The leafwise Laplacian of a singular foliation

Iakovos Androulidakis

Department of Mathematics,
University of Athens

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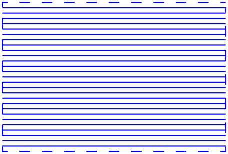
Summary

1.1 Definition: Foliation (regular)

Viewpoint 1:

Partition to connected submanifolds. Local picture: 

In other words: There is an open cover of M by foliation charts of the form $\Omega = U \times T$, where $U \subseteq \mathbb{R}^p$ and $T \subseteq \mathbb{R}^q$.

T is the transverse direction and U is the longitudinal or leafwise direction.

The change of charts is of the form $f(u,t) = (g(u,t), h(t))$.

Viewpoint 2:

Frobenius theorem

Equivalently consider the unique $C_c^\infty(M)$-module $\mathcal{F}$ of vector fields tangent to leaves.

Fact: $\mathcal{F} = C_c^\infty(M, F)$ and $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$.

Examples

M: compact manifold.

1. Orbits of (some) Lie group actions on M. Vector fields: image of infinitesimal action $\mathfrak{g} \to \mathcal{X}(M)$.
2. Poisson geometry: Symplectic foliation on M by Hamiltonian vector fields (not always regular). Determines the Poisson structure completely...

For the moment, focus on regular examples:

3. X nowhere vanishing vector field of $M \rightsquigarrow$ action of $\mathbb{R}$ on M.
4. Irrational rotation on torus T^2 ("Kronecker flow"). $\mathbb{R}$ injected as a dense leaf.
5. "Horocyclic" foliation:
   - Let Γ cocompact subgroup of $SL(2,\mathbb{R})$. Put $M = SL(2,\mathbb{R})/\Gamma$.
   - $\mathbb{R}$ is embedded in $SL(2,\mathbb{R})$ by $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, t \in \mathbb{R}$.
   - Therefore $\mathbb{R}$ acts on M. Action is ergodic, $\exists$ dense leaves.

Laplacians

Kronecker foliation on $M = T^2$: Vector field $X = \frac{d}{dx} + \theta \frac{d}{dy}$. $L = \mathbb{R}$

Two Laplacians:

- $\Delta_L = -\frac{d^2}{dx^2}$ acting on $L^2(\mathbb{R})$
- $\Delta_M = -X^2$ acting on $L^2(M)$

By Fourier:

- $\Delta_L \rightsquigarrow$ mult. by ξ^2 on $L^2(\mathbb{R})$. Spectrum: $[0, +\infty)$.
- $\Delta_M \rightsquigarrow$ mult. by $(n + \theta k)^2$ on $L^2(\mathbb{Z}^2)$. Spectrum dense in $[0, +\infty)$.

Laplacians

More generally M compact, (M, F) regular foliation. Each leaf is a complete Riemannian manifold.

Lie algebra $\mathcal{F} = C^\infty(M, F)$ acts on $C^\infty(H(F))$ by unbounded multipliers. The algebra generated is the algebra of differential operators.

- Laplacian $\Delta = \sum X_i^2$ as an unbounded multiplier of $C^*(M, \mathcal{F})$.

$L^2(L), L^2(M)$ are representations of $C^*(M, \mathcal{F})$. We get:

Laplacian Δ_L acting on $L^2(L)$

Laplacian Δ_M acting on $L^2(M)$

Statement of 2+1 theorems

Theorem 1 (Connes, Kordyukov)

Δ_M and Δ_L are essentially self-adjoint.

Also true (and more interesting)

- for $\Delta_M + f, \Delta_L + f$ where f is a smooth function on M. (Schroedinger operators, conformal geometry, etc.)
- more generally for every leafwise elliptic (pseudo-)differential operator.

Not trivial because:

- Δ_M not elliptic (as an operator on M).
- L not compact.

Theorem 2 (Kordyukov)

If L is dense + amenability, Δ_M and Δ_L have the same spectrum.

Connes

In many cases, one can predict the possible gaps in the spectrum.

Basic tools

1 $C^*(M,F)$: (Renault) Completion of a convolution algebra of kernels

$$k(\gamma) = \int_{\gamma_1\gamma_2=\gamma} k_1(\gamma_1)k_2(\gamma_2)$$

where $\gamma, \gamma_1, \gamma_2 \in H(F)$.

2 Pseudodifferential calculus on $H(F)$ (Connes, Monthubert-Pierrot, Nistor-Weinstein-Xu)

$$0 \to C^*(M,F) \to \Psi^*(M,F) \to C(SF^*) \to 0$$

Proposition (Connes)

- Negative order pseudodifferential operators $\in C^*(M,F)$
- Zero order pseudodifferential operators: multipliers of $C^*(M,F)$.

Theorem (Vassout)

Elliptic operators of positive order are regular unbounded multipliers (in the sense of Baaj-Woronowicz: $graph(D) \oplus graph(D)^\perp$ is dense).

Proof of theorems 1 and 2

Theorem 1

Δ_M and Δ_L are essentially self-adjoint.

- $L^2(M)$ and $L^2(L)$: representations of the foliation C^*-algebras.
- (Baaj, Woronowicz): Every representation extends to regular multipliers.

image of the adjoint = adjoint of the image

Theorem 2 (Kordyukov)

If all leaves L are dense + amenability assumptions, Δ_M and Δ_L have the same spectrum.

- (Fack and Skandalis): Leaves dense $\Rightarrow$ all representations are faithful.
- Every injective morphism of C^*-algebras is isometric and isospectral.

Examples for the Spectrum (Connes)

Horocyclic foliation: no gaps in the spectrum

Let the "$ax+b$" group act on a compact manifold M.
e.g. $M = SL(2,\mathbb{R})/\Gamma$ where Γ discrete co-compact group.
Leaves = orbits of the "$x+b$" group (assume it is minimal).

The spectrum of the Laplacian is an interval $[m, +\infty)$

Proof

- gaps in spectrum $\longrightarrow$ projections in $C^*(M,F)$.
- $\exists$ invariant measure by $ax+b \Longrightarrow$ trace on $C^*(M,F)$ faithful (Fack-Skandalis).
- The "ax" subgroup $\longrightarrow$ action of $\mathbb{R}_+^*$ which scales the trace.
- Image of K_0 countable subgroup of $\mathbb{R}$, invariant under $\mathbb{R}_+^*$ action.

Similarly, Kronecker flow: Image of the trace $\mathbb{Z}+\theta\mathbb{Z}$

Can be a Cantor type set

Analytic index

Baum-Connes predicts $K_0(C^*(M,\mathcal{F}))$: Assembly map is a kind of analytic index...

Analytic index can be obtained in 2 ways:

1. $\Psi^*(M,\mathcal{F})$ extension, mapping cones, etc...

$$\rightsquigarrow [\mathrm{Ind}] \in KK(C_0(F^*); C^*(M;\mathcal{F}))$$

2. Tangent groupoid $\mathcal{G} = F \times \{0\} \cup H(\mathcal{F}) \times (0,1]$:

   - $0 \to C^*(G) \otimes C_0((0,1]) \to C^*(\mathcal{G}) \to C_0(A^*G) \to 0$
   - $[\mathrm{Ind}] = [ev_1] \otimes [ev_0^{-1}] \in KK(C_0(F^*); C^*(M;\mathcal{F}))$

3.1 Almost injective algebroids

Recall Frobenius theorem

Δ only depends on the bundle $F \subset TM$ of vector fields tangent to the leaf.

Serre-Swan Theorem

Bundles = Finitely generated projective $C^\infty(M)$-modules.

$$\boxed{E \longleftrightarrow C^\infty(M; E)}$$

Debord's setting

$\mathcal{A}$: finitely generated projective sub-module of $C^\infty(M; TM)$, stable under brackets.

Equivalently:

Lie algebroid with anchor $A_x \to T_x M$, injective in a dense set.
Image F_x. Dimension lower semi-continuous.

Example: $\mathcal{A} = < X >$ s.t. $\mathrm{Int}\{X = 0\}$ is empty.

Almost injective algebroids II

Theorem (Debord, Pradines- Bigonnet, Crainic-Fernandes)

Every almost injective algebroid is integrable.

So it comes from a Lie groupoid; Whence

- C^*-algebra
- pseudodifferential calculus
- Elliptic operators: regular multipliers

Furthermore, well-defined Laplacian

- Theorems 1 and 2: Exactly same proof
- Theorem 3: No gaps for a manifold with conic singularities obtained using a finite covolume subgroup of $SL(2, \mathbb{R})$

3.2 Stefan-Sussmann foliations

Definition (Stefan, Sussmann, A-Skandalis)

A (singular) foliation is a finitely generated sub-module $\mathcal{F}$ of $C^\infty(M; TM)$, stable under brackets.

No longer projective. Fiber $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$: upper semi-continuous dimension.

One may still define leaves (Stefan-Sussmann).

Actually: Different foliations may yield same partition to leaves

Examples

1. $\mathbb{R}$ foliated by 3 leaves: $(-\infty, 0), \{0\}, (0, +\infty)$.
   $\mathcal{F}$ generated by $x^n \frac{\partial}{\partial x}$. Different foliation for every n.
2. $\mathbb{R}^2$ foliated by 2 leaves: $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$.
   No obvious best choice. $\mathcal{F}$ given by the action of a Lie group

$$GL(2, \mathbb{R}), SL(2, \mathbb{R}), \mathbb{C}^*$$

Constructions of A-Skandalis

In this general setting, one may still construct:

- a holonomy groupoid. Extremely singular...
- The foliation C^*-algebra (and its representations...)
- The cotangent "bundle": Not a bundle since dimension of fibres not constant. But $\mathcal{F}^*$: nice locally compact space.
- The pseudodifferential calculus... complicated...
  1. $0 \to C^*(M, \mathcal{F}) \to \Psi^*(M, \mathcal{F}) \to C_0(\mathcal{F}^*) \to 0$
  2. Elliptic operators of positive order are regular unbounded multipliers

And also

- Analytic index (element of $KK(C_0(\mathcal{F}^*); C^*(M, \mathcal{F}))$)
- tangent groupoid + defines same KK element.

Holonomy transformations I: Regular case

$\mathcal{F}$ sections of F involutive subbundle of TM.

$\gamma : [0,1] \to M$ path on a leaf, S_x, S_y transversals at $x = \gamma(0), y = \gamma(1)$

For any t, extend $\dot{\gamma}(t)$ to a time-dependent v.f $Z_t \in \mathcal{F}$

Define $\Gamma : S_x \times [0,1] \to M$ following the flow of Z_t on points of S_x.
(Assume $\Gamma(q,1) \subseteq S_y$).

Define holonomy of γ the germ at x of

$$hol_\gamma : S_x \to S_y \quad q \mapsto \Gamma(q,1)$$

Does not depend on choice of Z_t. Get maps

- $\{$homotopy classes of paths $\gamma\} \mapsto GermAut_{\mathcal{F}}(S_x; S_y)$ (holonomy)
- $\{$homotopy classes of paths $\gamma\} \mapsto Iso(T_x S_x; T_y S_y)$ (linear holonomy)

Holonomy transformations II: Singular case

Take $M = \mathbb{R}$, $\mathcal{F} = \langle x\frac{\partial}{\partial x} \rangle$ and $x = y = 0$.

Transversal = neighborhood of 0 in $\mathbb{R}$.

Constant path $\gamma(t) = 0$ admits many extensions, e.g.

1. flow of zero vector field: $\Gamma : S_0 \times [0,1] \to S_0, \quad (x,t) \mapsto x$;
2. flow of $x\frac{\partial}{\partial x}$: $\Gamma(x,t) = e^t x$

Observation 1

Different choices of Γ differ by the flow of $X \in \mathcal{F}(x) = \{X \in \mathcal{F} : X(x) = 0\}$.

Hence $\Gamma(\cdot, 1)$ gives a class in $\frac{GermAut_{\mathcal{F}}(S_x,S_x)}{\exp(\mathcal{F}(x))}$

Observation 2 (A-Zambon)

Not linearizable! To make it so, must consider $\frac{GermAut_{\mathcal{F}}(S_x,S_x)}{\exp(I_x\mathcal{F})}$.

Bi-submersions

Take $x \in M$, put $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$. Then $\dim(\mathcal{F}_x) = n < \infty$

- Take $X_1, \dots, X_n \in \mathcal{F}$ generating $\mathcal{F}$.
- Find $U \subset \mathbb{R}^n \times M$ neighborhood of $(x, 0)$ where $t : U \to M$ is defined:

$$t(\lambda_1, \dots, \lambda_n, y) = \exp_y\left(\sum_{i=1}^{n} \lambda_i X_i\right)$$

- Put $s = pr_2$. Then $s, t : U \to M$ submersions and U foliated by

$$s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F}) = C^\infty(U; \ker ds) + C^\infty(U; \ker dt)$$

(Leaves: $s^{-1}(L) \cap t^{-1}(L)$ where L leaf of $\mathcal{F}$.)

A bisection b of s, t carries a holonomy h s.t. $h_*(\mathcal{F}) \subseteq \mathcal{F}$:

$$h = t|_b \circ (s|_b)^{-1}$$

Bi-submersions (U, t, s) as such, provide a stable way to keep track of holonomies near the identity.

Holonomy groupoid

Holonomy groupoid $H(\mathcal{F})$ can be realized as a quotient of a collection of bi-submersions covering M...

Whence $\sharp : U \to H(\mathcal{F})$ is a smooth cover of an open subset of $H(\mathcal{F})$.

1. (Almost) regular case: $H(\mathcal{F})$ usual holonomy groupoid.
2. $\mathcal{F} = \rho(AG)$: $H(\mathcal{F})$ is a quotient of G.
3. $\mathcal{F} =< X >$ s.t. X has non-periodic integral curves around $\partial\{X = 0\}$:
$$H(\mathcal{F}) = H(X)|_{\{X\neq 0\}} \cup \mathrm{Int}\{X = 0\} \cup (\mathbb{R} \times \partial\{X = 0\})$$
4. action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$:
$$H(\mathcal{F}) = (\mathbb{R}^2 \setminus \{0\})^2 \cup SL(2,\mathbb{R}) \times \{0\}$$
topology: Let $x \in \mathbb{R}^2 \setminus \{0\}$. Then $(\frac{x}{n}, \frac{x}{n}) \in H(\mathcal{F})$ converges to every g in stabilizer group of x... namely to every point of $\mathbb{R}$!

Generalization of Theorem 1

A-Skandalis

Using bi-submersions can construct $C^*(\mathcal{F})$ and longitudinal pseudodifferential calculus!

Outline:

- U is a smooth manifold, so it has lots of smooth functions.
- Just like $H(\mathcal{F}) = (\coprod_i U_i)/\sim$, the convolution algebra is a quotient

$$\mathcal{A} = \oplus_i C_c^\infty(U_i)/\sim \quad \text{(use densities...)}$$

Theorem 1 (A-Skandalis)

Let M be a smooth compact manifold. Let $X_1, \ldots, X_N \in C^\infty(M; TM)$ be smooth vector fields such that $[X_i, X_j] = \sum_{k=1}^N f_{ij}^k X_k$.

Then $\Delta = \sum X_i^* X_i$ is essentially self-adjoint (both in $L^2(M)$ and $L^2(L)$).

Proof

This operator is indeed a regular unbounded multiplier of our C^*-algebra.

Generalization of Theorem 2

Theorem (Skandalis)

Assume that the (dense open) set $\Omega \subset M$ where leaves have maximal dimension is Lebesgue measure 1. Assume the restriction of all leaves to Ω are dense in Ω. Assume that the holonomy groupoid of the restriction of $\mathcal{F}$ to Ω is Hausdorff and amenable. Then Δ_M and Δ_L have the same spectrum.

Proof

- The C^*-algebra $C^*(\Omega, \mathcal{F})$ is simple (Fack-Skandalis) and sits as a two-sided ideal in $C^*(M, \mathcal{F})$.
- $L^2(L)$ and $L^2(M)$ are faithful representations of $C^*(\Omega, \mathcal{F}) \Rightarrow$ weakly equivalent.
- The natural representations of $C^*(M, \mathcal{F})$ to $L^2(L)$ and $L^2(M)$ are extensions to multipliers of faithful representations of $C^*(\Omega, \mathcal{F})$. They are weakly equivalent.

The singular extension of the foliation to the closure M of Ω is used to

What about the spectrum?

Need to know the "shape" of $K_0(C^*(M, \mathcal{F}))$.

Note that for singular foliations:

1. in many cases the holonomy groupoid is longitudinally smooth and restricts to a nice groupoid.
2. leaves of a given dimension:
   locally closed subsets $\longrightarrow$ decomposition series for the C^*-algebra.

Questions

- Is this always the case?
- Give then a formula for the K-theory: Baum Connes conjecture...

Answers...

1. A - M. Zambon: Longitudinal smoothness controlled by "essential isotropy groups" attached to each leaf. When discrete, $H(\mathcal{F})$ longitudinally smooth.
2. Conjecture: Baum-Connes true for $\mathcal{F}$ iff true for each stratum.

Papers

[1] I. A. and G. Skandalis. The holonomy groupoid of a singular foliation. *J. Reine Angew. Math.*, 2009.

[2] I. A. and G. Skandalis. Pseudodifferential Calculus on a singular foliation. *J. Noncomm. Geom.*, 2011.

[3] I. A. and G. Skandalis. The analytic index of elliptic pseudodifferential operators on singular foliations. *J.* K*-theory*, 2011.

[4] I. A. and M. Zambon. Smoothness of holonomy covers for singular foliations and essential isotropy. `arXiv:1111.1327`

[5] I.A. and M. Zambon. Holonomy transformations for singular foliations. `arXiv:1205.6008`