# Hypoellipticity and the Helffer-Nourrigat conjecture

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# Hypoellipticity 1

**Definition**

A linear differential operator

$$D : C^\infty(M) \to C^\infty(M)$$

is *hypoelliptic* if, for every distribution $u$

$$Du \text{ smooth } \Rightarrow u \text{ smooth}$$

- $\partial_x$ on $\mathbb{R}$ is hypoelliptic but on $\mathbb{R}^2$ isn't.

**Sobolev's lemma**

Consider the Hilbert spaces

- $H^0(M) = L^2_{loc}(M)$
- $H^{k+1}(M) = \{f \in H^k(M) : \partial_{x_1}(f), \dots, \partial_{x_n}(f) \in H^k(M)\}$

Then

$$C^\infty(M) = \bigcap_{k \in \mathbb{N}} H^k(M)$$

# Hypoellipticity 2

Let $\phi : \mathbb{N} \to \mathbb{N}$ with $\phi(k) \to +\infty$ as $k \to +\infty$.
If, for any distribution $u$,

$$Du \in H^k(M) \Rightarrow u \in H^{\phi(k)}(M)$$

Then Sobolev's lemma implies $D$ is hypoelliptic.

Observation: If $D$ has order $\ell$ then $\phi(k) \leqslant k + \ell$.

**Definition**

If $\phi(k) = k + \ell$ then $D$ is *elliptic*.

**Corollary**

Every elliptic operator is hypoelliptic.

# Elliptic regularity

**Theorem (Kohn, Nirenberg, Hörmander,...)**

Let $D$ be of order $\ell$ on $M$. TFAE:

- for any $k \in \mathbb{N}$ and any distribution $u$,

$$Du \in H^k(M) \Rightarrow u \in H^{k+\ell}(M)$$

- for any $(x, \xi) \in T^*M \backslash \{0\}$, $\sigma_D(x, \xi) \neq 0$.

If $M$ is compact, the above is equivalent to

- for any $k \in \mathbb{N}$, $D : H^{k+\ell}(M) \to H^k(M)$ is Fredholm.

Example: Kolmogorov's operator on $M = \mathbb{R}^2$

$$D = \partial_x^2 + x^2 \partial_y^2$$

$D$ not elliptic, but *hypoelliptic*.

Proof: Hoermander's "sums of squares theorem". Uses calculus of variations...

# Folland and Stein Sobolev spaces ('70s)

- $\tilde{H}^0(\mathbb{R}^2) = L^2_{loc}(\mathbb{R}^2)$
- $\tilde{H}^{k+1}(\mathbb{R}^2) = \{f \in \tilde{H}^k(\mathbb{R}^2) : \partial_x(f), x\partial_y(f) \in \tilde{H}^k(\mathbb{R}^2)\}$

We have

$$[\partial_x, x\partial_y] = \partial_y$$

So $\tilde{H}^2(\mathbb{R}^2) \subseteq H^1(\mathbb{R}^2)$. By recurrence

$$\tilde{H}^{2k}(\mathbb{R}^2) \subseteq H^k(\mathbb{R}^2)$$

whence

$$\bigcap_k \tilde{H}^k(\mathbb{R}^2) = \bigcap_k H^k(\mathbb{R}^2) = C^\infty(\mathbb{R}^2)$$

**Theorem (Folland and Stein) $D = \partial_x^2 + x^2\partial_y^2$**

For any $u$ and any $k$, $Du \in \tilde{H}^k(\mathbb{R}^2)$ implies $u \in \tilde{H}^{k+2}(\mathbb{R}^2)$.

**Maximal hypoelliticity, heuristically**

D is maximally hypoelliptic, if we can find Sobolev spaces such that D satisfies the best possible regularity condition.

# General Sobolev spaces

Take vector fields $X_1, \dots, X_m$ and define:

- $\tilde{H}^0(\mathbb{R}^2) = L^2_{loc}(\mathbb{R}^2)$
- $\tilde{H}^{k+1}(\mathbb{R}^2) = \{f \in \tilde{H}^k(\mathbb{R}^2) : X_1(f), \dots, X_k(f) \in \tilde{H}^k(\mathbb{R}^2)\}$

For a Sobolev lemma we need Hörmander's condition:

For any $x \in M$,

$$X_i(x), [X_i, X_j](x), [[X_i, X_j], X_k](x), \dots$$

spans $T_x M$.

# Main theorem

**Theorem (A, Omar Mohsen, Robert Yuncken)**

Let $X_1, \dots, X_m$ vector fields on $M$, satisfying Hörmander's condition and $D$ an order $\ell$ differential operator. TFAE:

1. for any $k \in \mathbb{N}$ and distribution $u$, $Du \in \tilde{H}^k(M) \Rightarrow u \in \tilde{H}^{k+\ell}(M)$
2. for any $x \in M$, $\pi \in \mathcal{T}_x^* \subseteq \widehat{G}_x$ (set of unitary irreducible representations), $\tilde{\sigma}(D, x, \pi)$ is invertible.

If $M$ is compact, the above is equivalent to

3. for any $k \in \mathbb{N}$, $D : \tilde{H}^{k+\ell}(M) \to \tilde{H}^k(M)$ is left invertible modulo compact operators.

Until 2022 a conjecture by Helffer and Nourrigat (1979). In 1985 they proved (1) $\Rightarrow$ (2) (full generality) and (2) $\Rightarrow$ (1) when $G_x$ has rank 2. Special cases (sums of squares and their powers) proved by Rothschild and Stein (1976).

We say $D$ is maximally hypoelliptic if it satisfies the above.
Obviously, maximally hypoelliptic implies hypoelliptic.

# Group $G_x$: Algebraic construction

$$\mathcal{F}^1 = C^\infty(M)X_1 + \ldots + C^\infty(M)X_m$$
$$\mathcal{F}^2 = \mathcal{F}^1 + \sum_{i,j} C^\infty(M)[X_i, X_j]$$
$$\mathcal{F}^3 = \mathcal{F}^2 + \sum_{i,j,k} C^\infty(M)[[X_i, X_j], X_k]$$
$$\vdots$$
$$\mathcal{F}^N = \mathfrak{X}(M)$$

Localization: $\frac{\mathcal{F}^i}{I_x \mathcal{F}^i}$ where $I_x = \{f \in C^\infty(M) : f(x) = 0\}$

Get graded nilpotent Lie algebra:

$$\mathfrak{g}_x = \oplus_{i=1}^N \frac{\mathcal{F}^i}{\mathcal{F}^{i-1} + I_x \mathcal{F}^i}$$

$G_x$ is the simply connected nilpotent Lie group which integrates $\mathfrak{g}_x$.

# Group $G_x$: Explanation of algebraic construction

Consider the Lie filtration

$$\mathcal{F}^1 \subseteq \mathcal{F}^2 \subseteq \ldots \subseteq \mathcal{F}^N = \mathcal{X}(M), \qquad [\mathcal{F}^i, \mathcal{F}^j] \subseteq \mathcal{F}^{i+j}$$

Associated grading: $gr(\mathcal{F}) = \mathcal{F}^1 \oplus \frac{\mathcal{F}^2}{\mathcal{F}^1} \oplus \ldots \oplus \frac{\mathcal{F}^N}{\mathcal{F}^{N-1}}$

Localization at $x$:

$$\mathfrak{g}_x = \frac{gr(\mathcal{F})}{I_x \, gr(\mathcal{F})}$$

**Example**

$$\mathcal{F}^1 = \langle \partial_x, x\partial_y \rangle \subseteq \mathcal{F}^2 = \langle \partial_x, \partial_y \rangle$$

$\mathfrak{g}_{(x,y)} = \mathbb{R}^2$ if $x \neq 0$ and $\mathfrak{g}_{(0,y)} = \mathbb{R}^3$.

$G_{(x,y)} = \mathbb{R}^2$ if $x \neq 0$ and $G_{(x,0)} = H^3$ (Heisenberg group).

# Group $G_x$ as holonomy

On $M \times \mathbb{R}$ we have the singular foliation:

$$\mathcal{F} = t\mathcal{F}^1 + t^2\mathcal{F}^2 + \ldots + t^N\mathcal{F}^N$$

Lie algebra:

$$\mathfrak{g}_x = \frac{\mathcal{F}}{I_{(x,0)}\mathcal{F}}$$

Holonomy groupoid:

$$H(\mathcal{F}) = \left(\bigcup_{x\in M} G_x \times \{0\}\right) \coprod (M \times M \times \mathbb{R}^*) \rightrightarrows M \times \mathbb{R}$$

**Theorem A, Skandalis**

$H(\mathcal{F})$ has a $C^*$-algebra $C^*(\mathcal{F})$. At $t = 0$ it is the field of $C^*$-algebras $C^*(G_x)$. On $\mathbb{R}^*$ it is $K(L^2(M))$.

**Observation**

$C^*(\mathcal{F})$ is <u>not</u> a continuous field of $C^*$-algebras over $\mathbb{R}$.

# Order of a differential operator

Any differential operator can be written $P(X_1, \dots, X_m)$ where $P$ is a noncommutative polynomial. That's thanks to Hörmander's condition and the fact that

$$[X_i, X_j] = X_i X_j - X_j X_i$$

is a polynomial in $X_i, X_j$.

**Definition**

The Hörmander order of $D$ is the minimum degree of $P$ such that $D = P(X_1, \dots, X_m)$.

Remark: Hörmander order $>$ classical order

**Example**

Take $D = \partial_x^2 + x^2 \partial_y^2$. Then $\partial_x$ and $x\partial_y$ have Hörmander order 1 but $\partial_y$ has Hörmander order 2.

# Order of a differential operator

Another example: $X = x^2\partial_x$, $Y = x\partial_x$, $Z = \partial_x$. Filtration

$$\mathcal{F}^\bullet : \langle X \rangle \subseteq \langle Y \rangle \subseteq \langle Z \rangle$$

Put $D = XZ - Y^2$. Order:

- In $\mathcal{F}^\bullet$, $\operatorname{ord}(X) = 1, \operatorname{ord}(Z) = 3, \operatorname{ord}(Y) = 2$, so $\operatorname{ord}(XZ - Y^2) = 3$.
- Calculation: $D = -Y$. So $D$ has Hörmander order 2.
- The group at zero is $\mathbb{R}^3$.

**Proposition**

If $D$ has Hörmander order $\ell$, then for any $k$,

$$D : \tilde{H}^{k+\ell}(M) \to \tilde{H}^k(M)$$

is bounded.

# Principal symbol

Let $\pi : G_x \to B(H)$ irreducible unitary representation. Derivative

$$d\pi : \mathfrak{g}_x \to (C^\infty(\pi))$$

where $C^\infty(\pi)$ are the smooth vectors. (In $L^2$, Schwarz functions.)

**Definition**

The symbol of $D = P(X_1, \ldots, X_m)$ is

$$\tilde{\sigma}(x, D, \pi) = P_{\mathtt{highest-Hoerm-order}}(d\pi(X_1), \ldots, d\pi(X_m))$$

Remark: If $\mathfrak{g}_x = T_x M$ then $\pi = e^{i\langle \cdot, \xi \rangle}$, so $d\pi(X) = i\xi(X)$.

**Theorem A-Mohsen-Yuncken**

There is $\mathcal{T}_x^* \subseteq \hat{G}_x$ such that $\tilde{\sigma}$ is well defined for every $\pi \in \mathcal{T}_x^*$.

We call $\mathcal{T}_x^*$ the Helffer-Nourrigat tangent cone.

**Proposition (Helffer-Nourrigat, A-Mohsen-Yuncken)**

$\mathcal{T}_x^*$ is closed under coadjoint orbits.

# First explanation for $\mathcal{T}^*_x$: Differential operators of the filtration

**Recall PBW isomorphism for Lie groupoid $\mathcal{G} \rightrightarrows M$**

The following maps are isomorphisms:

- $U(A\mathcal{G}) \to \Gamma(\mathcal{S}ymm(A\mathcal{G})), D \mapsto \sigma(\tau(D))$
- $\tau : U(A\mathcal{G}) \to \mathrm{Diff}(\mathcal{G})$

Given $\mathcal{F}^\bullet$, consider smallest filtration:

$$0 \subseteq C^\infty(M) \subseteq \mathrm{Diff}_{\mathcal{F}^1}(M) \subseteq \ldots \subseteq \mathrm{Diff}_{\mathcal{F}^{N-1}}(M) \subseteq \mathrm{Diff}(M)$$

such that $\mathcal{F}^i \subseteq \mathrm{Diff}_{\mathcal{F}^i}(M)$ and

- $\mathrm{Diff}_{\mathcal{F}^i}(M)\mathrm{Diff}_{\mathcal{F}^j}(M) \subseteq \mathrm{Diff}_{\mathcal{F}^{i+j}}(M)$

Formal symbols: $\Sigma^i = \frac{\mathrm{Diff}_{\mathcal{F}^i}(M)}{\mathrm{Diff}_{\mathcal{F}^{i-1}}(M)}$. ($C^\infty(M)$-module.)

Symbol map for every $p \in M$:

$$\mathrm{Diff}_{\mathcal{F}^i}(M) \xrightarrow{\sigma^i_p} \frac{\mathrm{Diff}_{\mathcal{F}^i}(M)}{\mathrm{Diff}_{\mathcal{D}^{i-1}}(M) + I_p\mathrm{Diff}_{\mathcal{F}^i}(M)}$$

# First explanation for $\mathcal{T}_x^*$: Differential operators of the filtration

Example: $M = \mathbb{R}$, $\mathcal{F}^1 = \langle x^2 \partial_x \rangle$, $\mathcal{F}^2 = \langle \partial_x \rangle$. Take $P = x\partial_x$.

$$\sigma_p^2 : \mathrm{Diff}_{\mathcal{F}^2}(\mathbb{R}) \to \frac{\mathrm{Diff}_{\mathcal{F}_2}(\mathbb{R})}{\mathrm{Diff}_{\mathcal{F}^1}(\mathbb{R}) + I_p \mathrm{Diff}_{\mathcal{F}^2}(\mathbb{R})}$$

- $P$ lives in $I_0 \mathrm{Diff}_{\mathcal{F}^2}(\mathbb{R})$, so $\sigma_0^2(P) = 0$.
- About $p \neq 0$ we can divide by $x$ and $x^2$, so rhs vanishes.

<u>Conclusion:</u> $\sigma_p^2(P) = 0$ for every $p$, but $P \notin \mathrm{Diff}_{\mathcal{F}^1}(M)$.

Note:

$$G_p = \begin{cases} \mathbb{R} \oplus 0, p \neq 0 \\ \mathbb{R} \oplus \mathbb{R}, p = 0 \end{cases}$$

# The issue with the order of P in the filtration

Surjective map: $\mathcal{U}(\mathfrak{gr}(\mathcal{F}^\bullet)) \to \oplus_i \Sigma^i$. Localization at $p$:

$$\mathcal{U}(\mathfrak{gr}(\mathcal{F})_p) \to \oplus_i \frac{\mathrm{Diff}^i_{\mathcal{F}}(M)}{\mathrm{Diff}^{i-1}_{\mathcal{F}}(M) + I_p \mathrm{Diff}^i_{\mathcal{F}}(M)} \tag{1}$$

Does principal symbol at $p$ live in $\mathcal{U}(\mathfrak{gr}(\mathcal{F})_p)$? Namely, is the natural surjection above injective?

**No!**

Reason: Singularities! Injective when $\mathcal{F}^\bullet$ constant rank.

**Helffer-Nourrigat tangent cone $\mathcal{T}^*_p$ (algebraic):**

Representations of $G_p$ which vanish on ker of (1)

# The topological viewpoint of the Helffer-Nourrigat tangent cone

Singular foliation $(M, \mathcal{F}) \rightsquigarrow$ locally compact space $\mathcal{F}^* = \coprod_{p \in M} \mathcal{F}_p^*$.

Adiabatic foliation $(M \times \mathbb{R}, \mathcal{F})$:

$$\mathcal{F}^* = (T^*M \times \mathbb{R}^*) \sqcup \left( \left( \bigcup_{p \in M} \mathfrak{g}_p^* \right) \times \{0\} \right)$$

Singularities $\rightsquigarrow T^*M \times \mathbb{R}^*$ not dense in $\left( \left( \bigcup_{p \in M} \mathfrak{g}_p^* \right) \times \{0\} \right)$.

**Helffer-Nourrigat tangent cone $\mathcal{T}_p^*$:**

1. Definition: $\mathcal{T}_p^* = \overline{\pi^{-1}(M \times \mathbb{R}^*)} \cap \mathfrak{g}_p^* \subseteq \mathfrak{g}_p^*$
2. $Ad^*$-invariant. Orbit method implies $\mathcal{T}_p^*$ closed subset of unitary dual $\widehat{G}_p$.

# The topological viewpoint of the Helffer-Nourrigat tangent cone: Calculation method

Orbit method says

Irreducible unitary representations of $G_p$ correspond bijectively to $Ad^*(G_p)$-orbits in $\mathfrak{g}_p^*$

An element $\pi$ lives in $\mathcal{T}_p^*$ if there exists $t_n \in \mathbb{R}_+^*$ and $x_n \in M$ and $\xi_n \in T_{x_n}^* M$ such that $x_n \to p, \quad t_n \to 0$ such that:

Recall

$$\mathfrak{g}_x = \oplus_{i=1}^N \frac{\mathcal{F}^i}{\mathcal{F}^{i-1} + I_x \mathcal{F}^i}$$

For $\pi = (\pi_1, \ldots, \pi_N)$ we have:

- for every $i$, $\pi_1(X_i) = \lim_n t_n \xi_n(X_i(x_n))$
- for every $i, j$ $\pi_i([X_i, X_j]) = \lim_n t_n^2 \xi_n([X_i, X_j](x_n))$
- for every $i, j, k$...

# The topological viewpoint of the Helffer-Nourrigat tangent cone: Examples

- (Mohsen) $\mathcal{T}_p^*$ is the Helffer-Nourrigat characteristic set.
- If $p$ regular point, $\mathcal{T}_p^* = \mathfrak{g}_p^*$.
- $M = \mathbb{R}$ and

$$\mathcal{F}^1 = \langle x^2 \partial_x \rangle \subseteq \mathcal{F}^2 = \langle x \partial_x \rangle \subseteq \mathcal{F}^3 = \mathcal{X}(\mathbb{R})$$

Then

$$G_p = \begin{cases} \mathbb{R} \oplus 0 \oplus 0, p \neq 0 \\ \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}, p = 0 \end{cases}$$

We find the cone

$$\mathcal{T}_{p=0}^* = \{(\xi_1, \xi_2, \xi_3) : \xi_1 \xi_3 = \xi_2^2\}$$

## Example

Start from $\partial_x, x^k\partial_y$ in $\mathbb{R}^2$ and descend to $S^1 \times S^1$: $\partial_x, (\sin(x))^k\partial_y$. Consider the operator

$$D = (\partial_x^2 + ((\sin(x))^k\partial_y)^2)^{\frac{k+1}{2}} + ig(x,y)\partial_y$$

where $g : S^1 \times S^1 \to \mathbb{R}$ smooth and non-vanishing. Hörmander order is $k+1$. Symbol:

- If $\sin(x) \neq 0$, $G_{(x,y)} = T_{(x,y)}M$ and our symbol is the classical symbol.
- If $\sin(x) = 0$ the Lie algebra of $G_x$ is generated by

$$\partial_x, x^k\partial_y, x^{k-1}\partial_y, \ldots, \partial_y$$

with Lie bracket

$$[\partial_x, x^j\partial_y] = jx^{j-1}\partial_y$$

The two representations $\pi_\pm : G_x \to B(L^2\mathbb{R})$ are:

$$\partial_x \mapsto (f \mapsto \partial_t f)$$
$$x^j\partial_y \mapsto (f \mapsto \pm it^j f)$$

# Example

Evaluating $D$ at $\pi_{\pm}$ gives

$$(\partial_t^2 - t^{2k})^{\frac{k+1}{2}} \mp g(x,y)\mathrm{id}_{L^2\mathbb{R}}$$

Schrödinger type operator, compact resolvent, diagonalisable, so:

$D$ and $D^*$ maximally hypoelliptic iff

$$g(0,y), g(\pi,y) \notin spec(\partial_t^2 - t^{2k})^{\frac{k+1}{2}}$$

(If $k = 1$ this is a harmonic oscillator.)

**Thank you!**