Integration of Singular Foliations and Usage

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Penn State, November 2014

Summary

Integration 1: In the spirit of Sophus Lie

1. $\mathfrak{g} = \mathbb{R}$, $G = (\mathbb{R}, +)$ or $G = S^1$:

$$0 \longrightarrow (\mathbb{Z}, +) \to (\mathbb{R}, +) \xrightarrow{\exp} S^1 \to 0$$

2. M connected, $A = TM$, $\mathscr{G} = \Pi(M)$ or $\mathscr{G} = M \times M$:

$$M \times \pi_1(M) \longrightarrow \Pi(M) \xrightarrow{(s,t)} M \times M$$

3. $A = F \leqslant TM$ involutive distribution with constant rank.
$\mathscr{G} = \mathrm{Mon}(F) = \bigcup_L \Pi(L)$ or $\mathscr{G} = \mathrm{Hol}(F)$:

$$\text{holonomy map} : \mathrm{Mon}(F) \to \mathrm{Hol}(F)$$

4. $A \to M$ Lie algebroid. Crainic and Fernandes constructed topological groupoid $W(A)$ with connected and simply connected s-fibers. Smooth iff certain obstruction vanishes. When smooth, $\mathscr{A}(W(A)) = A$.

Integration 1: In the spirit of Sophus Lie

Integrability in a smooth sense

Given Lie algebroid $A \to M$, find:

- topological groupoid $\mathscr{G}(A)$ over M
- a kind of Lie functor such that $\mathscr{A}(\mathscr{G}(A)) = A$

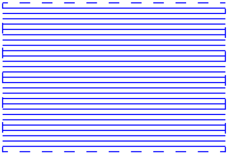
Serre-Swan thm: $A \equiv \Gamma A$ and ΓA projective $C^{\infty}(M)$-module.

So integration also means: "Recover ΓA from $\mathscr{G}(A)$"

Case study: Regular foliation

Viewpoint 1:

Partition to connected submanifolds. Local picture:



In other words: There is an open cover of M by foliation charts of the form $\Omega = U \times T$, where $U \subseteq \mathbb{R}^p$ and $T \subseteq \mathbb{R}^q$.

T is the transverse direction and U is the longitudinal or leafwise direction.

The change of charts is of the form $f(u,t) = (g(u,t), h(t))$.

Viewpoint 2:

Frobenius theorem

Equivalently consider the unique $C_c^\infty(M)$-module $\mathcal{F}$ of vector fields tangent to leaves.

Fact: $\mathcal{F} = C_c^\infty(M, F)$ and $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$.

Examples

M: compact manifold.

1. Orbits of (some) Lie group actions on M. Vector fields: image of infinitesimal action $\mathfrak{g} \to \mathfrak{X}(M)$.
2. Poisson geometry: Symplectic foliation on M by Hamiltonian vector fields (almost never regular!). Determines the Poisson structure...
3. X nowhere vanishing vector field of $M \rightsquigarrow$ action of $\mathbb{R}$ on M.
4. Irrational rotation on torus T^2: "Kronecker" flow of $X = \frac{d}{dx} + \theta \frac{d}{dy}$. $\mathbb{R}$ injected as a dense leaf.
5. "Horocyclic" foliation:
   - Let Γ cocompact subgroup of $SL(2, \mathbb{R})$. Put $M = SL(2, \mathbb{R})/\Gamma$.
   - $\mathbb{R}$ is embedded in $SL(2, \mathbb{R})$ by $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, $t \in \mathbb{R}$.
   - Therefore $\mathbb{R}$ acts on M. Action is ergodic, $\exists$ dense leaves.

Holonomy

We wish to put a smooth structure on the equivalence relation

$$\{(x, y) \in M^2 : L_x = L_y\}$$

What is the dimension of this manifold?

$p + q$ degrees of freedom for x; then p degrees of freedom for y.

- Let $x \in W = U \times T$ and $x' \in W' = U' \times T'$
- Nhd of (x, x') should be $U \times U' \times T$
- Need identification of T with T'

Definition

A holonomy of $(M, \mathcal{F})$ is a diffeomorphism

$$h : T \to T'$$

such that $t, h(t)$ live in same leaf (for all $t \in T$).

Fact: Holonomies form a pseudogroup.

Examples of holonomies

- Small Holonomies: Charts $W = U \times T$ and $W' = U' \times T'$. In $W \cap W'$ have
$$(u', t') = (g(u,t), h(t))$$
Map $h = h_{W',W}$ is a holonomy.

- Path Holonomies: Let $\gamma : [0,1] \to M$ smooth path in L.
Cover γ with $W_i = U_i \times T_i (1 \leqslant i \leqslant n)$. Take
$$h(\gamma) = h_{W_n, W_{n-1}} \circ \ldots \circ h_{W_2, W_1}$$

Definition

The holonomy of the path γ is the germ of $h(\gamma)$.

Fact: Path holonomy depends only on the homotopy class of the path!

The holonomy groupoid

Definition

$H(F) = \{(x, y, h(\gamma))\}$, where γ: path in L joining x to y

- Manifold structure. If $W = U \times T$ and $W' = U' \times T'$ are charts and $h : T \to T'$ path-holonomy, get chart

$$\Omega_h = U' \times U \times T$$

- Groupoid structure. $t(x, y, h) = x$, $s(x, y, h) = y$ and $(x, y, h)(y, z, k) = (x, z, h \circ k)$.

$H(F)$ is a Lie groupoid. Its Lie algebroid is F. Its orbits are the leaves.

$H(F)$ is the smallest possible smooth groupoid over F.

Integration 2: By operator algebras

Lie algebra $\mathcal{F} = \Gamma(M, F)$ acts by unbounded multipliers on $C_c^\infty(H(F))$. Generates algebra of differential operators P.

Fourier transform: P acting on $f \in C_c^\infty(H(F)))$ is:

$$(Pf)(x, y) = \int \exp(i\langle \phi(x, z), \xi \rangle) \alpha(x, \xi) \chi(x, z) f(z, y) d\xi dz$$

Where

- ϕ the phase: through a local diffeomorphism defined on an open subset $\widetilde{\Omega} \simeq U \times U \times T \subset G$ (where $\Omega = U \times T$ is a foliation chart). $\phi(x, z) = x - z \in F_x$;
- χ the cut-off function: χ smooth, $\chi(x, x) = 1$ on (a compact subset of) Ω, $\chi(x, z) = 0$ for $(x, z) \notin \widetilde{\Omega}$;
- $\alpha \in C^\infty(F^*)$ a polynomial on ξ called the symbol of P.

Generalized to any Lie groupoid (Nistor, Weisntein, Xu).

The convolution algebra of $\mathcal{F}$ (Connes, Renault)

For $f, g \in C_c^\infty(H(F))$:

- we put $f^*(x) = \overline{f(x^{-1})}$
- we want to form $f * g$ by a formula

$$f * g(x) = \int_{yz=x} f(y)g(z)$$

In other words, we want to have an integration along the fibers of the composition $H(F) \times_{s,t} H(F) \to H(F)$.
Use either Haar systems or half densities.

Proposition

The above involution and product make $C_c^\infty(H(F))$ a $*$-algebra.

The C^*-algebra

J. Renault proves:

- The (continuous) $*$-representations of the $*$-algebra $C_c^\infty(H(F))$ are in one to one correspondence with unitary representations of the groupoid.
- An L^1-estimate shows that, for $f \in C_c^\infty(H(F))$, the supremum $f \mapsto \sup_\pi ||\pi(f)||$ over all such representations π is bounded.

$$||f||_1 = \sup_u max\{\int_{H(F)^u} |f(x)| d\lambda^u(x), \int_{H(F)_u} |f(x)| d\lambda_u(x)\}$$

Definition

- The full C^*-algebra $C^*(\mathcal{F})$ of $\mathcal{F}$ is the completion of $C_c^\infty(H(F))$ w.r.t the norm $f \mapsto \sup_\pi ||\pi(f)||$.
- Left-regular representation ρ_u on $L^2(H(F)_u)$. The reduced C^*-algebra $C_r^*(\mathcal{F})$ of $\mathcal{F}$ is the completion w.r.t $f \mapsto \sup_u ||\rho_u(f)||$.

Conclusion: $\Gamma(M, F)$ can be recovered from unbounded multipliers of

Integration by operator algebras

Conclusion: $\Gamma(M, F)$ can be recovered from unbounded multipliers of $C^*(M, F)$.

Recall: Integrability in a smooth sense

Given Lie algebroid $A \to M$, find:

- topological groupoid $\mathscr{G}(A)$ over M;
- a kind of Lie functor such that $\mathscr{A}(\mathscr{G}(A)) = A$.

Integrability by operator algebras

Given Lie algebroid $A \to M$, find:

- topological groupoid $\mathscr{G}(A)$ over M;
- a C^*-functor such that $\Gamma(M, A)$ sits in unbounded multipliers of $C^*(\mathscr{G}(A))$.

Debord's setting

Almost regular foliations

Submodule $\mathcal{A}$ of $\Gamma(M, TM)$ such that:

- finitely generated projective
- stable under brackets

Serre-Swan theorem

Bundles = finitely generated projective $C^\infty(M)$-modules, so:

- $\mathcal{A}$ is a the module of sections of a Lie algebroid A;
- Anchor map $A_x \to T_x M$ injective on a dense open subset;
- Image of anchor map is F_x;
- Dimension lower semi-continuous.

Example: $\mathcal{A} = \langle X \rangle$, where interior of $\{X = 0\}$ is empty.

Examples of almost regular foliations

1. $\mathcal{A} = \langle X \rangle$, where interior of $\{X = 0\}$ is empty. *e.g.* $X = x\frac{\partial}{\partial x}$.

2. Poisson bivector $\Pi : T^*\mathbb{R}^2 \to T\mathbb{R}^2$ given by $x dx \wedge dy$.

$$\mathcal{A} = \Omega^1(\mathbb{R}^2) = \mathrm{Im}(\Pi)$$

3. Log-symplectic manifolds...

Integrability:

- Crainic-Fernandes: $\mathcal{W}(A)$ Lie groupoid for every almost injective Lie algebroid $A \to TM$. So $\mathscr{A}(\mathcal{W}(A)) = A$.
- Debord: Constructed smallest Lie groupoid $H(A)$ such that $\mathscr{A}(H(A)) = A$. Quotient of $\mathcal{W}(A)$.

Integration in Debord's setting

Main object of this integration: quasi-graphoids:
Pieces (V, t, s) of groupoids with s, t submersions and $(t, s) : V \to M \times M$ injective in dense set (so $t(s^{-1}(x))$ is a piece of a leaf). Three steps:

1. Local integration
2. Composition: $V_1 \times_{s,t} V_2$ (stable dimension).
3. Natural equivalence relation $\rightsquigarrow$ (non-Hausdorff) Lie groupoid $H(A)$.

Example: $S^1 \circlearrowright \mathbb{R}^2$ by rotations: $H(A) = S^1 \ltimes \mathbb{R}^2$.

Consequences: C^*-algebra, psdo calculus, elliptic operators, etc.

General Stefan, Sussmann foliations

Definition (Stefan, Sussmann, A., Skandalis)

A singular foliation is a submodule $\mathcal{F}$ of $C^\infty(M, TM)$ which is:

- locally finitely generated
- stable under brackets

No longer projective!

Two notions of fibers:

- F_x tangent to leaf through x: Image of $\mathcal{F}$ on $T_x M$. Lower semi-continuous. Continuous $\leftrightarrow$ regular.
- $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$. Upper semi-continuous. Continuous $\leftrightarrow$ almost regular.

$$0 \to \mathfrak{g}_x \to \mathcal{F}_x \xrightarrow{ev_x} F_x \to 0$$

$\mathfrak{g}_x = 0$ iff L_x has maximal dimension (regular leaf).

Examples

Actually, different foliations may yield same partition to leaves...

1. $\mathbb{R}$ foliated by 3 leaves: $(-\infty, 0), \{0\}, (0, +\infty)$.
   $\mathcal{F}$ generated by $x^n \frac{\partial}{\partial x}$. Different foliation for every n.

2. $\mathbb{R}^2$ foliated by 2 leaves: $\{0\}$ and $\mathbb{R}^2 \backslash \{0\}$.
   No obvious best choice. $\mathcal{F}$ given by the action of a Lie group
   $$GL(2, \mathbb{R}), SL(2, \mathbb{R}), \mathbb{C}^*$$

3. $\mathbb{R}^3$ foliated by spheres with center 0. Doesn't admit almost injective algebroid: tangent to sphere non-trivial, no extension to 0.

4. $\langle X \rangle$ where interior of $\{X = 0\}$ is not empty.

e.g. $GL(2, \mathbb{R}) \circlearrowright \mathbb{R}^2$: fibers $\mathcal{F}_0 = \mathfrak{gl}(2, \mathbb{R})$ and $F_0 = \{0\}$.

Holonomy groupoid (Extremely singular!)

As dimension of $\mathcal{F}_x$ varies, no hope for quasi-graphoids. We give up dimension requirements.

Our main object: bi-submersions: (U, t, s) with $s, t : U \to M$ submersions and $t : s^{-1}(x) \to L_x$ submersion. $t(s^{-1}(x))$ piece of a leaf because

$$s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F}) = C^\infty(U; \ker ds) + C^\infty(U; \ker dt)$$

Same steps as in almost regular case

1. Local integration of vector fields $X_1, \ldots, X_n$ that form a basis of $\mathcal{F}_x$:

$$t(x, \vec{\lambda}) = \exp_x \left(\sum_{i=1}^{n} \lambda_i X_i \right), \quad (x, \vec{\lambda}) \in U \subset M \times \mathbb{R}^n$$

2. Composition: $U_1 \times_{t,s} U_2$ (dimension may rise)
3. Natural equivalence relation mixes dimensions $\Rightarrow$ Very singular groupoid...

Holonomy groupoid: Examples

1. $\mathcal{F} =< X >$ s.t. X has non-periodic integral curves around $\partial\{X = 0\}$:

$$H(\mathcal{F}) = H(X)|_{\{X \neq 0\}} \cup \mathrm{Int}\{X = 0\} \cup (\mathbb{R} \times \partial\{X = 0\})$$

2. $SL(2, \mathbb{R}) \circlearrowright \mathbb{R}^2$:

$$H(\mathcal{F}) = (\mathbb{R}^2 \backslash \{0\})^2 \cup SL(2, \mathbb{R}) \times \{0\}$$

topology: Let $x \in \mathbb{R}^2 \backslash \{0\}$. Then $(\frac{x}{n}, \frac{x}{n}) \in H(\mathcal{F})$ converges to every g in stabilizer group of x... namely to every point of $\mathbb{R}$!

3. $SO(3) \circlearrowright \mathbb{R}^3$: $H(\mathcal{F})$ quotient of $SO(3) \ltimes \mathbb{R}^3$.

Integration 1 for singular foliations

Theorem (A - Zambon)

By construction $\mathrm{H}(\mathcal{F})$ is a diffeological groupoid! (Souriau 1970's).

- There is a Lie functor for diffeological groupoids
- $\mathscr{A}(\mathrm{H}(\mathcal{F})) = \mathcal{F}$

Recall: Souriau used similar Lie functor to prove $\mathscr{A}(\mathrm{Diff}(\mathrm{M})) = \mathfrak{X}(\mathrm{M})$.

Holonomy map

$S^1 \circlearrowright \mathbb{R}^2$ Rotations: $\mathcal{F} = span_{C^\infty(\mathbb{R}^2)} < x\partial_y - y\partial_x >$. Projective!

Regular leaf $L = S^1$, transversal S. Get holonomy map

$$h : \pi_1(L) \to GermDiffeo(S)$$

Singular leaf $L = \{0\}$

- Take γ: constant path at origin.
- Transversal S_0: open neighborhood of origin in $\mathbb{R}^2$.

Realize γ either by integrating the zero vector field or $x\partial_y - y\partial_x$ at the origin. Get completely different diffeomorphisms of S_0!

Conclusion: Holonomy map not well defined on singularity!

Singular holonomy map

Let $(M, \mathcal{F})$ a singular foliation, L a leaf, $x, y \in L$ and S_x, S_y slices of L at x, y respectively.

Theorem (A-Zambon)

There is an injective map

$$\Phi_x^y : H_x^y \to \frac{\mathrm{GermAut}_{\mathcal{F}}(S_x, S_y)}{\exp(I_x\mathcal{F}) \mid_{S_x}}, h \mapsto \langle\tau\rangle$$

where τ is defined as

- pick any bi-submersion (U, t, s) and $u \in U$ with $[u] = h$
- pick any section $b : S_x \to U$ of s through u such that $(t \circ b)S_x \subseteq S_y$

and define $\tau = t \circ b : S_x \to S_y$.

Here bi-submersions are crucial!

Holonomy map and the Bott connection

If $\mathcal{F}$ is regular then $exp(I_x\mathcal{F}) \mid_{S_x} = \{Id\}$, so we recover the usual holonomy map.

Let L be a leaf. Recall $H(\mathcal{F})_L$ is a Lie groupoid (Debord).

1. Derivative of τ gives representation of $H(\mathcal{F})_L$:

$$\Psi_L : H(\mathcal{F})_L \to Iso(NL, NL)$$

2. Differentiating Ψ_L gives

$$\nabla^{L,\perp} : A_L \to Der(NL)$$

It's the Bott conection...

Linearization I

Vector field on M tangent to $L \rightsquigarrow$
Vector field Y_{lin} on NL, defined as follows:

$$Y_{lin} \text{ acts on the fibrewise constant functions as } Y\,|_L$$
$$Y_{lin} \text{ acts on } C^{\infty}_{lin}(NL) \equiv I_L/I_L^2 \text{ as } Y_{lin}[f] = [Y(f)].$$

Definition

The linearization of $\mathcal{F}$ at L is the foliation $\mathcal{F}_{lin}$ on NL generated by

$$\{Y_{lin} : Y \in \mathcal{F}\}$$

Lemma

Let L be a leaf. Then $\mathcal{F}_{lin}$ is the foliation induced by the Lie groupoid action Ψ_L of $H(\mathcal{F})_L$ on NL.

Linearization II

Definition

We say $\mathcal{F}$ is linearizable at L if there is a diffeomorphism mapping $\mathcal{F}$ to $\mathcal{F}_{lin}$.

For $\mathcal{F} = \langle X \rangle$ with X vanishing at $L = \{x\}$ linearizability means:

There is a diffeomorphism taking X to fX_{lin} for a non-vanishing function f.

This is a weaker condition than the linearizability of the vector field X!

Normal form around a (singular) leaf

Theorem (A-Zambon)

Let L_x leaf at $x \in M$. The following are equivalent:

1. $\mathcal{F}$ is linearizable about L and $H(\mathcal{F})_x^x$ compact.
2. There exists a tubular neighborhood U of L and a (Hausdorff) Lie groupoid $\mathscr{G} \to U$, proper at x, inducing the foliation $\mathcal{F}|_U$.

In that case:

- $\mathscr{G}$ can be chosen to be the transformation groupoid

$$H(\mathcal{F})|_L \ltimes_{\Psi_L} NL$$

- $(U, \mathcal{F}|_U)$ admits the structure of a singular Riemannian foliation.

C^*-algebra(s) (A - Skandalis)

- Building blocks for convolution algebra: $C_c^\infty(\mathcal{U})$.
- Form $\oplus_{i \in I} C_c^\infty(\mathcal{U}_i)$ where $(\mathcal{U}_i)$ family of bi-submersions (atlas)
- Natural quotient $\mathcal{D} = \oplus_{i \in I} C_c^\infty(\mathcal{U}_i)/\sim$
- Need densities!
- Completion(s) of $\mathcal{D}$ easy: Smooth s-fibers! (See C. Debord.)

Desintegration Theorem (A - Skandalis)

$*$-representations of $C^*(\mathcal{F})$ correspond to unitary representations of $H(\mathcal{F})$.

Cotangent "bundle"

$\mathcal{F}^* = \cup_{x \in M} \mathcal{F}^*_x$. Not a bundle because dimension varies.
Nice locally compact space though.

Example: $\mathrm{SO}(3) \circlearrowright \mathbb{R}^3$

$$\mathcal{F}^* = \cup_{\xi \in \mathbb{R}^3} \{x \in \mathbb{R}^3 : \langle x, \xi \rangle = 0\}$$

Pseudodifferential calculus: Idea

Submfd $V \leqslant \mathcal{U}$, vector field X, distribution

$$q_X : f \mapsto \int_V Xf$$

If X tangent to V,

$$\int_V Xf = -\int_V div(X) \cdot f$$

So up to zero order, q_X depends on image of X in NV.

Idea:

- Distributions on $\mathcal{U}$ smooth outside V, pseudodiff. singularity on V.
- Principal symbol on $\mathcal{F}^*$.

Pseudodifferential calculus: Formulas

$(\mathcal{U}, t, s)$ bi-submersion, $V \subset \mathcal{U}$ identity bisection, $N \to V$ normal bundle.

Take symbol $\alpha \in S^m_{cl,c}(V, N^*; \Omega^1 N^*)$.

Define $C^\infty(V)$-linear $P_\alpha : C^\infty_c(N; \Omega^1 N) \to C^\infty(V)$:

$$< P_\alpha, f > (x) = (2\pi)^{-k} \int_{N^*_x \times N_x} \alpha(x, \xi) e^{-i<u,\xi>} f(u)$$

Integrating on V gives distribution. P_α pseudodifferential kernel.

Generalized functions on $\mathcal{U}$ with pseudodifferential singularities on V

$$P = h + \chi \cdot P_\alpha \circ \phi$$

- $h \in C^\infty(\mathcal{U})$, $\phi : \mathcal{U}_1 \to N$ tubular neighborhood;
- χ smooth "bump function" s.t. $\chi_{|V} = 1, \quad \chi_{|\mathcal{U}_1^c} = 0$

Example: $q_X : f \mapsto \int_V Xf$.

Pseudodifferential calculus: Results

1. Elliptic operators: Can construct parametrix.

2. $$0 \to C^*(\mathcal{F}) \to \overline{\Psi^0(\mathcal{F})} \xrightarrow{\sigma} C_0(S^*\mathcal{F}) \to 0$$
   whence analytic index map

3. Difficulty: Operators of order $\leqslant -n$ for all n may not be smooth: *e.g.* $SO(3) \circlearrowright \mathbb{R}^3$ and 0-order symbol
   $$\alpha(x,\xi) = \begin{cases} e^{-\frac{1}{<x,\xi>^2}} & \text{out of} \quad \mathcal{F}^* \\ 0 & \text{in} \quad \mathcal{F}^* \end{cases}$$
   $\nexists$ order -1 operator with symbol α around $\mathcal{F}^*$!

The Laplacian

Theorem 1 (A-Skandalis)

Let M be a smooth compact manifold. Let $X_1, \ldots, X_N \in C^\infty(M; TM)$ be smooth vector fields such that $[X_i, X_j] = \sum_{k=1}^N f_{ij}^k X_k$.

Then $\Delta = \sum X_i^* X_i$ is essentially self-adjoint (both in $L^2(M)$ and $L^2(L)$).

Proof

This operator is indeed a regular unbounded multiplier of our C^*-algebra.

In Baaj-Woronowicz terminology: regular multiplers means:

- Δ is densely defined and closed.

  (graph Δ closed (right) submodule of $C^*(M, F) \times C^*(M, F)$).
- Δ has a densely defined (closed) adjoint Δ^*;
- $\mathrm{graph}\Delta \oplus (\mathrm{graph}\Delta)^\perp = C^*(M, F) \times C^*(M, F)$
  $((y, x) \in \mathrm{graph}\Delta^* \Leftrightarrow (x, -y) \in (\mathrm{graph}\Delta)^\perp)$

Motivation: Laplacian of Kronecker foliation

Kronecker foliation on $M = T^2$: $\mathcal{F} = \langle X = \frac{d}{dx} + \theta \frac{d}{dy} \rangle$. $L = \mathbb{R}$

Two Laplacians:

- $\Delta_L = -\frac{d^2}{dx^2}$ acting on $L^2(\mathbb{R})$
- $\Delta_M = -X^2$ acting on $L^2(M)$

By Fourier:

- $\Delta_L \rightsquigarrow$ mult. by ξ^2 on $L^2(\mathbb{R})$. Spectrum: $[0, +\infty)$.
- $\Delta_M \rightsquigarrow$ mult. by $(n + \theta k)^2$ on $L^2(\mathbb{Z}^2)$. Spectrum dense in $[0, +\infty)$.

Laplacians revisited

More generally M compact, (M, F) regular foliation.

Recall

- Lie algebra $\mathcal{F} = C^\infty(M, F)$ acts on $C^\infty(H(F))$ by unbounded multipliers.
- Laplacian $\Delta = \sum X_i^2$ as an unbounded multiplier of $C^*(M, \mathcal{F})$.

Fact: $L^2(L)$ representation of $C^*(M, \mathcal{F})$.

Proposition (Baaj, Woronowicz)

Every representation extends to regular multipliers.

Recover Laplacian Δ_L.

Statement of 2+1 theorems

Theorem 1 (Connes, Kordyukov)

Δ_M and Δ_L are essentially self-adjoint.

Also true (and more interesting)

- for $\Delta_M + f, \Delta_L + f$ where f is a smooth function on M. (Schroedinger operators, conformal geometry, etc.)
- more generally for every leafwise elliptic (pseudo-)differential operator.

Not trivial because:

- Δ_M not elliptic (as an operator on M).
- L not compact.

Theorem 2 (Kordyukov)

If L is dense + amenability, Δ_M and Δ_L have the same spectrum.

Connes

In many cases, one can predict the possible gaps in the spectrum.

Generalization of Connes' and Kordyukov's theorem

Theorem (A-Skandalis)

Assume that the (dense open) set $\Omega \subset M$ where leaves have maximal dimension is Lebesgue measure 1. Assume the restriction of all leaves to Ω are dense in Ω. Assume that the holonomy groupoid of the restriction of $\mathcal{F}$ to Ω is Hausdorff and amenable. Then Δ_M and Δ_L have the same spectrum.

Proof

The C^*-algebra $C^*(\Omega, \mathcal{F})$ is simple (Fack-Skandalis) and sits as a two-sided ideal in $C^*(M, \mathcal{F})$. $L^2(L)$ and $L^2(M)$ are faithful representations of $C^*(\Omega, \mathcal{F})$ $\Rightarrow$ weakly equivalent. The natural representations of $C^*(M, \mathcal{F})$ to $L^2(L)$ and $L^2(M)$ are extensions to multipliers of faithful representations of $C^*(\Omega, \mathcal{F})$. They are weakly equivalent.

The singular extension of the foliation to the closure M of Ω is used to prove Δ_M is regular. Furthermore, Δ_M depends on the way $\mathcal{F}$ is extended.

What about the spectrum?

Gaps in spectrum $\leftrightarrow$ Projections of $C^*(\mathcal{F})$

Need to know the "shape" of $K_0(C^*(\mathcal{F}))$. Baum-Connes assembly map...

Observation:

leaves of given dimension $\rightsquigarrow$ locally closed subsets $\rightsquigarrow$ filtration of $C^*(\mathcal{F})$...

Give formula for assembly map? Possible in some cases...

Thank you!