Singular foliations and their holonomy

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Foliations appear in many situations:

- Actions of Lie group(oid)s
- Poisson geometry...

Most foliations are singular.

Aim: understand the "space of leaves" $M/\mathcal{F}$.

Holonomy groupoid $H(\mathcal{F})$ best model for $M/\mathcal{F}$:

1. Desingularizes $\mathcal{F}$...
2. No unnecessary isotropy...

Noncommutative Geometry methods

If $\mathrm{H}(\mathcal{F})$ smooth, can attach $\mathrm{C}^*(\mathcal{F})$. Leaves correspond to ideals. If all leaves are dense, $\mathrm{C}^*(\mathcal{F})$ simple (Fack-Skandalis).

If $\mathrm{H}(\mathcal{F})$ smooth, can attach longitudinal pseudodifferential calculus. Replace leaves with operators... $\mathrm{C}^*(\mathcal{F})$ carries all info about this calculus.

Particularly, obtain a longitudinal Laplacian Δ as an essentially self-adjoint, unbounded multiplier of $\mathrm{C}^*(\mathcal{F})$. Also Scroedinger-type operators $\Delta + f$...

Gaps in spectrum correspond to projections of $\mathrm{C}^*(\mathcal{F})$. Calculations: K-theory, index theory, Baum-Connes map...

The singular case

$\mathrm{H}(\mathcal{F})$ very pathological.

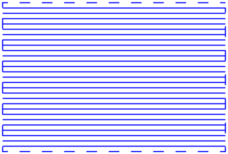
IA-Skandalis: Still were able to construct $C^*(\mathcal{F})$, PSDO calculus, Laplacian...

Constructions "acrobatic"... Mainly due to lack of left-regular representation. Need smoothness of s-fibers (IA-Zambon)

Summary

1.1 Definition: Foliation (regular)

Viewpoint 1:

Partition to connected submanifolds. Local picture: 

In other words: There is an open cover of M by foliation charts of the form $\Omega = U \times T$, where $U \subseteq \mathbb{R}^p$ and $T \subseteq \mathbb{R}^q$.

T is the transverse direction and U is the longitudinal or leafwise direction.

The change of charts is of the form $f(u,t) = (g(u,t), h(t))$.

Viewpoint 2:

Frobenius theorem

Consider the unique $C^\infty(M)$-module $\mathcal{F}$ of vector fields tangent to leaves.

Fact: $\mathcal{F} = C^\infty_c(M, F)$ and $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$.

Holonomy groupoid of a regular foliation
Holonomy

We wish to put a smooth structure on the equivalence relation

$$\{(x, y) \in M^2 : L_x = L_y\}$$

What is the dimension of this manifold?

$p + q$ degrees of freedom for x; then p degrees of freedom for y.

A neighborhood of (x, x') where $x \in W = U \times T$ and $x' \in W' = U' \times T'$ should be of the form $U \times U' \times T$: we need an identification of T with T'. (Here T, T' are local transversals.)

Definition

A holonomy of $(M, \mathcal{F})$ is a diffeomorphism $h : T \to T'$ such that $t, h(t)$ are in the same leaf for all $t \in T$.

Examples of holonomies

- Take $W = U \times T$. Then id_T is a holonomy.
- If h is a holonomy, h^{-1} is a holonomy.
- The composition of holonomies is a holonomy. (Holonomies form a pseudogroup.)
- If $W = U \times T$ and $W' = U' \times T'$, in their intersection $u' = g(u, t)$ and $t' = h(t)$ by definition of the chart changes. The map $h = h_{W',W}$ is a holonomy.
- Let $\gamma : [0, 1] \to M$ be a smooth path in a leaf.
  Cover γ by foliation charts $W_i = U_i \times T_i (1 \leqslant i \leqslant n)$. Consider the composition

$$h(\gamma) = h_{W_n,W_{n-1}} \circ \ldots \circ h_{W_2,W_1}$$

Definition

The holonomy of the path γ is the germ of $h(\gamma)$.

Proposition

The germ of h_γ depends only on the homotopy class of γ!

The holonomy groupoid

Definition

The holonomy groupoid is $H(F) = \{(x, y, h(\gamma))\}$ where γ is a path in a leaf joining x to y.

- Manifold structure. If $W = U \times T$ and $W' = U' \times T'$ are charts and $h : T \to T'$ path-holonomy, get chart

$$\Omega_h = U' \times U \times T$$

- Groupoid structure. $t(x, y, h) = x$, $s(x, y, h) = y$ and $(x, y, h)(y, z, k) = (x, z, h \circ k)$.

$H(F)$ is a Lie groupoid. Its Lie algebroid is F. Its orbits are the leaves.

$H(F)$ is the smallest possible smooth groupoid over $\mathcal{F}$.

Holonomy revisited

Starting from the projective module of vector fields $\mathcal{F}$ the notion of holonomy in the regular case is:

- Pick a path $\gamma : [0,1] \to L$ and $S_{\gamma(0)}, S_{\gamma(1)}$ small transversals of L.
- Path holonomy is (germ of) a local diffeomorphism $S_{\gamma(0)} \to S_{\gamma(1)}$ obtained by "sliding along γ in nearby leaves".
- Namely: Let $X \in \mathcal{F}$ whose flow at $\gamma(0)$ is γ. Now flow X at other points of $S_{\gamma(0)}$ until time 1.
  - $H(F) = \{\text{paths in leaves}\}/\{\text{path holonomy}\}$

Recall: Path holonomy depends only on the homotopy class of γ; get a map

$$h : \pi_1(L, x) \to GermAut_{\mathcal{F}}(S_x; S_x)$$

Its image is H_x^x. It's called the holonomy group of F.

Linearizes to a representation

$$dh : \pi_1(L, x) \to GL(N_x L)$$

Path holonomy in the singular case fails!

Orbits of the rotations action in $\mathbb{R}^2$: $\mathcal{F} = span < x\partial_y - y\partial_x >$.

- Take γ the constant path at the origin.
- A transversal S_0 is just an open neighborhood of the origin in $\mathbb{R}^2$.

Realize γ either by integrating the zero vector field or $x\partial_y - y\partial_x$ at the origin. We get completely different diffeomorphisms of S_0!

Here $\mathcal{F}$ is projective as well!

But there are lots of non-projective examples... Think of a vector field X whose interior of $\{x \in M : X(x) = 0\}$ is non-empty...

Holonomy map cannot be defined on $\pi_1(L)$ in the singular case... What about the holonomy groupoid?

Debord showed that a projective $\mathcal{F}$ always has a smooth holonomy groupoid.

Stability for regular foliations

Local Reeb stability theorem

If L is a compact embedded leaf and H_x^x is finite then nearby L the foliation F is isomorphic to its linearization.

Namely, around L the manifold looks like

$$\frac{\widetilde{L} \times \mathbb{R}^q}{\pi_1(L)}$$

$\pi_1(L)$ acts diagonally by deck transformations and linearized holonomy.

This is equal to

$$\frac{H_x \times N_x L}{H_x^x}$$

The action of H_x^x on $N_x L$ is the one that integrates the Bott connection

$$\nabla : F \to CDO(N), \quad (X, \langle Y \rangle) \to \langle [X, Y] \rangle$$

The singular case

- What is the notion of holonomy in the singular case?
- Is there any sense in which the holonomy groupoid of a singular foliation is smooth?
- When is a singular foliation isomorphic to its linearization?

Stefan-Sussmann foliations

Definition (Stefan, Sussmann, A-Skandalis)

A (singular) foliation is a finitely generated sub-module $\mathcal{F}$ of $C^\infty_c(M; TM)$, stable under brackets.

No longer projective. Fiber $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$: upper semi-continuous dimension.

One may still define leaves (Stefan-Sussmann).

Let L be a leaf and $x \in L$. There is a short exact sequence of vector spaces

$$0 \to \mathfrak{g}_x \to \mathcal{F}_x \stackrel{ev_x}{\to} T_x L \to 0$$

where ev_x is evaluation at x. Get a transitive Lie algebroid

$$A_L = \cup_{x \in L} \mathcal{F}_x, \quad \text{with} \quad \Gamma A_L = \mathcal{F}/I_L\mathcal{F}$$

"Regular" leaves = leaves of maximal dimension.

On regular leaves $\mathfrak{g}_x = 0$.

Examples

Actually: Different foliations may yield same partition to leaves

1. $\mathbb{R}$ foliated by 3 leaves: $(-\infty, 0), \{0\}, (0, +\infty)$.
   $\mathcal{F}$ generated by $x^n \frac{\partial}{\partial x}$. Different module $\mathcal{F}$ for every n.
   $\mathfrak{g}_0 = \mathbb{R}$ in every case.

2. If G acts linearly on a vector space V and $\mathcal{F}$ is the image of the infinitesimal action, then $\mathfrak{g}_0 = \mathit{Lie}(G)$.

3. $\mathbb{R}^2$ foliated by 2 leaves: $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$.
   No obvious best choice. $\mathcal{F}$ given by the action of a Lie group
   $$GL(2, \mathbb{R}), SL(2, \mathbb{R}), \mathbb{C}^*$$

Extra difficulty: Keep track of the choice of $\mathcal{F}$!

Bi-submersions

Need a stable way to keep track of (path) holomies associated with a particular choice of $\mathcal{F}$.

Answer: Bi-submersions. Think of them as covers of open subsets of the holonomy groupoid $H(\mathcal{F})$. Explicitly:

- Let $X_1, \ldots, X_n$ local generators of $\mathcal{F}$.
- Let $U \subseteq M \times \mathbb{R}^n$ a neighborhood where the map
$$t : U \to M, t(y, \xi) = exp(\sum_{i-1}^{n} \xi_i X_i)(y)$$
is defined.
- Put $s : U \to M$ the projection.
- The triple (U, t, s) is a path holonomy bi-submersion.

Indeed (U, t, s) keeps track of path holonomies near the identity:

$$\text{bisections of } (U, t, s) \rightsquigarrow \text{path holonomies}$$

Passing to germs

Cover M with a family $\{(U_i, t_i, s_i)\}_{i \in I}$. Let $\mathcal{U}$ be the family of all finite products of $\{(U_i, t_i, s_i)\}_{i \in I}$ and of their inverses.

Holonomy groupoid (A-Skandalis)

The holonomy groupoid is

$$H(\mathcal{F}) = \coprod_{U \in \mathcal{U}} U / \sim$$

where $U \ni u \sim u' \in U'$ iff there is a morphism of bi-submersions $f : U \to U'$ (defined near u) such that $f(u) = u'$.

$H(\mathcal{F})$ is a topological groupoid over M, usually not smooth.

Examples

1. (Almost) regular case: $H(\mathcal{F})$ usual holonomy groupoid.
2. Action of S^1 on $\mathbb{R}^2$ by rotations: H is the transformation groupoid $M \times S^1$.
3. $\mathcal{F} = \rho(AG)$: $H(\mathcal{F})$ is a quotient of G.
4. $\mathcal{F} =< X >$ s.t. X has non-periodic integral curves around $\partial\{X = 0\}$:

$$H(\mathcal{F}) = H(X)|_{\{X \neq 0\}} \cup Int\{X = 0\} \cup (\mathbb{R} \times \partial\{X = 0\})$$

5. action of $SL(2, \mathbb{R})$ on $\mathbb{R}^2$:

$$H(\mathcal{F}) = (\mathbb{R}^2 \setminus \{0\})^2 \cup SL(2, \mathbb{R}) \times \{0\}$$

topology: Let $x \in \mathbb{R}^2 \setminus \{0\}$. Then $(\frac{x}{n}, \frac{x}{n}) \in H(\mathcal{F})$ converges to every g in stabilizer group of x... namely to every point of $\mathbb{R}$!

Integrating A_L

All the previous examples have smooth s-fibers! Is this always the case?
Equivalently, is A_L always integrable?

Let G_x the connected and simply connected Lie group integrating $\mathfrak{g}_x$.
Near the identity, consider the map

$$\widetilde{\varepsilon}_x : G_x \to U_x^x, \quad exp_{\mathfrak{g}_x}(\sum_{i=1}^{n} \xi_i [X_i]) \mapsto exp(\sum_{i=1}^{n} \xi_i Y_i)$$

where $Y_i \in C^\infty(U; \ker ds)$ are vertical lifts of the X_is.
Composing with $\sharp : U_x^x \to H_x^x$ we get a morphism

$$\varepsilon_x : G_x \to H_x^x$$

$\ker \varepsilon_x$ is the essential isotropy group of the leaf L_x.

Theorem (A-Zambon)

The transitive Lie groupoid H_L is smooth and integrates A_L if and only if the essential isotropy group of L is discrete.

Relation with monodromy

Lemma (A-Zambon and Duistermaat-Kolk)

If $\ker \varepsilon_x$ is discrete then it lies in ZG_x.

It follows that $\varepsilon_x : G_x \to H_x^x$ is a cover of the Lie group H_x^x and

$$\ker \varepsilon_x = \pi_1(H_x^x)$$

Crainic and Fernandes showed that when A_L is integrable then there is an s-simply connected Lie groupoid Γ with $A\Gamma = A_L$.

"Monodromy group": $\mathcal{N}_x(A_L) = \ker(G_x \to \Gamma_x^x)$ induced by $\mathfrak{g}_x \to A_L$.

Integrating $\mathrm{Id} : A_L \to A_L$ provides a morphism $\Gamma_x^x \to H_x^x$ and ε factors as

$$G_x \to \Gamma_x^x \to H_x^x$$

whence

$$\mathcal{N}_x(A_L) \subset \ker \varepsilon$$

We do not yet know how the isotropy and monodromy groups are related in general...

A discreteness criterion

Essential isotropy is very hard to compute. However, we were able to show the following:

Let S_x be a slice to a leaf L_x (at x). There is a "splitting theorem" for $\mathcal{F}$, namely S_x is naturally endowed with a "transversal" foliation $\mathcal{F}_{S_x}$.

Theorem (A-Zambon)

Assume that for any time-dependent vector field $\{X_t\}_{t\in[0,1]} \in I_x\mathcal{F}_{S_x}$ there exists a vector field $Z' \in I_x\mathcal{F}_{S_x}$ and a neighborhood S' of x in S_x such that $exp(Z)\mid_{Z'}$ is the time-1 flow of $\{X_t\}_{t\in[0,1]}$.

Guess: Condition satisfied whenever $\mathcal{F}$ is closed as a Frechet space...

(This rules out the *extremely* singular cases...)

The holonomy map

Let $(M, \mathcal{F})$ a singular foliation, L a leaf, $x, y \in L$ and S_x, S_y slices of L at x, y respectively.

Theorem (A-Zambon)

There is a well defined map

$$\Phi_x^y : H_x^y \to \frac{\mathrm{GermAut}_{\mathcal{F}}(S_x, S_y)}{\exp(I_x\mathcal{F}) \mid_{S_x}}, h \mapsto \langle\tau\rangle$$

where τ is defined as

- pick any bi-submersion (U, t, s) and $u \in U$ with $[u] = h$
- pick any section $b : S_x \to U$ of s through u such that $(t \circ b)S_x \subseteq S_y$

and define $\tau = t \circ b : S_x \to S_y$.

It defines a morphism of groupoids

$$\Phi : H \to \cup_{x,y} \frac{\mathrm{GermAut}_{\mathcal{F}}(S_x, S_y)}{\exp(I_x\mathcal{F}) \mid_{S_x}}$$

Holonomy map and the Bott connection

Conjecture: Φ is injective.
(Proven at points x where $\mathcal{F}$ vanishes and for regular foliations.)

If $\mathcal{F}$ is regular then $exp(I_x\mathcal{F}) \mid_{S_x} = \{\mathrm{Id}\}$, so we recover the usual holonomy map.

Let L be a leaf with discrete essential isotropy.

1. The derivative of τ gives

$$\Psi_L : H_L \to Iso(NL, NL)$$

   Lie groupoid representation of H_L on NL;

2. Differentiating Ψ_L gives

$$\nabla^{L,\perp} : A_L \to Der(NL)$$

   It is the Bott conection...

All this justifies the terminology "holonomy groupoid"!

Linearization

Vector field on M tangent to $L \rightsquigarrow$
Vector field Y_{lin} on NL, defined as follows:

$$\begin{gathered} Y_{lin} \text{ acts on the fibrewise constant functions as } Y\,|_L \\ Y_{lin} \text{ acts on } C^\infty_{lin}(NL) \equiv I_L/I_L^2 \text{ as } Y_{lin}[f] = [Y(f)]. \end{gathered}$$

The linearization of $\mathcal{F}$ at L is the foliation $\mathcal{F}_{lin}$ on NL generated by

$$\{Y_{lin} : Y \in \mathcal{F}\}$$

Lemma

Let L be an embedded leaf such that $\ker \varepsilon$ is discrete. Then $\mathcal{F}_{lin}$ is the foliation induced by the Lie groupoid action Ψ_L of H_L on NL.

We say $\mathcal{F}$ is linearizable at L if there is a diffeomorphism mapping $\mathcal{F}$ to $\mathcal{F}_{lin}$.

For $\mathcal{F} = \langle X \rangle$ with X vanishing at $L = \{x\}$ linearizability means:

There is a diffeomorphism taking X to fX_{lin} for a non-vanishing function f.

This is a weaker condition than the linearizability of the vector field X!

Question: When is a singular foliation isomorphic to its linearization?

We don't know yet, but:

Proposition (A-Zambon)

Let L_x embedded leaf with discrete essential isotropy. Assume H^x_x compact.

The following are equivalent:

1. $\mathcal{F}$ is linearizable about L
2. there exists a tubular neighborhood U of L and a (Hausdorff) Lie groupoid $G \to U$, proper at x, inducing the foliation $\mathcal{F}\,|_U$.

In that case:

- G can be chosen to be the transformation groupoid of the action Ψ_L of H_L on NL.

- $(U, \mathcal{F}\,|_U)$ admits the structure of a singular Riemannian foliation.

Papers

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[2] I. A. and M. Zambon Smoothness of holonomy covers for singular foliations and essential isotropy. `arXiv:1111.1327`

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[4] I. A. and G. Skandalis. Pseudodifferential Calculus on a singular foliation. *J. Noncomm. Geom.*, 2011.

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[6] I.A. Laplacians and spectrum for singular foliations. (Submitted.)

Thank you Alberto!