Integration of Singular Foliations and Usage

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Integration 1: In the spirit of Sophus Lie

1. \( g = \mathbb{R}, \ G = (\mathbb{R}, +) \) or \( G = S^1 \):

\[ 0 \longrightarrow (\mathbb{Z}, +) \longrightarrow (\mathbb{R}, +) \overset{\exp}{\longrightarrow} S^1 \longrightarrow 0 \]

2. \( M \) connected, \( A = TM, \ \mathcal{G} = \Pi(M) \) or \( \mathcal{G} = M \times M \):

\[ M \times \pi_1(M) \longrightarrow \Pi(M) \overset{(s,t)}{\longrightarrow} M \times M \]

3. \( A = F \leq TM \) involutive distribution with constant rank.

\[ \mathcal{G} = \text{Mon}(F) = \bigcup_{L} \Pi(L) \) or \( \mathcal{G} = \text{Hol}(F) \):

holonomy map \( : \text{Mon}(F) \rightarrow \text{Hol}(F) \)

4. \( A \rightarrow M \) Lie algebroid. Crainic and Fernandes constructed topological groupoid \( W(A) \) with connected and simply connected \( s \)-fibers. Smooth iff certain obstruction vanishes. When smooth, \( \mathcal{A}(W(A)) = A \).
Introduction

Integration 1: In the spirit of Sophus Lie

Integrability in a smooth sense

Given Lie algebroid $A \to M$, find:

- topological groupoid $\mathcal{G}(A)$ over $M$
- a kind of Lie functor such that $\mathcal{A}(\mathcal{G}(A)) = A$

Serre-Swan thm: $A \equiv \Gamma A$ and $\Gamma A$ projective $C^\infty(M)$-module.

So integration also means: ”Recover $\Gamma A$ from $\mathcal{G}(A)$”
Case study: Regular foliation

**Viewpoint 1:**

Partition to connected submanifolds. Local picture:

In other words: There is an open cover of $\mathcal{M}$ by foliation charts of the form $\Omega = \mathcal{U} \times \mathcal{T}$, where $\mathcal{U} \subseteq \mathbb{R}^p$ and $\mathcal{T} \subseteq \mathbb{R}^q$.

$\mathcal{T}$ is the transverse direction and $\mathcal{U}$ is the longitudinal or leafwise direction.

The change of charts is of the form $f(u, t) = (g(u, t), h(t))$.

**Viewpoint 2:**

**Frobenius theorem**

Equivalently consider the unique $C_c^\infty(\mathcal{M})$-module $\mathcal{F}$ of vector fields tangent to leaves.

**Fact:** $\mathcal{F} = C_c^\infty(\mathcal{M}, \mathcal{F})$ and $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$. 
Examples

\( M \): compact manifold.

1. Orbits of (some) Lie group actions on \( M \). Vector fields: image of infinitesimal action \( g \to \mathcal{X}(M) \).

2. Poisson geometry: Symplectic foliation on \( M \) by Hamiltonian vector fields (almost never regular!). Determines the Poisson structure...

3. \( X \) nowhere vanishing vector field of \( M \xrightarrow{\sim} \) action of \( \mathbb{R} \) on \( M \).

4. Irrational rotation on torus \( T^2 \): ”Kronecker” flow of \( X = \frac{d}{dx} + \theta \frac{d}{dy} \). \( \mathbb{R} \) injected as a dense leaf.

5. ”Horocyclic” foliation:
   - Let \( \Gamma \) cocompact subgroup of \( \text{SL}(2, \mathbb{R}) \). Put \( M = \text{SL}(2, \mathbb{R})/\Gamma \).
   - \( \mathbb{R} \) is embedded in \( \text{SL}(2, \mathbb{R}) \) by \( \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \), \( t \in \mathbb{R} \).
   - Therefore \( \mathbb{R} \) acts on \( M \). Action is ergodic, \( \exists \) dense leaves.
Holonomy

We wish to put a smooth structure on the equivalence relation

\[ \{(x, y) \in M^2 : L_x = L_y\} \]

What is the dimension of this manifold?

\( p + q \) degrees of freedom for \( x \); then \( p \) degrees of freedom for \( y \).

- Let \( x \in W = U \times T \) and \( x' \in W' = U' \times T' \)
- Nhd of \((x, x')\) should be \( U \times U' \times T \)
- Need identification of \( T \) with \( T' \)

**Definition**

A **holonomy** of \((M, \mathcal{F})\) is a diffeomorphism

\[ h : T \to T' \]

such that \( t, h(t) \) live in same leaf (for all \( t \in T \)).

**Fact:** Holonomies form a pseudogroup.
Examples of holonomies

- **Small Holonomies:** Charts $W = U \times T$ and $W' = U' \times T'$. In $W \cap W'$ have

  $$(u', t') = (g(u, t), h(t))$$

  Map $h = h_{W', W}$ is a holonomy.

- **Path Holonomies:** Let $\gamma : [0, 1] \rightarrow M$ smooth path in $L$. Cover $\gamma$ with $W_i = U_i \times T_i (1 \leq i \leq n)$. Take

  $$h(\gamma) = h_{W_n, W_{n-1}} \circ \ldots \circ h_{W_2, W_1}$$

**Definition**

The **holonomy of the path** $\gamma$ is the germ of $h(\gamma)$.

**Fact:** Path holonomy depends only on the homotopy class of the path!
The holonomy groupoid

**Definition**

\[ H(F) = \{ (x, y, h(\gamma)) \}, \] where \( \gamma \): path in \( L \) joining \( x \) to \( y \)

- **Manifold structure.** If \( W = U \times T \) and \( W' = U' \times T' \) are charts and \( h: T \to T' \) path-holonomy, get chart

  \[ \Omega_h = U' \times U \times T \]

- **Groupoid structure.** \( t(x, y, h) = x, s(x, y, h) = y \) and \( (x, y, h)(y, z, k) = (x, z, h \circ k) \).

\( H(F) \) is a **Lie groupoid**. Its **Lie algebroid** is \( F \). Its orbits are the leaves. \( H(F) \) is the **smallest possible smooth** groupoid over \( F \).
Integration 2: By operator algebras

Lie algebra \( \mathcal{F} = \Gamma(\mathcal{M}, F) \) acts by unbounded multipliers on \( C_c^\infty(\mathcal{H}(F)) \). Generates algebra of differential operators \( P \).

Fourier transform: \( P \) acting on \( f \in C_c^\infty(\mathcal{H}(F)) \) is:

\[
(Pf)(x, y) = \int \exp(i\langle \phi(x, z), \xi \rangle) \alpha(x, \xi)\chi(x, z)f(z, y) d\xi dz
\]

Where

- \( \phi \) the phase: through a local diffeomorphism defined on an open subset \( \tilde{\Omega} \cong U \times U \times T \subset G \) (where \( \Omega = U \times T \) is a foliation chart).
  \[ \phi(x, z) = x - z \in F_x; \]

- \( \chi \) the cut-off function: \( \chi \) smooth, \( \chi(x, x) = 1 \) on (a compact subset of) \( \Omega \), \( \chi(x, z) = 0 \) for \( (x, z) \notin \tilde{\Omega} \);

- \( \alpha \in C^\infty(F^*) \) a polynomial on \( \xi \), called the symbol of \( P \).

Generalized to any Lie groupoid (Nistor, Weinstein, Xu).
The convolution algebra of \( \mathcal{F} \) (Connes, Renault)

For \( f, g \in C_c^\infty(\text{H}(F)) \):

- we put \( f^*(x) = f(x^{-1}) \)
- we want to form \( f \ast g \) by a formula

\[
    f \ast g(x) = \int_{yz=x} f(y)g(z)
\]

In other words, we want to have an integration along the fibers of the composition \( \text{H}(F) \times_{s,t} \text{H}(F) \to \text{H}(F) \).

Use either Haar systems or half densities.

**Proposition**

The above involution and product make \( C_c^\infty(\text{H}(F)) \) a \(*\)-algebra.
The C*-algebra

J. Renault proves:

- The (continuous) ∗-representations of the ∗-algebra $C^\infty_c(H(F))$ are in one to one correspondence with unitary representations of the groupoid.
- An $L^1$-estimate shows that, for $f \in C^\infty_c(H(F))$, the supremum $f \mapsto \sup_\pi \|\pi(f)\|$ over all such representations $\pi$ is bounded.

$$\|f\|_1 = \sup \max_U \left\{ \int_{H(F)^U} |f(x)| d\lambda^U(x), \int_{H(F)^U} |f(x)| d\lambda_U(x) \right\}$$

Definition

- The **full** C*-algebra $C^*(\mathcal{F})$ of $\mathcal{F}$ is the completion of $C^\infty_c(H(F))$ w.r.t. the norm $f \mapsto \sup_\pi \|\pi(f)\|$.
- Left-regular representation $\rho_u$ on $L^2(H(F)^U)$. The **reduced** C*-algebra $C^*_r(\mathcal{F})$ of $\mathcal{F}$ is the completion w.r.t. $f \mapsto \sup_U \|\rho_U(f)\|$.

Conclusion: $\Gamma(M, F)$ can be recovered from unbounded multipliers of $\mathcal{F}$. 
Integration by operator algebras

**Conclusion:** $\Gamma(M,F)$ can be recovered from unbounded multipliers of $\text{C}^*(M,F)$.

**Recall: Integrability in a smooth sense**

Given Lie algebroid $A \to M$, find:

- topological groupoid $G(A)$ over $M$;
- a kind of Lie functor such that $\mathcal{A}(G(A)) = A$.

**Integrability by operator algebras**

Given Lie algebroid $A \to M$, find:

- topological groupoid $G(A)$ over $M$;
- a $\text{C}^*$-functor such that $\Gamma(M,A)$ sits in unbounded multipliers of $\text{C}^*(G(A))$. 
Debord’s setting

Almost regular foliations

Submodule \( \mathcal{A} \) of \( \Gamma(M, TM) \) such that:
- finitely generated projective
- stable under brackets

Serre-Swan theorem

Bundles = finitely generated projective \( C^\infty(M) \)-modules, so:
- \( \mathcal{A} \) is a the module of sections of a Lie algebroid \( A \);
- Anchor map \( A_x \to T_x M \) injective on a dense open subset;
- Image of anchor map is \( F_x \);
- Dimension lower semi-continuous.

Example: \( \mathcal{A} = \langle X \rangle \), where interior of \( \{ X = 0 \} \) is empty.
Examples of almost regular foliations

1. $\mathcal{A} = \langle X \rangle$, where interior of $\{X = 0\}$ is empty. e.g. $X = x \frac{\partial}{\partial x}$.

2. Poisson bivector $\Pi : T^*\mathbb{R}^2 \to T\mathbb{R}^2$ given by $xdx \wedge dy$.

$$\mathcal{A} = \Omega^1(\mathbb{R}^2) = \text{Im}(\Pi)$$

3. Log-symplectic manifolds...

Integrability:

- Crainic-Fernandes: $W(A)$ Lie groupoid for every almost injective Lie algebroid $A \to TM$. So $\mathcal{A}(W(A)) = A$.

- Debord: Constructed smallest Lie groupoid $H(A)$ such that $\mathcal{A}(H(A)) = A$. Quotient of $W(A)$. 
Integration in Debord’s setting

Main object of this integration: quasi-graphoids:
Pieces \((V, t, s)\) of groupoids with \(s, t\) submersions and \((t, s): V \to M \times M\) injective in dense set (so \(t(s^{-1}(x))\) is a piece of a leaf). Three steps:

1. Local integration
2. Composition: \(V_1 \times_{s,t} V_2\) (stable dimension).
3. Natural equivalence relation \(\sim\) (non-Hausdorff) Lie groupoid \(H(A)\).

Example: \(S^1 \subset \mathbb{R}^2\) by rotations: \(H(A) = S^1 \times \mathbb{R}^2\).

Consequences: \(C^*\)-algebra, psdo calculus, elliptic operators, etc.
General Singular case

General Stefan, Sussmann foliations

Definition (Stefan, Sussmann, A., Skandalis)

A singular foliation is a submodule $F$ of $C^\infty(M, TM)$ which is:

- locally finitely generated
- stable under brackets

No longer projective!

Two notions of fibers:

- $F_x$ tangent to leaf through $x$: Image of $F$ on $T_x M$. Lower semi-continuous. Continuous $\leftrightarrow$ regular.

- $F_x = F/I_x F$. Upper semi-continuous. Continuous $\leftrightarrow$ almost regular.

\[ 0 \rightarrow g_x \rightarrow F_x \xrightarrow{ev_x} F_x \rightarrow 0 \]

$g_x = 0$ iff $L_x$ has maximal dimension (regular leaf).
Examples

Actually, different foliations may yield same partition to leaves...

1. \( \mathbb{R} \) foliated by 3 leaves: \((-\infty, 0), \{0\}, (0, +\infty)\).
   \( \mathcal{F} \) generated by \( x^n \frac{\partial}{\partial x} \). Different foliation for every \( n \).

2. \( \mathbb{R}^2 \) foliated by 2 leaves: \( \{0\} \) and \( \mathbb{R}^2 \backslash \{0\} \).
   No obvious best choice. \( \mathcal{F} \) given by the action of a Lie group
   \[ \text{GL}(2, \mathbb{R}), \text{SL}(2, \mathbb{R}), \mathbb{C}^* \]

3. \( \mathbb{R}^3 \) foliated by spheres with center 0. Doesn’t admit almost injective
   algebroid: tangent to sphere non-trivial, no extension to 0.

4. \( \langle X \rangle \) where interior of \( \{X = 0\} \) is not empty.

\[ \text{e.g. } \text{GL}(2, \mathbb{R}) \subset \mathbb{R}^2 : \text{fibers } \mathcal{F}_0 = \text{gl}(2, \mathbb{R}) \text{ and } F_0 = \{0\}. \]
Holonomy groupoid (Extremely singular!)

As dimension of $\mathcal{F}_x$ varies, no hope for quasi-graphoids. We give up dimension requirements.

Our main object: bi-submersions: $(U, t, s)$ with $s, t : U \to M$ submersions and $t : s^{-1}(x) \to L_x$ submersion. $t(s^{-1}(x))$ piece of a leaf because

$$s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F}) = C^\infty(U; \ker ds) + C^\infty(U; \ker dt)$$

Same steps as in almost regular case

1. Local integration of vector fields $X_1, \ldots, X_n$ that form a basis of $\mathcal{F}_x$:

$$t(x, \tilde{\lambda}) = \exp_x \left( \sum_{i=1}^{n} \lambda_i X_i \right), \quad (x, \tilde{\lambda}) \in U \subset M \times \mathbb{R}^n$$

2. Composition: $U_1 \times_{t,s} U_2$ (dimension may rise)

3. Natural equivalence relation mixes dimensions $\Rightarrow$ Very singular groupoid...
Holonomy groupoid: Examples

1. \( \mathcal{F} = \langle X \rangle \) s.t. \( X \) has non-periodic integral curves around \( \partial \{ X = 0 \} \):

\[
H(\mathcal{F}) = H(X)|_{\{ X \neq 0 \}} \cup \text{Int}\{ X = 0 \} \cup (\mathbb{R} \times \partial \{ X = 0 \})
\]

2. \( \text{SL}(2, \mathbb{R}) \rhd \mathbb{R}^2 \):

\[
H(\mathcal{F}) = (\mathbb{R}^2 \setminus \{ 0 \})^2 \cup \text{SL}(2, \mathbb{R}) \times \{ 0 \}
\]

**topology:** Let \( x \in \mathbb{R}^2 \setminus \{ 0 \} \). Then \( \left( \frac{x}{n}, \frac{x}{n} \right) \in H(\mathcal{F}) \) converges to every \( g \) in stabilizer group of \( x \)... namely to every point of \( \mathbb{R} \)!

3. \( \text{SO}(3) \rhd \mathbb{R}^3 \): \( H(\mathcal{F}) \) quotient of \( \text{SO}(3) \ltimes \mathbb{R}^3 \).
Integration 1 for singular foliations

Theorem (A - Zambon)

By construction $H(\mathcal{F})$ is a \textit{diffeological} groupoid! (Souriau 1970’s).
\begin{itemize}
  \item There is a Lie functor for diffeological groupoids
  \item $\mathcal{A}(H(\mathcal{F})) = \mathcal{F}$
\end{itemize}

Recall: Souriau used similar Lie functor to prove $\mathcal{A}(\text{Diff}(M)) = \mathcal{X}(M)$. 

Holonomy map

\( S^1 \subset \mathbb{R}^2 \) Rotations: \( \mathcal{F} = \text{span}_{\mathcal{C}^\infty(\mathbb{R}^2)} \langle x\partial_y - y\partial_x \rangle \). Projective!

Regular leaf \( L = S^1 \), transversal \( S \). Get holonomy map

\[ h : \pi_1(L) \to \text{GermDiffeo}(S) \]

Singular leaf \( L = \{0\} \)

- Take \( \gamma \): constant path at origin.
- Transversal \( S_0 \): open neighborhood of origin in \( \mathbb{R}^2 \).

Realize \( \gamma \) either by integrating the zero vector field or \( x\partial_y - y\partial_x \) at the origin. Get completely different diffeomorphisms of \( S_0 \)!

Conclusion: Holonomy map \textbf{not} well defined on singularity!
Singular holonomy map

Let \((M, \mathcal{F})\) a singular foliation, \(L\) a leaf, \(x, y \in L\) and \(S_x, S_y\) slices of \(L\) at \(x, y\) respectively.

**Theorem (A-Zambon)**

There is an injective map

\[ \Phi^y_x : H^y_x \rightarrow \frac{\text{GermAut}_\mathcal{F}(S_x, S_y)}{\exp(I_x \mathcal{F})} |_{S_x}, \, h \mapsto \langle \tau \rangle \]

where \(\tau\) is defined as

- pick any bi-submersion \((U, t, s)\) and \(u \in U\) with \([u] = h\)
- pick any section \(b : S_x \rightarrow U\) of \(s\) through \(u\) such that \((t \circ b)S_x \subseteq S_y\)

and define \(\tau = t \circ b : S_x \rightarrow S_y\).

Here bi-submersions are crucial!
Holonomy map and the Bott connection

If $\mathcal{F}$ is regular then $\exp(I_{x}\mathcal{F})|_{S_{x}} = \{\text{Id}\}$, so we recover the usual holonomy map.

Let $L$ be a leaf. Recall $H(\mathcal{F})_{L}$ is a Lie groupoid (Debord).

1. Derivative of $\tau$ gives representation of $H(\mathcal{F})_{L}$:

$$\Psi_{L} : H(\mathcal{F})_{L} \to \text{Iso}(NL, NL)$$

2. Differentiating $\Psi_{L}$ gives

$$\nabla^{L,\perp} : A_{L} \to \text{Der}(NL)$$

It’s the Bott connection...
Linearization I

Vector field on $M$ tangent to $L \rightsquigarrow$
Vector field $Y_{\text{lin}}$ on $NL$, defined as follows:

\[ Y_{\text{lin}} \text{ acts on the fibrewise constant functions as } Y \big|_L \]
\[ Y_{\text{lin}} \text{ acts on } C^\infty_{\text{lin}}(NL) \equiv I_L/I^2_L \text{ as } Y_{\text{lin}}[f] = [Y(f)]. \]

Definition

The \textbf{linearization of } $\mathcal{F}$ \textbf{at } $L$ is the foliation $\mathcal{F}_{\text{lin}}$ on $NL$ generated by

\[ \{Y_{\text{lin}} : Y \in \mathcal{F}\} \]

Lemma

Let $L$ be a leaf. Then $\mathcal{F}_{\text{lin}}$ is the foliation induced by the Lie groupoid action $\Psi_L$ of $H(\mathcal{F})_L$ on $NL$. 
Linearization II

**Definition**

We say $\mathcal{F}$ is **linearizable at** $L$ if there is a diffeomorphism mapping $\mathcal{F}$ to $\mathcal{F}_{\text{lin}}$.

For $\mathcal{F} = \langle X \rangle$ with $X$ vanishing at $L = \{x\}$ linearizability means:

There is a diffeomorphism taking $X$ to $fX_{\text{lin}}$ for a non-vanishing function $f$.

This is a **weaker** condition than the linearizability of the vector field $X$!
Normal form around a (singular) leaf

Theorem (A-Zambon)

Let $L_x$ leaf at $x \in M$. The following are equivalent:

1. $\mathcal{F}$ is linearizable about $L$ and $H(\mathcal{F})_x$ compact.
2. There exists a tubular neighborhood $U$ of $L$ and a (Hausdorff) Lie groupoid $\mathcal{G} \to U$, proper at $x$, inducing the foliation $\mathcal{F}|_U$.

In that case:

- $\mathcal{G}$ can be chosen to be the transformation groupoid
  \[ H(\mathcal{F})|_L \ltimes \psi_L NL \]
- $(U, \mathcal{F}|_U)$ admits the structure of a singular Riemannian foliation.
C*-algebra(s) (A - Skandalis)

- Building blocks for convolution algebra: $C_c^\infty(\mathcal{U})$.

- Form $\bigoplus_{i \in I} C_c^\infty(\mathcal{U}_i)$ where $(\mathcal{U}_i)$ family of bi-submersions (atlas)

- Natural quotient $\mathcal{D} = \bigoplus_{i \in I} C_c^\infty(\mathcal{U}_i)/\sim$

- Need densities!

- Completion(s) of $\mathcal{D}$ easy: Smooth s-fibers! (See C. Debord.)

Desintegration Theorem (A - Skandalis)

*-representations of $C^*(\mathcal{F})$ correspond to unitary representations of $H(\mathcal{F})$. 
Cotangent ”bundle”

\[ \mathcal{F}^* = \bigcup_{x \in \mathcal{M}} \mathcal{F}_x^*. \] Not a bundle because dimension varies.

Nice locally compact space though.

Example: \( \text{SO}(3) \subset \mathbb{R}^3 \)

\[ \mathcal{F}^* = \bigcup_{\xi \in \mathbb{R}^3} \{ x \in \mathbb{R}^3 : \langle x, \xi \rangle = 0 \} \]
Pseudodifferential calculus: Idea

Submfd $V \subseteq U$, vector field $X$, distribution

$$q_X : f \mapsto \int_V Xf$$

If $X$ tangent to $V$,

$$\int_V Xf = - \int_V \text{div}(X) \cdot f$$

So up to zero order, $q_X$ depends on image of $X$ in $NV$.

Idea:

- Distributions on $U$ smooth outside $V$, pseudodiff. singularity on $V$.
- Principal symbol on $\mathcal{F}^*$. 
Pseudodifferential calculus: Formulas

\((U, t, s)\) bi-submersion, \(V \subset U\) identity bisection, \(N \to V\) normal bundle.

Take symbol \(\alpha \in S^m_{cl,c}(V, N^*; \Omega^1 N^*)\).

Define \(C^\infty(V)\)-linear \(P_\alpha : C^\infty_c(N; \Omega^1 N) \to C^\infty(V)\):

\[
< P_\alpha, f > (x) = (2\pi)^{-k} \int_{N_x^* \times N^*_x} \alpha(x, \xi) e^{-i < u, \xi >} f(u)
\]

Integrating on \(V\) gives distribution. \(P_\alpha\) pseudodifferential kernel.

Generalized functions on \(U\) with pseudodifferential singularities on \(V\)

\[
P = h + \chi \cdot P_\alpha \circ \phi
\]

- \(h \in C^\infty(U)\), \(\phi : U_1 \to N\) tubular neighborhood;
- \(\chi\) smooth “bump function” s.t. \(\chi|_V = 1\), \(\chi|_{U_1^c} = 0\)

Example: \(q_\chi : f \mapsto \int_V \chi f\).
Pseudodifferential calculus: Results

1. Elliptic operators: Can construct parametrix.

2. \[
0 \to C^*(\mathcal{F}) \to \Psi^0(\mathcal{F}) \overset{\sigma}{\to} C_0(S^*\mathcal{F}) \to 0
\]
whence analytic index map

3. Difficulty: Operators of order \(\leq -n\) for all \(n\) may not be smooth:
e.g. \(\text{SO}(3) \subset \mathbb{R}^3\) and 0-order symbol
\[
\alpha(x, \xi) = \begin{cases} 
    e^{\frac{1}{<x, \xi>^2}} & \text{out of } \mathcal{F}^* \\
    0 & \text{in } \mathcal{F}^*
\end{cases}
\]
\(\neq\) order \(-1\) operator with symbol \(\alpha\) around \(\mathcal{F}^*\)!
The Laplacian

**Theorem 1 (A-Skandalis)**

Let $M$ be a smooth compact manifold. Let $X_1, \ldots, X_N \in C^\infty(M; TM)$ be smooth vector fields such that $[X_i, X_j] = \sum_{k=1}^{N} f_{ij}^k X_k$.

Then $\Delta = \sum X_i^* X_i$ is essentially self-adjoint (both in $L^2(M)$ and $L^2(L)$).

**Proof**

This operator is indeed a regular unbounded multiplier of our $C^*$-algebra.

In Baaj-Woronowicz terminology: **regular multipliers** means:

- $\Delta$ is densely defined and closed.
  
  $(\text{graph } \Delta \text{ closed (right) submodule of } C^*(M, F) \times C^*(M, F))$.

- $\Delta$ has a densely defined (closed) adjoint $\Delta^*$;

- $\text{graph} \Delta \oplus (\text{graph} \Delta)^\perp = C^*(M, F) \times C^*(M, F)$
  
  $((y, x) \in \text{graph} \Delta^* \iff (x, -y) \in (\text{graph} \Delta)^\perp)$
Motivation: Laplacian of Kronecker foliation

Kronecker foliation on $M = T^2$: $\mathcal{F} = \langle X = \frac{d}{dx} + \theta \frac{d}{dy} \rangle$. $L = \mathbb{R}$

Two Laplacians:
- $\Delta_L = -\frac{d^2}{dx^2}$ acting on $L^2(\mathbb{R})$
- $\Delta_M = -X^2$ acting on $L^2(M)$

By Fourier:
- $\Delta_L \sim$ mult. by $\xi^2$ on $L^2(\mathbb{R})$. Spectrum: $[0, +\infty)$.
- $\Delta_M \sim$ mult. by $(n + \theta k)^2$ on $L^2(\mathbb{Z}^2)$. Spectrum dense in $[0, +\infty)$. 
Laplacians revisited

More generally $M$ compact, $(M, F)$ regular foliation.

Recall

- Lie algebra $\mathcal{F} = C^\infty(M, F)$ acts on $C^\infty(H(F))$ by unbounded multipliers.
- Laplacian $\Delta = \sum X_i^2$ as an unbounded multiplier of $C^*(M, \mathcal{F})$.

**Fact:** $L^2(L)$ representation of $C^*(M, \mathcal{F})$.

**Proposition (Baaj, Woronowicz)**

Every representation extends to regular multipliers.

Recover Laplacian $\Delta_L$. 
Statement of 2+1 theorems

Theorem 1 (Connes, Kordyukov)

\[ \Delta_M \text{ and } \Delta_L \text{ are essentially self-adjoint.} \]

Also true (and more interesting)

- for \( \Delta_M + f, \Delta_L + f \) where \( f \) is a smooth function on \( M \). (Schrodinger operators, conformal geometry, etc.)
- more generally for every leafwise elliptic (pseudo-)differential operator.

Not trivial because:

- \( \Delta_M \) not elliptic (as an operator on \( M \)).
- \( L \) not compact.

Theorem 2 (Kordyukov)

If \( L \) is dense + amenability, \( \Delta_M \) and \( \Delta_L \) have the same spectrum.

Connes

In many cases, one can predict the possible gaps in the spectrum.
**Generalization of Connes’ and Kordyukov’s theorem**

**Theorem (A-Skandalis)**

Assume that the (dense open) set $\Omega \subset M$ where leaves have maximal dimension is Lebesgue measure 1. Assume the restriction of all leaves to $\Omega$ are dense in $\Omega$. Assume that the holonomy groupoid of the restriction of $\mathcal{F}$ to $\Omega$ is Hausdorff and amenable. Then $\Delta_M$ and $\Delta_L$ have the same spectrum.

**Proof**

The $C^*$-algebra $C^*(\Omega, \mathcal{F})$ is simple (Fack-Skandalis) and sits as a two-sided ideal in $C^*(M, \mathcal{F})$. $L^2(L)$ and $L^2(M)$ are faithful representations of $C^*(\Omega, \mathcal{F})$ ⇒ weakly equivalent. The natural representations of $C^*(M, \mathcal{F})$ to $L^2(L)$ and $L^2(M)$ are extensions to multipliers of faithful representations of $C^*(\Omega, \mathcal{F})$. They are weakly equivalent.

The singular extension of the foliation to the closure $M$ of $\Omega$ is used to prove $\Delta_M$ is regular. Furthermore, $\Delta_M$ depends on the way $\mathcal{F}$ is extended.
What about the spectrum?

Gaps in spectrum $\leftrightarrow$ Projections of $C^*(\mathcal{F})$

Need to know the ”shape” of $K_0(C^*(\mathcal{F}))$. Baum-Connes assembly map...

Observation:

leaves of given dimension $\rightsquigarrow$ locally closed subsets $\rightsquigarrow$ filtration of $C^*(\mathcal{F})$...

Give formula for assembly map? Possible in some cases...

Thank you!