μ_r and μ_s:

$$\mu_r(a)^{\mathrm{t}} = \mu_s(a^{\mathrm{t}})$$

It follows that the transpose of a proper r-fibred distribution is a proper s-fibred distribution, and so by Proposition 4.5, transposition gives an anti-isomorphism of convolution algebras ${}^{\mathrm{t}} : \mathcal{E}'_{r,s}(\mathcal{F}) \to \mathcal{E}'_{s,r}(\mathcal{F})$.

Although properness is a crucial property for kernels of pseudodifferential operators (see [vEY], as well as Section 8 below), it can be difficult to check directly. We therefore conclude this section by giving a convenient sufficient condition for the properness of a distribution using the wavefront set. This result is due to Lescure-Manchon-Vassout [LMV17]. Of course, the idea that wavefront sets can be used to detect smooth and non-smooth directions in distributions is Hörmander's.

Proposition 7.6. *Let S be a bisection in a bisubmersion U, and suppose that $a \in \mathcal{D}'(U)$ is a distribution with*

- *proper support,*
- *singular support contained in S,*
- *wavefront set contained in $TS^\perp = \{\eta \in T^*U : (\eta, \xi) = 0$ for $\xi \in TS\}$.*

Then a is a proper distribution.

Proof. We claim that $TS^{\perp} \cap \ker(dr)^{\perp} = \emptyset = TS^{\perp} \cap \ker(ds)^{\perp}$. For if $\eta \in T^*U_x$ lies in $TS_x^{\perp} \cap \ker(dr)_x^{\perp}$ then $(\eta, \xi) = 0$ for all $\xi \in \ker(dr)_x + TS_x = TU_x$, and hence $\eta = 0$. Similarly with s in place of r. The result now follows from [LMV17, Proposition 2.9]. □

8. The action on generalized functions

According to Section 5, the properly supported r-fibred distributions act on $C^\infty(M)$ and $C_c^\infty(M)$, while properly supported s-fibred distributions act on $\mathcal{E}'(M)$ and $\mathcal{D}'(M)$. If we fix a nowhere vanishing smooth density μ on M, then by Lemma 7.3 we obtain an algebra isomorphism $\mu_s^{-1} \circ \mu_r : \mathcal{E}'_{r,s}(U) \to \mathcal{E}'_{s,r}(U)$, and hence an action of $\mathcal{E}'_{r,s}(E)$on all four of the above spaces. However, the action of $\mathcal{E}'_{r,s}(E)$ on $\mathcal{D}'(M)$ which is obtained in this way depends upon the choice of μ. To obtain a canonical action, we need to work with generalized functions instead of distributions.

Remark 8.1. This issue wouldn't arise if we had followed the operator algebraists' strategy of using half-densities throughout. We have chosen to avoid this because we will ultimately want to apply these results to classical PDE problems.

Definition 8.2. Let $|\Omega|$ denote the bundle of 1-densities on M. We write $C^{-\infty}(M)$ for the continuous linear dual of $C_c^\infty(M; |\Omega|)$, and refer to its elements as *generalized functions* on M. We also write $C_c^{-\infty}(M) = C^\infty(M; |\Omega|)^*$ for the compactly supported generalized functions.

The space of smooth functions $C^\infty(M)$ admits a canonical embedding as a dense subspace of $C^{-\infty}(M)$. Given a choice of nowhere vanishing 1-density μ on M, we obtain a linear isomorphism

$$C^\infty(M) \to C^\infty(M; |\Omega|); \qquad f \mapsto f\mu,$$

and this extends by density to an isomorphism

$$C^{-\infty}(M) \to \mathcal{D}'(M)$$

which we denote formally by $k \mapsto k\mu$ for $k \in C^{-\infty}(M)$.

Definition 8.3. Fix a nowhere-vanishing smooth density μ on M. Let $a \in \mathcal{E}'_{r,s}(U)$ be a proper r-fibred density on the bisubmersion U, which means there is $\tilde{a} \in \mathcal{E}'_{s,r}(U)$ with $\mu_s(\tilde{a}) = \mu_r(a)$. We define the operator $\mathrm{Op}(a)$ on $C^{-\infty}(M)$ by the formula

$$(\mathrm{Op}(a)k)\mu = (\widetilde{\mathrm{Op}}(\tilde{a}))(k\mu)$$

where $k \in C^{-\infty}(U)$ and $\widetilde{\mathrm{Op}}$ is the representation of $\mathcal{E}'_{s,r}(U)$ on $\mathcal{D}'(M)$ defined at the end of Section 5.

Proposition 8.4. *The map* $\mathrm{Op} : \mathcal{E}'_{r,s}(\mathbf{U}) \to \mathbb{L}(C^{-\infty}(M))$ *defined above is independent of the choice of smooth nowhere-vanishing density* μ. *It induces a continuous linear representation of* $\mathcal{E}'_{r,s}(\mathcal{F})$ *on generalized functions which extends the representation* Op *on* $C^\infty(M)$.

Proof. Let $k \in C^\infty(M)$. If we use the definition of $\mathrm{Op}(a)k$ from Definition 8.3, we get, for all $f \in C^\infty(M)$,

$$\begin{aligned}
((\mathrm{Op}(a)k)\mu, f) &= (\widetilde{\mathrm{Op}}(\tilde{a})(k\mu), f)\\
&= (k\mu, \mathrm{Op}(\tilde{a}^{\mathrm{t}})f)\\
&= \int_{x\in M} (\tilde{a}^{\mathrm{t}}, r_U^* f)(x)k(x)\mu(x)\\
&= \int_{x\in M} (\tilde{a}^{\mathrm{t}}, (r_U^* f)(s_U^* k))(x)\mu(x)\\
&= (\mu_s(\tilde{a}), (r_U^* f)(s_U^* k)),
\end{aligned}$$

where the second last equality uses the $C^\infty(M)$-linearity of $\tilde{a}^t$ as an r-fibred distribution. On the other hand, using the definition of $\mathrm{Op}(a)k$ from Propostion 5.1 gives

$$\begin{aligned}
((\mathrm{Op}(a)k)\mu, f) &= \int_{x\in M} (a, s_U^* k)(x)\mu(x)f(x)\\
&= \int_{x\in M} (a, (r_U^* f)(s_U^* k))(x)\mu(x)\\
&= (\mu_s(\tilde{a}), (r_U^* f)(s_U^* k)),
\end{aligned}$$

where the second equality uses the $C^\infty(M)$-linearity of a as an r-fibred distribution. This proves that the two definitions of $\mathrm{Op}(a)k$ agree when $k \in C^\infty(M)$. The definition of $\mathrm{Op}(a)k$ from Proposition 5.1 clearly does not depend on the choice of μ, so neither does Definition 8.3 in this case. By density, the same must be true for all $k \in C^{-\infty}(M)$.

For $a \in \mathcal{E}'_{r,s}(U)$ and $b \in \mathcal{E}'_{r,s}(V)$, Lemma 7.3 plus the fact that $\widetilde{\mathrm{Op}}$ is an algebra representation of $\mathcal{E}_{s,r}(\mathcal{F})$ gives

$$(\mathrm{Op}(a*b)k)\mu = \widetilde{\mathrm{Op}}(\tilde{a}*\tilde{b})(k\mu) = \widetilde{\mathrm{Op}}(\tilde{a})\widetilde{\mathrm{Op}}(\tilde{b})(k\mu) = (\mathrm{Op}(a)\,\mathrm{Op}(b)k)\mu,$$

so Op is indeed a representation. □

Therefore, representation Op defines an action of $\mathcal{E}'_{r,s}(\mathcal{F})$ on each of the four spaces $C^\infty(M)$, $C^\infty_{\mathrm{c}}(M)$, $C^{-\infty}(M)$ and $C^{-\infty}_{\mathrm{c}}(M)$.

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