Almost regular Poisson manifolds and their holonomy groupoids

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Summary

1. Desingularization
2. Almost regular Poisson manifolds
3. Holonomy groupoid
4. Examples

Symplectic vs Poisson

Darboux's theorem

Symplectic manifolds simple: Locally $(\mathbb{R}^{2n}, (p_1, \ldots, p_n, q_1, \ldots, q_n))$ with

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$$

Weinstein's splitting theorem

Poisson manifolds complicated: Locally $\mathbb{R}^{2n+k}$ with

$$\Pi = \sum_{i=1}^{n} dp_i \wedge dq_i + \sum_{i,j=1}^{k} \{x_i, x_j\} dx_i \wedge dx_j$$

Here n, k are not constant.

But both have very rich dynamics:

$$\Pi : T^*M \to TM, \quad df \mapsto X_f, \text{ where } X_f(g) = -\{f, g\}.$$

T^*M Lie algebroid with bracket $[df, dg] = d\{f, g\}$.

- Π isomorphism iff (M, Π) symplectic.
- In general Π doesn't have constant rank.

Desingularization and integrability

Let (M, Π) Poisson manifold.

Symplectic realization (A. Weinstein)

Try to desingularize (M, Π). Namely find:

- Symplectic manifold (Σ, ω)
- Submersion $t : \Sigma \to M$ which is a Poisson map.

Usually addressed as an integration problem:

Theorem (Karasev, Weinstein, Zakrzewski, Hector, Dazord)

Lie groupoid $\Sigma \rightrightarrows M$ such that $Lie(\Sigma) = T^*M$. Then:

- (Σ, ω) symplectic manifold;
- graph of multiplication Lagrangian in $\Sigma \times \Sigma \times \bar{\Sigma}$;
- $s : \Sigma \to M$ Poisson and $t : \Sigma \to M$ anti-Poisson.

Notice that SR desingularizes both Poisson str Π and symplectic foliation.

Local Lie groupoid Σ as above always exists!

Overview of integration (Cattaneo, Felder, Crainic, Fernandes)

Put $A = T^*M$. Space of A-paths:

$$P(A) = \{\text{Lie algebroid morphisms } \alpha : TI \to A\}$$

Banach manifold ($\dim = \infty$)

Monodromy groupoid:

$$\mathscr{G}(A) = P(A)/A\text{-homotopy}$$

$\mathscr{G}(A)$ topological groupoid, s-simply connected. Smoothness depends on behaviour of $\pi_2(L)$...

When $\mathscr{G}(A)$ smooth, put $\Sigma = \mathscr{G}(A)$.

- In some cases we know integrability obstruction vanishes, e.g.:
  - $\dim(M) = 2$
  - anchor Π injective in open dense subset of M (C. Debord).

(My) trouble 1 with integrability... (A & P. Antonini)

Wish to convey NCG methods in Poisson geometry: Longitudinal pseudodifferential calculus, index theory, etc.

Need a desingularizing Lie groupoid around.

Local Lie groupoids admit no C^*-algebra, hence no K-theory. Same for gpds with $\dim = \infty$.

- Is T^*M quotient of an integrable Lie algebroid A?
- Dimension of A dictated by $\pi_2(L)$...

(My) trouble 2 with integrability... (A. & M. Zambon)

Integrating T^*M to find symplectic realizations of (M, π) means we double the dimension of M a priori.

- Maybe there exists a symplectic realization $\Sigma \to M$ such that $\dim\Sigma < 2\dim(M)$.
- If so, what is the smallest $\dim(\Sigma)$? Is it linked with the topology of (M, Π)?
- "Desingularization" could also mean lifting (M, Π) to, e.g. a regular Poisson groupoid. Namely a Poisson groupoid $(G, \hat{\Pi}) \rightrightarrows (M, \Pi)$ such that $\hat{\Pi}$ is a regular Poisson structure.
- Maybe in this setting the dimension doesn't need to double.

Many more ways to make sense of "nicer" Poisson structures and what "desingularization" might mean...

A naive hierarchy of Poisson structures

Let $\Pi : T^*M \to TM$ a Poisson structure.

- Symplectic: Π has full rank everywhere (isomorphism).
- Regular: Π has constant rank everywhere.
- Π has full rank in open dense subset.
  Example: $\Pi : T^*\mathbb{R}^2 \to T\mathbb{R}^2$, $\quad \Pi = y\partial_x \wedge \partial_y$.
  Zeros of Π is $N = x-$ axis. $\mathbb{R}^2 \backslash N$ dense in $\mathbb{R}^2$.
- Π has constant rank in open dense subset.
  Example: $\Pi : T^*\mathbb{R}^3 \to T\mathbb{R}^3$, $\quad \Pi = z\partial_x \wedge \partial_y$.
  Lie-Poisson structure on $\mathfrak{h}^*$ where H: Heisenberg group.
- None of the above...

Conclusion

Too many Poisson structures around!

Need a hierarchy. (Better in terms of Algebra...)

Symplectic foliation

Determines Poisson structure completely! Algebraic viewpoint than simply partition to symplectic leaves:

Definition - Proposition

Symplectic foliation $\mathcal{F}$ is the $C^{\infty}(M)$-submodule of $\mathcal{X}_c(M)$ generated by $\Pi(\Omega^1_c(M))$. It is:

1. locally finitely generated;
2. $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$.

- Stefan-Sussmann thm says that it integrates to immersed subanifolds (symplectic leaves).
- Dimension of leaves may jump!

Symplectic foliation

Module $\mathcal{F}$ carries more information than the partition to leaves:
Put $A_x^{\mathcal{F}} = \mathcal{F}/I_x\mathcal{F}$. Then

$$0 \to \mathfrak{g}_x \to A_x^{\mathcal{F}} \xrightarrow{ev_x} T_xL \to 0$$

Fact: L regular iff $A_x^{\mathcal{F}} = T_xL$.

Serre-Swan

- $\mathcal{F}$ projective $\Leftrightarrow A^{\mathcal{F}} = \bigcup_{x\in M} A_x^{\mathcal{F}}$ Lie algebroid.
- $\Gamma(A^{\mathcal{F}}) = \mathcal{F}$
- Anchor $ev : A^{\mathcal{F}} \to TM$ can be injective in open dense subset of M.

Holonomy groupoid

A-Skandalis

- There always exists a topological holonomy groupoid $H(\mathcal{F})$;
- Fibers $H(\mathcal{F})_x$ smooth; differentiate to $A_x^{\mathcal{F}}$.
- $H(\mathcal{F})$ Lie groupoid $\Leftrightarrow$ $\mathcal{F}$ projective

Construction: Let $\mathscr{G} \rightrightarrows (M, \mathcal{F})$. Then $H(\mathcal{F})$ quotient of $\mathscr{G}$ as follows:

$$g_1 \equiv g_2 \textbf{ iff}$$

$\exists$ nhds U_1, U_2 and $\phi : U_1 \to U_2$ commuting with s, t and $\phi(g_1) = g_2$

Example: $\mathcal{F} = \mathfrak{X}_c(M)$. Integrates to $\Pi(M)$. Quotient

$$(s, t) : \Pi(M) \to M \times M$$

$A^{\mathcal{F}} = TM$.

Examples

- $M = \mathbb{R}$ and $\mathcal{F} = \langle x\partial_x \rangle$. Leaves: $(-\infty, 0), \{0\}, (0, +\infty)$.
  - $A_x^{\mathcal{F}} = \mathbb{R}$ for all $x \in \mathbb{R}$.
  - $T_x L = \mathbb{R}$ for all $x \in \mathbb{R} \setminus \{0\}$.
  - So $A^{\mathcal{F}} = \bigcup_{x \in \mathbb{R}} A_x^{\mathcal{F}}$ is a Lie algebroid. Anchor map $ev : A^{\mathcal{F}} \to TM$, injective in $\mathbb{R} \setminus \{0\}$.
  - Holonomy groupoid: $(\mathbb{R}, +) \ltimes \mathbb{R} \rightrightarrows (\mathbb{R}, \mathcal{F})$. Quotient

$$H(\mathcal{F}) = \mathbb{R}\setminus\{0\} \times \mathbb{R}\setminus\{0\} \bigcup (\mathbb{R}, +) \times \{0\}$$

- $\Pi : T^*\mathbb{R}^2 \to T\mathbb{R}^2$ given by $\Pi = y\partial_x \wedge \partial_y$.
  - Zeros of Π is $N = x -$ axis. $M \setminus N$ dense in M.
  - $\Omega^1(M)$ coincides with $\mathcal{F}$ as a $C^\infty(M)$-module. Get two integrations of T^*M:
    - $\Sigma \rightrightarrows M$ symplectic and s-simply connected (Crainic-Fernandes);
    - Holonomy groupoid $H(\mathcal{F})$.
    - Anchor map integrates to surjection $\Sigma \to H(\mathcal{F})$, so $H(\mathcal{F})$ is a terminal object" "adjoint"

Log-symplectic manifolds and blow-ups

Definition

(M^{2n}, π) Log-symplectic if $\wedge^n \pi$ is a section of the line bundle $\wedge^{2n} TM$ transverse to the zero section.
Then, the zeros of $\wedge^n \pi$ form a smooth "exceptional" hypersurface N.

Whence π has full rank in $M \backslash N$, which is dense.

So Log-symplectic manifolds are integrable.

Gualtieri, Li (2013)

For every Log-symplectic (M, π) there exists a symplectic groupoid $\Sigma_0 \rightrightarrows M$ which is a terminal object ("adjoint").
(Uses Melrose's blow-up construction...)

Blow-up groupoid is holonomy groupoid

Proof.

- Put
  - Σ the s-simply connected symplectic groupoid integrating T^*M.
  - $H(\mathcal{F})$ the holonomy groupoid of $\mathcal{F}$.

  Anchor map integrates to surjective morphism $\Sigma \to H(\mathcal{F})$.
- Since $\Pi : T^*M \to TM$ is injective in a dense subset, the $C^\infty(M)$ modules $\Omega^1(M)$ and $\mathcal{F}$ coincide.
- So both Σ and $H(\mathcal{F})$ integrate T^*M. Kernel of $\Sigma \to H(\mathcal{F})$ discrete.
- So symplectic structure of Σ inherited to $H(\mathcal{F})$.
- A priori, $H(\mathcal{F})$ is "adjoint". So coincides with blow-up by Gualtieri and Li.

□

Almost regular Poisson manifolds

Definition (A-Zambon)

A Poisson manifold (M, π) is almost regular iff $\mathcal{F}$ is a projective module.

Remarkably, if (M, π) is a Poisson manifold, the property of "being almost regular" depends only on the partition of M to immersed leaves. (And not the symplectic structure of these leaves.)

Theorem (A-Zambon)

Let $M_{reg} \subset M$ the set where Π has maximal rank. Then, (M, π) is almost regular iff

1. M_{reg} is dense in M and
2. There is a distribution D on M such that $D_x = T_x L$ for all $x \in M_{reg}$, where L is the symplectic leaf at x.

Idea of proof

Proof.

- Serre-Swan $\Rightarrow A^{\mathcal{F}} = \bigcup_{x \in M} A_x^{\mathcal{F}}$ Lie algebroid.
- So kernel $\mathfrak{h}$ of extension
$$0 \to \mathfrak{h} \to T^*M \xrightarrow{\Pi} A^{\mathcal{F}} \to 0$$
has constant rank.
- Put D_x^0 the annihilator of $\mathfrak{h}_x$. At regular points, we have $A_x^{\mathcal{F}} = T_x L$, so $D_x^0 = T_x L$.

□

Exercises

- $\mathfrak{h}$ is a bundle of abelian(!) Lie algebras.
- $\mathfrak{h}_x = \{\xi \in T_x^* M : \text{ local extn } \tilde{\xi} \in \Omega^1(M) \text{ with } \tilde{\xi}|_L = 0 \text{ for any leaf } L\}$

Examples 1

- $M_{reg} = M$: regular Poisson manifold.
- $D = TM$: Π has full rank in a dense subset: Log-symplectic.

- What about Π: constant rank in dense subset?

Counterexample: Lie algebra dual $\mathfrak{g}^*$ of $SU(2)$. Symplectic leaves are coadjoint orbits. Concentric spheres in $\mathbb{R}^3$. Now count dimensions in

$$0 \to \mathfrak{h} \to T^*M \xrightarrow{\Pi} A^{\mathcal{F}} \to 0$$

  - If $x \neq 0$, $A^{\mathcal{F}}_x = T_x S^2 = \mathbb{R}^2$, so $\dim(\mathfrak{h}_x) = 1$
  - $\mathfrak{h}_0 = 0$: every 1-form which annihilates the spheres must vanish at 0.

Every D_x is the annihilator of $\mathfrak{h}_x$. So D can't have constant rank.

Examples 2

Lemma

Let (M, π) *almost regular.*

1. $f \in C^{\infty}(M)$ *is Casimir iff* f *constant along leaves of* D*.*
2. *Let* f *Casimir s.t.* $supp(f)$ *dense in* M*. Then* $(M, f\pi)$ *almost regular with distribution* D*.*

- Let (N, π_N) symplectic. Then $(N \times \mathbb{R}, \pi_N)$ regular Poisson.
- $t \in C^{\infty}(N \times \mathbb{R})$ Casimir $(supp(f) = N \times \mathbb{R}^*)$. So $(N \times \mathbb{R}, t\pi_N)$ almost regular.
- Put $N = \mathbb{R}^2$, $\pi_N = dx \wedge dy$. Get $(\mathbb{R}^3, z dx \wedge dy)$.
- Exactly Lie-Poisson structure on $Lie(H)^*$, where H: Heisenberg group.

Lie bialgebroid

Proposition

Let (M, π) almost regular. Since M_{reg} is dense in M, by continuity we have:

- the distribution D is unique
- D is involutive
- D integrates to a regular foliation by Poisson submanifolds $(P, \pi_P = \pi|_P)$.
- $A^{\mathcal{F}} = T^*M/D^0 = D^*$ almost regular Lie algebroid defining $\mathcal{F}$.

Since $[\pi_P, \pi_P] = 0$, $D^* = A^{\mathcal{F}}$ is also a Lie algebroid and

$$(D, D^*)$$

is a triangular Lie bialgebroid.

Proposition

The holonomy groupoid $H(D)$ is a Poisson groupoid with $\overleftarrow{\pi} - \overrightarrow{\pi}$.

The holonomy groupoid

Also (D^*, D) is a Lie bialgebroid.

Let Γ the s-simply connected Lie groupoid integrating D^*. It is the union of the universal covers of $H(\mathcal{F})_x$.

So Γ is a Poisson groupoid (canonical).

Theorem (A-Zambon)

1. The Poisson structure of Γ descends to $H(\mathcal{F})$ and makes it a Poisson groupoid.
2. Symplectic leaves are $H(\mathcal{F})|_P$ where (P, π_P) leaf of D. All leaves as such regular.
3. $H(\mathcal{F})|_P$ symplectic groupoid for (P, π_P).

Proof.

$H(\mathcal{F}) = \Gamma/K$ where K discrete. Now use M_{reg} is dense in M... □

Examples 1: The extreme ones

Since (D, D^*) arises from r-matrix π, we have a Lie algebroid morphism

$$\sharp : D^* \to D$$

Integrates to anti-Poisson morphism of Lie groupoids

$$H(\mathcal{F}) \to H(D)$$

- $\underline{M_{reg} = M}$ regular Poisson. Then $H(\mathcal{F}) \simeq H(D)$.
- $\underline{D = TM}$ Log-symplectic. Then $H(D) = M \times M$. We get
$$(s, t) : H(\mathcal{F}) \to M \times M$$
- Also for Log-symplectic, since $D = TM$ its foliation has a single leaf, the entire Poisson manifold M.
Whence $H(\mathcal{F})$ symplectic groupoid.

Examples 2: Heisenberg-Poisson

Let (V, π) Poisson vector space (so $\pi \in \wedge^2 V$).

1. Symplectic groupoid: $V^* \ltimes V$, where:
   - $(V^*, +)$ acts on V by $(\xi, v) \mapsto v + \pi^\sharp(\xi)$.
   - symplectic form Ω_π: $((\xi, v), (\xi', v)) \mapsto \langle \xi', v \rangle + \langle \xi, v' \rangle + \pi(\xi, \xi')$
2. Let $\pi = \omega$ symplectic. Symplectic groupoid $(V \times V, \omega \times (-\omega))$.

   $$(t, s) : V^* \ltimes V \to V \times V$$

   isomorphism of symplectic groupoids.

Weinstein, Hawkins

Consider $V \times \mathbb{R}$ with $\Pi = t\omega$. It is the linear Heisenberg-Poisson manifold associated with (V, ω). It's an almost regular Poisson manifold.

Similarly we may start from any symplectic manifold... Also put $f(t)$ instead of t for $f \in C^\infty(\mathbb{R})$ such that $\{t \in \mathbb{R} : f(t) \neq 0\}$ dense in $\mathbb{R}$.

Examples 2: Heisenberg-Poisson

Analysis of (D^*, D):

1. $H(\mathcal{F}) = V^* \ltimes (V \times \mathbb{R})$ as Poisson groupoids, where:
   - $(V^*, +)$ acts on $V \times \mathbb{R}$ by $(\xi, (v, t)) \mapsto (v + t\pi^\sharp(\xi), t)$
   - Symplectic leaves: $V^* \ltimes (V \times \{t\})$ with $\Omega_{t\pi}$.

2. $H(D) = V \times V \times \mathbb{R}$ as Poisson groupoids, where:
   - Poisson tensor at (v_1, v_2, t) is $-t\pi_{v_1} + t\pi_{v_2}$
   - $V \times V \times \{t\}$ subgroupoid of $H(D)$ with pair groupoid structure.

3. Anti-Poisson map

$$(s, t) : V^* \ltimes (V \times \mathbb{R}) \to V \times V \times \mathbb{R} \qquad (\xi, v, t) \mapsto (v + \pi^\sharp(\xi), v, t)$$

Not an isomorphism (problem at $t = 0$).

Examples 2: Heisenberg-Poisson

Isomorphism $V^* \ltimes V \simeq V \times V$ turns $\mathrm{H}(\mathcal{F})$ to Connes' tangent groupoid:

$$\mathrm{H}(\mathcal{F}) = \mathsf{T}V \times \{0\} \cup V \times V \times \mathbb{R}^*$$

Take $(V, \omega) = (\mathbb{R}^2, dx \wedge dy)$. Then $(V \times \mathbb{R}, t\omega)$ is the Lie algebra dual of the Heisenberg group.

Symplectic groupoids for Heisenberg-Poisson manifolds were constructed by Weinstein using a "double explosion" technique. Generalized by Hawkins.

Presumably, all cases as such can be treated through foliation theory...

The case of $SU(2)$

Put $M = Lie(SU(2))^*$. Recall Lie-Poisson structure on M not almost regular.

But $\mathcal{F}$ is quite close to projectivity: Kernel of anchor map $\Pi : \Omega^1(M) \to \mathcal{F}$ generated by

$$\alpha = rdr = xdx + ydy + zdz$$

Projective resolution:

$$0 \to \Gamma_c(M \times \mathbb{R}) \xrightarrow{\alpha} \Gamma_c(T^*M) \xrightarrow{\Pi} \mathcal{F} \to 0$$

C. Laurent-Gengoux and S. Lavau

Every projective resolution of a singular foliation $\mathcal{F}$ admits an L_∞-structure, unique up to quasi-isomorphism.

Towards a hierarchy of Poisson structures

A hierarchy of Poisson structures? In terms of algebra...

- Distinguish Poisson structures using the projective dimension of their symplectic foliation. A hierarchy of singularities really...
- Almost regular Poisson structures: Class of zeroth projective dimension.
- Desingularization: Integrate the L_∞-algebroid associated with the projective resolution.

Some questions:

- Determine completely the class of Poisson structures which have a given projective dimension.
- Can we do NCG in a class as such?

Thank you Nicola!