# The Geometry behind Analysis

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# Galois theory

K: field of characteristic zero, $f \in K[x]$ polynomial

F: splitting field of $f$.

- F radical extension of K iff
  1. $F = K[u_1, \ldots, u_n]$
  2. some power of $u_1$ lies in K
  3. for each $i \geqslant 2$, some power of $u_i$ lies in $K(u_1, \ldots, u_{i-1})$
- $f(x) = 0$ is solvable by radicals if there is a radical extension F and a splitting field E of f s.t. $K \subset E \subset F$.

  Namely, F contains all roots of $f(x)$.
- $Aut_K F =$ Galois group (abelian) of $f$;

# Galois theory

**Theorem**

$f(x) = 0$ is solvable by radicals iff $G = \mathrm{Aut}_K F$ is solvable.

Namely, if there is a (finite) chain

$$\langle e \rangle = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

**Sophus Lie, ca 1870**

Can we solve differential equations this way?

# Differential equations

- Too many of them... ODEs, PDEs, linear, non-linear, etc...
- Solutions depend on initial conditions...
- Usually solved with "cookbook" methods...

**Example: Heat equation**

$$u_t = u_{xx}$$

Fourier transform $\rightsquigarrow$ Fundamental Source Solution

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}, \quad t > 0, \quad -\infty < x < \infty$$

- Initial condition: Trigonometric function (is enough...)
- Superposition principle: Linear combinations of solutions are solutions.

# Summary

1. Solving equations
   - Case study: The heat equation
   - Lie, Jacobi and Differential Equations
   - Back to Galois
2. Understanding the space of solutions
   - Indices and the Atiyah-Singer theorem
   - Use of the AS: The quest for positive scalar curvature
   - Symmetries behind the Atiyah-Singer theorem
3. Foliations
   - Foliations and their symmetries
   - Use of foliations: Spectrum calculations
4. Two open questions

# Heat equation: Geometric formulation

Heat equation: $u_t - u_{xx} = 0$

- Independent variables: $t, x$
- Dependent variable: $u$

1st jet: $X \times U = \mathbb{R}^3$ with coordinates $(x, t, u)$.

Heat equation is 2nd order $\rightsquigarrow$ 2nd jet: $X \times U^{(2)} = \mathbb{R}^8$

- coordinates $(x, t, u, u_x, u_t, u_{tx}, u_{tt}, u_{xx})$.
- Natural projection: $\pi : X \times U^{(2)} \to X \times U$.

Solutions: Put $\Delta(x, u^{(2)}) = u_t - u_{xx}$. Solution is $u : X \to U$ s.t:

- graph $\Gamma_u \subset X \times U$ and
- submanifold of $X \times U^{(2)}$ defined by $S_{\Delta,u} = \Delta^{-1}(0)$

satisfy $\pi(S_{\Delta,u}) = \Gamma_u$

# Symmetries of a differential equation

**Definition**

Let $\Delta$: $n$th order differential equation. A symmetry group of $\Delta$ is a local Lie group $G$ such that:

- $G$ acts on open $M \subseteq X \times U$
- $u$: solution $\Rightarrow$ $g \cdot u$: solution, for all $g \in G$.

**Theorem**

$G$ symmetry group for $\Delta$ iff $\mathfrak{g}^{(n)}$ tangent to $S_\Delta$.

Question: Say $\Delta$ admits $\mathfrak{g}_\Delta$ as a group of symmetries.

How does knowledge of $\mathfrak{g}_\Delta$ simplify the solution of $\Delta(x, u^{(n)}) = 0$?

# Symmetries of heat equation

$G_1$ $(x+\varepsilon, t, u)$

translation on $x$-axis

$G_2$ $(x, t+\varepsilon, u)$

translation on $t$-axis

$G_3$ $(x, t, e^{\varepsilon}u)$

positive multiple of solution is solution

$G_4$ $(e^{\varepsilon}x, e^{2\varepsilon}t, u)$

well-known scaling symmetry

$G_5$ $(x+2\varepsilon t, t, ue^{-\varepsilon x-\varepsilon^2 t})$

Galilean boost to a moving frame

$G_6$ $\left(\frac{x}{1-4\varepsilon t}, \frac{t}{1-4\varepsilon t}, u\sqrt{1-4\varepsilon t}\cdot exp\left(\frac{-\varepsilon x^2}{1-4\varepsilon t}\right)\right)$

$G_a$ $(x, t, u+\varepsilon a(x,t))$, where $a$: solution

superposition principle

# Fundamental Source Solution from Symmetries

- From $G_6$, if $u(x,t)$ is a solution then another solution is

$$v(x,t) = \frac{1}{\sqrt{1-4\varepsilon t}} e^{\frac{-\varepsilon x^2}{1-4\varepsilon t}} \cdot u\left(\frac{x}{\sqrt{1-4\varepsilon t}}, \frac{t}{\sqrt{1-4\varepsilon t}}\right)$$

- Any constant $c$ is a solution, so get solution

$$v_1 = \frac{c}{\sqrt{1-4\varepsilon t}} e^{\frac{-\varepsilon x^2}{1-4\varepsilon t}}$$

- Put $c = \sqrt{\frac{\varepsilon}{\pi}}$ and get

$$v_2 = \frac{1}{\sqrt{4\pi\left(t+\frac{1}{4\varepsilon}\right)}} exp\left(\frac{-x^2}{e^{4\left(t+\frac{1}{4\varepsilon}\right)}}\right)$$

- Apply $G_2$ and "right" translate $v_2$ by $\frac{1}{4\varepsilon}$ in $t$. Obtain

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

# Sophus Lie and Line Geometry

Let $\Delta$: tetrahedron

Each line $\ell \in P^3(\mathbb{C})$ meets $\Delta$ in 4 points $p_1, p_2, p_3, p_4$.

$\mathcal{T}_{\ell_0}$ = tetrahedral line complex = lines $\ell$ whose cross-ratio of 4 points is the same as those of $\ell_0$.

$\mathcal{B}$ = projective transformations fixing vertices of $\Delta$ (coordinate changes...)

**Fact**

$\mathcal{T}_{\ell_0}$ = orbit of $\ell_0$ by $\mathcal{B}$-action.

# Lie's "Idée Fixe"

Pick a point $p$ and choose a tetrahedral line complex $\mathcal{T}$.

Put $C(p)$ = all lines in $\mathcal{T}$ passing from $p$.

**Problem**

Determine all surfaces $S$ such that at each point $p \in S$ the tangent plane $T_p S$ meets the cone $C(p)$ in exactly one straight line.

Equivalent to solving:

$$f(x, y, z, p, q) = 0, \quad p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

**Fact**: Any $T \in \mathcal{B}$ takes a solution surface into another.

**Theorem (S. Lie, ca 1870)**

- If $f(x, y, z, p, q) = 0$ admits 3 commuting infinitesimal projective transformations, then it can be transformed to $f(P, Q) = 0$
- 2 commuting transformations $\rightsquigarrow f(Z, P, Q) = 0$.
- 1 commuting transformation $\rightsquigarrow f(X, Y, P, Q) = 0$.

# Enter Poisson brackets...

(Jacobi-)Poisson bracket: Put $(x, p) = (x_1, \ldots, x_n, p_1, \ldots, p_n)$.

- For $G, H \in C^\infty(\mathbb{R}^{2n})$ define

$$\{G, H\} = \sum_{i=1}^{n} \left( \frac{\partial G}{\partial p_i} \frac{\partial H}{\partial x_i} - \frac{\partial G}{\partial x_i} \frac{\partial H}{\partial p_i} \right)$$

**Theorem (Jacobi 1830s - Generalized by Adolf Meyer and Sophus Lie, 1872.)**

A pde $F_1(x, p) = 0$ can be integrated if functions $F_2, \ldots, F_n$ of the $2n$ variables $(x, p)$ can be determined such that

- $F_1, \ldots, F_n$ are functionally independent;
- $\{F_i, F_j\} = 0$ for all $i, j$.

# Towards Galois...

Whence, integration of $F_1(x, p) = 0$ reduces to determining one solution to each of the following systems of ODEs:

- ▷ 1 system of order $2n - 2$
- ▷ 2 systems of order $2n - 4$

⋮

- ▷ $n - 1$ systems of order 2

These systems somehow play the role of the auxiliary polynomial equations associated to the decomposition series of the Galois group...

# Jacobi's problem

## Jacobi's problem

Suppose $f = F_1, \dots, F_r$ are $r$ functionally independent solutions to $\{F_1, f\} = 0$, such that the bracketing produces no more solutions. Namely,

$$\{F_i, F_j\} = \sum_{k=1}^{r} \Omega_{i,j}^{k} F_k$$

How does knowledge of $F_1, \dots, F_r$ simplify the solution of $F_1(x, p) = 0$?

In terms of the Idée Fixe:

Given that the pde $F_1(x, p) = 0$ admits $\mathfrak{g}_F$ as a group of symmetries, how does knowledge of $\mathfrak{g}_F$ simplify the problem of solving $F_1(x, p) = 0$?

# The result

**Theorem**

Let $\Delta(x, u^{(n)}) = 0$ an ode of order $n$. If $\Delta$ admits an $n$-dimensional group of symmetries $\mathfrak{g}_\Delta$ which is solvable, the general solution to $\Delta$ can be found by quadratures alone.

- $\mathfrak{g}$ as above is solvable if there is a chain of subalgebras

$$\{0\} = \mathfrak{g}^{(0)} \subseteq \mathfrak{g}^{(1)} \subseteq \ldots \subseteq \mathfrak{g}^{(n-1)} \subseteq \mathfrak{g}^{(n)} = \mathfrak{g}$$

  such that $\dim \mathfrak{g}^{(k)} = k$ and $[\mathfrak{g}^{(k-1)}, \mathfrak{g}^k] \subseteq \mathfrak{g}^{(k-1)}$ for all $1 \leqslant k \leqslant n$.

- Equivalently: There is a basis $\{v_1, \ldots, v_n\}$ of $\mathfrak{g}$ such that

$$[v_i, v_j] = \sum_{k=1}^{n} c_{ij}^k v_k \quad \text{whenever } i < j$$

# Aside: Noether's theorem

**Theorem (Emmy Noether, 1915)**

Let $X = \sum f_i \frac{\partial}{\partial q^i}$ symmetry of a Lagrangian system $(M, L)$.
An integral of motion is

$$I(q, \dot{q}) = \sum \frac{\partial L}{\partial \dot{q}_i} f_i$$

"If you want to find conservation laws, first detect if there are any (infinitesimal) symmetries around..."

# Linear operators

H: Hilbert space, $L : H \to H$ linear operator (*e.g.* differential operator...)

$$Lf = g$$

- Existence: Given $g$, is there $f$ such that $Lf = g$?
- Uniqueness: Given $f, g$ such that $Lf = g$, to what extent is $f$ unique?

- $\dim(\mathrm{coker}L)) = \dim(\mathrm{ker}L^*)$ measures to what extent $Lf = g$ can fail to have a solution.

  If $Lf = g$ has a solution for any $g$ then $\mathrm{Im}L = 0$
- $\dim(\mathrm{ker}L)$ measures to what extent $Lf = g$ fails to have a unique solution

  If $Lf = g$ always has a unique solution (when it has a solution) then $\mathrm{Ker}L = 0$.

# Fredholm index

Let $L$ Fredholm: $\mathrm{Im}L$ closed subspace and $\dim(\ker L), \dim(\mathrm{coker}L) < \infty$.

- ▷ Equivalently, $L$ is invertible modulo compact operators: There is $P$ such that

$$LP = 1 + Q_1, \quad PL = 1 + Q_2$$

where $Q_1, Q_2$ compact operators.

- ▷ Elliptic ((Pseudo)differential...) operators on compact $M$ are Fredholm.

**Definition**

$$\mathrm{Ind}(L) = \dim(\ker L) - \dim(\mathrm{coker}L) \in \mathbb{Z}$$

$$\begin{array}{ccc} \mathrm{Ell}(M) & \xrightarrow{\mathrm{Ind}} & \mathbb{Z} \\ \downarrow \sigma & \nearrow \mathrm{Ind}_{an} & \\ K^0(T^*M) & & \end{array}$$

# Toeplitz operators and winding number

- $H = H^2(S^1) = \{f : S^1 \to \mathbb{C} \text{ s.t. } f(\theta) = \sum_{n=0}^{\infty} a_n e^{in\theta}\} \subset L^2(S^1)$
- Projection $P : L^2(S^1) \to H^2(S^1)$
- Given $\phi : S^1 \to \mathbb{C}$ continuous, define

$$T_\phi : H \to H, \quad T_\phi(f) = P(\phi \cdot f)$$

**Proposition**

- $||T_\phi|| = ||\phi||_\infty$, so $T_\phi \in B(H)$.
- If $\phi \neq 0$ then $T_\phi$ is Fredholm.

**Theorem**

$$\mathrm{Ind}(T_\phi) = W(\phi)$$

# Atiyah-Singer

$M$: compact manifold, $D$ (pseudo)differential operator.

**Theorem(Atiyah-Singer)**

$$\mathrm{Ind}_{an}(D) = (-1)^n \int_{T^*M} \mathrm{ch}(\sigma(D)) \wedge \mathrm{Td}(TM \oplus \mathbb{C})$$

Corollaries

- Riemann-Roch thm
- Gauss-Bonet thm

# Use of AS: Positive scalar curvature 1

## Definition

Let $(M, g)$ Riemannian manifold. The scalar curvature $scal_g : M \to \mathbb{R}$ gives at each point the average of sectional curvatures. Formally,

$$scal_g = tr_g(Ric)$$

**Qn**: Given $M$, what are the possibilities for $scal_g$ as $g$ varies?

## Gauss-Bonnet

$X$ compact surface with Riemannian metric

$$\int_X scal(x)dvol(x) = \pi\,\chi(X)$$

(For surfaces $scal = 2 \cdot K_{Gauss}$)

So, if $X$ admits $g$ with psc then $X = S^2$ or $X = \mathbb{R}P^2$.

# Use of AS: Positive scalar curvature 2 - Dirac operator

$M$ compact spin manifold (need 2nd Stiefel-Whitney class to vanish...)

**Proposition**

- There is a Dirac operator $\mathrm{D}$ around (acting on the bundle of spinors). In flat space, $\mathrm{D}$ is a square root of the vector Laplacian.
- $\mathrm{D}$ is elliptic, hence Fredholm.

**Apply AS on $\mathrm{D}^+$:**

$$\mathrm{Ind}_{an}(\mathrm{D}) = \hat{A}(M)$$

- $\hat{A}(M)$ is a topological invariant of $M$. Can be computed without ever solving differential equations!
- $\hat{A}(M)$ does not depend on the metric. ($\mathrm{D}$ does!)

# Use of AS: Positive scalar curvature 3

**Theorem**

If scalar curvature is everywhere positive, then $\mathrm{D}^2$ is strictly positive, whence $\mathrm{D}$ is invertible (truly, not just up to compact operators).

Consequences:

- If $M$ has psc then $\mathrm{Ind}(\mathrm{D}) = 0$.
- $\hat{\mathrm{A}}(M) \neq 0$ is an obstruction to psc!

# Symmetries and the index 1

"External" symmetries of $M$: Lie Groupoid $M \times M$:

$$s(x,y) = y,\ t(x,y) = x$$
$$(x,y)\cdot(y,z) = (x,z)$$
$$(x,y)^{-1} = (y,x)$$
$$1_x = (x,x), \text{ so } M \subset M \times M \text{ diagonally.}$$

Lie functor: $Lie(M \times M) = \cup_{y \in M} T_{1_y}(s^{-1}(y)) = \cup_{y \in M} T_y M = TM$.
Groupoid multiplication differentiates to Lie bracket of vector fields!

$C^*$-functor: $C^*(M \times M) = \mathcal{K}(L^2(M))$

# Symmetries and the index 2

$$
\begin{array}{ccc}
\mathrm{Ell}(M) & \xrightarrow{\mathrm{Ind}} & K_0(\mathcal{K}(L^2(M))) = \mathbb{Z} \\
\Big\downarrow \sigma & \nearrow \mathrm{Ind}_{an} & \\
K^0(\mathrm{Lie}(M \times M)^*) & &
\end{array}
$$

# Symmetries and the index 3

- $TM$ with addition on each fiber. It's a groupoid. Units = zero section
- Tangent groupoid $\mathcal{G}$:

$$TM \times \{0\} \coprod (M \times M) \times (0,1]$$

Units: $M \times [0,1]$. Topology: $(x_n, y_n, t) \to \xi_x$ iff $x_n \to x$ and $\frac{x_n - y_n}{t} \to \xi$ as $t \to 0$.

$C^*$-functor:

- For $C^*(TM) = C_0(T^*M)$
- $C^*(\mathcal{G})$ is the extension:

$$0 \to K(L^2(M)) \otimes C_0((0,1]) \to C^*(\mathcal{G}) \overset{ev_0}{\to} C_0(T^*M) \to 0$$

**Theorem (Guillemin-Sternberg)**

$$\mathrm{Ind}_{an} = [ev_1] \circ [ev_0]^{-1}$$

$\mathrm{Ind}_{top}$ arises from a kind of tangent groupoid as well (Debord & Lescure).

# Foliations

**Definition**

$(M, \mathcal{F})$ finitely generated $C^\infty(M)$-submodule of $\Gamma(TM)$ with $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$.

- Stefan-Sussmann: $(M, \mathcal{F})$ partitions $M$ to (immersed) submanifolds. Dimension may jump.
- Constant dimension $\Rightarrow \mathcal{F}$ projective. *e.g.* $M = T^2$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $\mathcal{F} = \langle \frac{\partial}{\partial x} + \theta \frac{\partial}{\partial y} \rangle$
- $\mathcal{F}$ projective $\Rightarrow$ Dimension constant in a dense (open) subset. *e.g.* $M = \mathbb{R}$ and $\mathcal{F} = \langle x \frac{\partial}{\partial x} \rangle$.

Foliations appear in the study of Lie group actions, in Poisson geometry, and lots of other fields...

# Symmetries of foliations

For *any* singular foliation, A-Skandalis constructed:

- Holonomy groupoid $H(\mathcal{F})$. Very singular...
- $C^*(\mathcal{F})$, representations...
- The cotangent bundle $\mathcal{F}^*$: locally compact space.
- Pseudodifferential caclulus and longitudinal Laplacian.
- Analytic index (element of $KK(C_0(\mathcal{F}^*); C^*(M, \mathcal{F}))$)
- tangent groupoid + defines same KK element.

Example: $\mathcal{F} =< X >$ s.t. $X$ has non-periodic integral curves around $\partial\{X = 0\}$:

$$H(\mathcal{F}) = H(X)|_{\{X \neq 0\}} \cup \mathrm{Int}\{X = 0\} \cup (\mathbb{R} \times \partial\{X = 0\})$$

# Laplacian of $\mathbb{R}$ and Kronecker foliation

Kronecker foliation on $M = T^2$: $\mathcal{F} = \langle X = \frac{d}{dx} + \theta \frac{d}{dy} \rangle$. $L = \mathbb{R}$

Two Laplacians:

- $\Delta_L = -\frac{d^2}{dx^2}$ acting on $L^2(\mathbb{R})$
- $\Delta_M = -X^2$ acting on $L^2(M)$

By Fourier:

- $\Delta_L \rightsquigarrow$ mult. by $\xi^2$ on $L^2(\mathbb{R})$. Spectrum: $[0, +\infty)$.
- $\Delta_M \rightsquigarrow$ mult. by $(n + \theta k)^2$ on $L^2(\mathbb{Z}^2)$. Spectrum dense in $[0, +\infty)$.

**Theorem (Connes, Kordyukov)**

$(M, \mathcal{F})$ with constant dimension. If $L$ is a dense leaf, then $\Delta_L$ and $\Delta_{\mathcal{F}}$ are isospectral.

# What about spectrum calculation?

Gaps in spectrum $\longleftrightarrow$ projections of $C^*(\mathcal{F})$ $\longleftrightarrow$ elements of $K(C^*(\mathcal{F}))$

Need to know shape of $K(C^*(\mathcal{F}))$.

Predicted by Baum-Connes assembly map: Kind of analytic index map.

# Spectrum Calculation: Example

## Horocyclic foliation: Spectrum has no gaps

Consider the action of the "$ax+b$"-group on a compact manifold $M$.
*e.g.* $M = SL(2,\mathbb{R})/\Gamma$ where $\Gamma$ discrete co-compact group.
Leaves = orbits of "$x+b$" subgroup (dense).

Spectrum of Laplacian is an interval $[m, +\infty)$

## Proof: We show $C^*(M,F)$ has no projections.

- $\exists$ $ax+b$-invariant measure of $M \Longrightarrow$ get trace of $C^*(M,F)$. Faithful because $C^*(M,F)$ simple (Fack-Skandalis).
- "$ax$" subgroup induces $\mathbb{R}_+^*$-action on $C^*(M,F)$ which scales the trace.
- Image of $K_0$ is a countable subgroup of $\mathbb{R}$, invariant with respect $\mathbb{R}_+^*$-action.

# Laplacians of singular foliations

**Theorem 1**

$M$ compact manifold, $X_1, \ldots, X_N \in C^\infty(M; TM)$ such that

$$[X_i, X_j] = \sum f_{ij}^k X_k$$

Then $\Delta = \sum X_i^* X_i$ is essentially self-adjoint (both in $L^2(M)$ and $L^2(L)$).

**Theorem 2**

Assume that:

- the (dense open) set $\Omega \subset M$ where leaves have maximal dimension has Lebesgue measure 1.
- the restriction of all leaves to $\Omega$ are dense in $\Omega$.
- the holonomy groupoid of the restriction of $\mathcal{F}$ to $\Omega$ is Hausdorff and amenable.

Then $\Delta_M$ and $\Delta_L$ have the same spectrum.

# Question 1

Spectrum Calculation: Need to know the "shape" of $K_0(C^*(\mathcal{F}))$.

leaves of given dimension $\rightsquigarrow$ locally closed subsets $\rightsquigarrow$ filtration of $C^*(\mathcal{F})$

**Work in progress (IA-Skandalis)**

Give a formula for the K-theory. Baum-Connes conjecture...

# Question 2

**Setting:** Algebraic sets of an affine variety. (They are quite singular objects...)

- Understand their structure.
- Obtain topological invariants.

**Work in progress (IA-Higson)**

**Qn**: Are there any appropriate groupoids around?

# Thank you!