Absorbing Boundary Conditions Derived Based on Pauli Matrices Algebra

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***Abstract*—In this letter, we demonstrate that a set of absorbing boundary conditions (ABCs) for numerical simulations of waves, proposed originally by Engquist and Majda and later generalized by Trefethen and Halpern, can alternatively be derived with the use of Pauli matrices algebra. Hence a novel approach to the derivation of one-way wave equations in electromagnetics is proposed. That is, the classical wave equation can be factorized into two two-dimensional wave equations with first-order time derivatives. Then, using suitable approximations, not only Engquist and Majda ABCs can be obtained, but also generalized ABCs proposed by Trefethen and Halpern, which are applicable to simulations of radiation problems.**

***Index Terms*—Electromagnetic field theory, Computational electromagnetics, Maxwell equations, Finite difference methods.**

I. Introduction

In 1977, Engquist and Majda proposed a set of absorbing boundary conditions (ABCs) for numerical simulations of waves [1], [2]. These are partial-differential equations, also called one-way wave equations, allowing for wave propagation only in a certain direction. When such an ABC is applied at the boundary of a computational domain, a one-way wave equation numerically absorbs the outgoing waves. Therefore, Engquist and Majda ABCs were subsequently discretized by Mur [3] and applied to the finite-difference time-domain (FDTD) method [4] at the boundaries of the computational domain in simulations of radiation problems. Subsequently, these ABCs were generalized by Trefethen and Halpern [5], [6] to reduce further reflections from local ABCs owing to the optimal choice of their parameters. Later on, Higdon [7], [8] proposed a set of differential operators perfectly annihilating the outgoing waves for the assumed incidence angles, which generalizes Trefethen and Halpern as well as higher-order ABCs. These analytical ABCs, based on one-way wave equations, have been widely applied to FDTD simulations for many years before the technique of perfectly matched layer (PML) [9] became available.

Although the topic of analytical ABCs [10] may currently seem not to be so important, we investigated it and found that the factorization of the wave equation with the use of Pauli matrices algebra [11] allows one to decompose it into two two-dimensional (2-D) wave equations with first-order time derivatives. Such factorization was already found useful for an alternative derivation of the Dirac equation in quantum mechanics [12]. Then, using suitable approximations (i.e., the wave incidence is almost normal to ABC, or the field vector and the wavevector are not orthogonal within ABC as it can occur in FDTD), one-way wave equations, which are applicable as ABCs, can be derived. Hence, in this letter, we describe novel aspects of one-way wave equations in electromagnetics. Then, we derive a general set of ABCs, already proposed by Trefethen and Halpern for numerical simulations of wave propagation, which includes Engquist and Majda ABCs as special cases. It is worth noting that in the seminal paper of Engquist and Majda [2], the Authors cannot provide physical reasoning resulting in the obtained ABCs. In our approach, we employ wave-equation factorization and approximations which allow for derivation of their ABCs.

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II. Preliminaries

Let us introduce some notation. We consider the electric- and magnetic-field vectors defined in a Cartesian coordinate system as $\mathbf{E} = [E_x \quad E_y \quad E_z]$ and $\mathbf{H} = [H_x \quad H_y \quad H_z]$, respectively. We develop a strategy for the derivation of analytical ABCs which includes Engquist and Majda ABCs as special cases. Because these ABCs were originally proposed for fluid simulations, we keep our considerations general and consider the functions: U, U_a and U_b. Depending on the simulation scenario in electromagnetics, these functions are the electric- and magnetic-field components tangential to ABCs.

For the function of space $g\colon \mathbb{R}^3 \to \mathbb{C}$ (i.e., $g(\mathbf{r})$ where $\mathbf{r} = [\,x \quad y \quad z\,]$), the spatial Fourier transformation is defined

$$\mathcal{F}(g)(\mathbf{k}) = \int_{\mathbb{R}^3} e^{-i\mathbf{k}\cdot\mathbf{r}} g(\mathbf{r}) \mathrm{d}^3 r \tag{1}$$

where $\mathbf{k} = [\,k_x \quad k_y \quad k_z\,]$ is the wavevector. For the sake of brevity, the same symbols are employed to denote the Fourier transforms in the wavevector domain.

In our analysis, the Pauli matrices [11] are employed

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2}$$

For the set of matrices (2), one obtains $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix. Let us define the Pauli vector $\boldsymbol{\sigma} = [\sigma_x \quad \sigma_y \quad \sigma_z]$. Then, one can calculate the dot product of the wavevector $\mathbf{k}$ and the Pauli vector $\boldsymbol{\sigma}$

$$\mathbf{k}\cdot\boldsymbol{\sigma} = k_x\sigma_x + k_y\sigma_y + k_z\sigma_z = \begin{bmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{bmatrix}. \tag{3}$$

Let us consider the 3-D wave equation in the Cartesian space

$$GU = (\partial_{xx} + \partial_{yy} + \partial_{zz} - \partial_{\tau\tau})U = 0 \tag{4}$$

where $U = U(x, y, z, \tau)$ is the scalar solution, $\tau = ct$, $t \in \mathbb{R}$ denotes the time variable, and c denotes the wave-propagation velocity (i.e., the velocity of light). We denote the order of differentiation as follows: $\partial_{xz\tau} = \partial_x \partial_z \partial_\tau$. However, we also assume that the solutions to the wave equation (4) and the considered problem of ABC are smooth functions, for which the order of partial derivatives can be changed. Because we use mixed partial derivatives up to the third order, based on Schwartz's theorem, it is sufficient that these functions are in $\mathcal{C}^3$ (i.e., all the third-order partial derivatives exist for them and are continuous). Because we assume that the order of partial derivatives can be changed, and $\tau = ct$, our derivations are valid only in a homogeneous domain where c does not depend on either a spatial or a temporal variable.

III. Engquist and Majda ABCs

Let us consider ABC on the wall $z = 0$ (refer to Fig. 1) for the 2-D wave equation

$$GU = (\partial_{xx} + \partial_{zz} - \partial_{\tau\tau})U = 0 \tag{5}$$

where $U = U(x, z, t)$ is the scalar solution which we want to absorb at the boundary. In the considered 2-D domain Ω, the

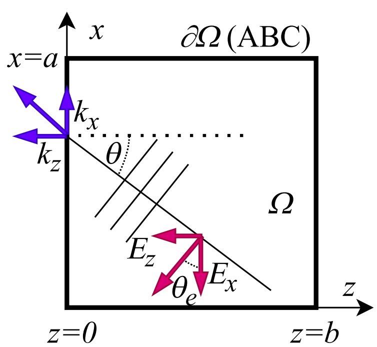

Fig. 1. Incidence of plane wave at boundary $z = 0$ of domain Ω.

function U is either the field component E_x or H_y tangential to ABC. Then, (5) can be *symbolically* factorized

$$GU = G^+ G^- U = 0 \tag{6}$$

where

$$G^+ = \partial_z + \sqrt{\partial_{\tau\tau} - \partial_{xx}} \qquad G^- = \partial_z - \sqrt{\partial_{\tau\tau} - \partial_{xx}}. \tag{7}$$

The operators G^+ and G^- describe the one-way wave propagation towards $+z$ and $-z$ respectively. Hence G^- provides ABC across $z = 0$. Using 1st-, 2nd- and 3rd-order approximations for the square root, one obtains, respectively, the following one-way wave equations for the $-z$ direction:

$$G^-|_{z=0} U = (\partial_z - \partial_\tau)|_{z=0} U = 0 \tag{8}$$

$$G^-|_{z=0} U = \left(\partial_{z\tau} - \partial_{\tau\tau} + \frac{1}{2}\partial_{xx} \right)\Big|_{z=0} U = 0 \tag{9}$$

$$G^-|_{z=0} U = \left(\partial_{z\tau\tau} - \partial_{\tau\tau\tau} - \frac{1}{4}\partial_{zxx} + \frac{3}{4}\partial_{\tau xx} \right)\Big|_{z=0} U = 0. \tag{10}$$

Equation (8) stems from the factorization of 1-D wave equation along the z direction. Equations (9)–(10) can be considered as subsequent differentiations of (8) with respect to the time τ, which include the terms increasing the ABC accuracy.

In the next step, the one-way wave equations (9)–(10) of Engquist and Majda were generalized by Trefethen and Halpern [5], [6]. Their approach relies on the approximation of the square root in G^+ and G^- formulas by a rational function. Then, one obtains the 2nd- and 3rd-order ABCs as follows:

$$G^-|_{z=0} U = (\partial_{z\tau} - p_0 \partial_{\tau\tau} - p_2 \partial_{xx})|_{z=0} U = 0 \tag{11}$$

$$G^-|_{z=0} U = (q_0 \partial_{z\tau\tau} - p_0 \partial_{\tau\tau\tau} + q_2 \partial_{zxx} - p_2 \partial_{\tau xx})|_{z=0} U = 0. \tag{12}$$

The coefficients p_0, p_2, q_0, q_2 are real numbers depending on incidence angles for which ABC provides an exact absorption. The values of these angles can be found in [4, Tables 6.1 and 6.2]. In general $p_0 \approx 1$ and $q_0 \approx 1$ because the first two terms in (11)–(12) correspond to the ones in the one-way wave equation (8) differentiated with respect to the time τ.

Finally, Higdon's operators

$$G^-|_{z=0} U = \prod_{l=0}^{L} (\partial_z - \cos\alpha_l \partial_\tau)|_{z=0} U = 0 \tag{13}$$

generalize (8)–(12) and exactly absorb the plane waves propagating at specific angles α_l $(l = 0, ..., L)$. Substituting $L = 2$ in (13), one obtains the coefficients p_0 and p_2 in the 2nd-order Trefethen and Halpern ABC (11) (cf. [4, Eq. (6.52c)])

$$p_0 = \frac{1 + \cos\alpha_1 \cos\alpha_2}{\cos\alpha_1 + \cos\alpha_2} \quad p_2 = -\frac{1}{\cos\alpha_1 + \cos\alpha_2}. \tag{14}$$

IV. Factorization Based on Pauli Matrices

Let us consider the two field components U_a and U_b, which satisfy the wave equation (4). We assume that U_a is the main field component which we want to absorb at the boundary, whereas U_b is employed to help with this task. Both field components U_a and U_b are part of the same solution to Maxwell's equations. For the transverse-magnetic (TM) propagation in Fig. 1, U_a and U_b correspond to E_x and E_z respectively. Then, one can write

$$(G\mathbf{I}) \begin{bmatrix} U_a \\ U_b \end{bmatrix} = 0 \tag{15}$$

where $\mathbf{I}$ is the identity matrix of size 2×2. Then, transforming (15) into the wavevector domain, one obtains

$$\left(\mathbf{I}\partial_{\tau\tau} + \mathbf{I}k^2 \right) \begin{bmatrix} U_a \\ U_b \end{bmatrix} = 0 \tag{16}$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$. This equation can be written as

$$\left(\mathbf{I}\partial_\tau + ik\sqrt{\mathbf{I}} \right) \left(\mathbf{I}\partial_\tau - ik\sqrt{\mathbf{I}} \right) \begin{bmatrix} U_a \\ U_b \end{bmatrix} = 0. \tag{17}$$

The set of solutions to $\sqrt{\mathbf{I}}$ is infinite; hence, in our derivations, only the solutions obtainable with the use of Pauli matrices are considered. Since $(\mathbf{k} \cdot \boldsymbol{\sigma})^2 = (\mathbf{k} \cdot \boldsymbol{\sigma})(\mathbf{k} \cdot \boldsymbol{\sigma}) = k^2\mathbf{I}$, one obtains

$$\left(\mathbf{I}\partial_\tau + i\mathbf{k} \cdot \boldsymbol{\sigma} \right) \left(\mathbf{I}\partial_\tau - i\mathbf{k} \cdot \boldsymbol{\sigma} \right) \begin{bmatrix} U_a \\ U_b \end{bmatrix} = 0. \tag{18}$$

Because the operators $(\mathbf{I}\partial_\tau + i\mathbf{k}\cdot\boldsymbol{\sigma})$ and $(\mathbf{I}\partial_\tau - i\mathbf{k}\cdot\boldsymbol{\sigma})$ commute, one can change their order in (18). Then, if either

$$(\mathbf{I}\partial_\tau + i\mathbf{k}\cdot\boldsymbol{\sigma})\begin{bmatrix}U_a\\U_b\end{bmatrix} = 0 \quad \text{or} \quad (\mathbf{I}\partial_\tau - i\mathbf{k}\cdot\boldsymbol{\sigma})\begin{bmatrix}U_a\\U_b\end{bmatrix} = 0 \tag{19}$$

is satisfied, then the wave equation (18) is satisfied as well. Such a factorization strategy can be applied to other classical equations in physics (e.g., the diffusion equation), but it can result in factors with fractional-order time derivatives [13].

V. ABCs Based on Derived Factorization

Let us consider ABC for the 2-D wave propagation as in Fig. 1. Then, $k_y = 0$ and (19) implies that the following two 2-D matrix wave equations are satisfied at ABC:

$$\begin{bmatrix}\partial_\tau + ik_z & ik_x\\ ik_x & \partial_\tau - ik_z\end{bmatrix}\begin{bmatrix}U_a\\U_b\end{bmatrix} = 0 \tag{20}$$

$$\begin{bmatrix}\partial_\tau - ik_z & -ik_x\\ -ik_x & \partial_\tau + ik_z\end{bmatrix}\begin{bmatrix}U_a\\U_b\end{bmatrix} = 0. \tag{21}$$

We employed the orientation of axes in the considered problem of ABC to ultimately obtain spatial derivatives without complex coefficients in differential equations. Therefore, we can respectively transfer (20)–(21) into the spatial domain

$$\begin{bmatrix}\partial_\tau + \partial_z & \partial_x\\ \partial_x & \partial_\tau - \partial_z\end{bmatrix}\begin{bmatrix}U_a\\U_b\end{bmatrix} = 0 \tag{22}$$

$$\begin{bmatrix}\partial_\tau - \partial_z & -\partial_x\\ -\partial_x & \partial_\tau + \partial_z\end{bmatrix}\begin{bmatrix}U_a\\U_b\end{bmatrix} = 0. \tag{23}$$

In (22), the function U_a satisfies the one-way wave equation in the $+z$ direction, corrected by the term $\partial_x U_b$, whereas the function U_b satisfies the one-way wave equation in the $-z$ direction, corrected by the term $\partial_x U_a$. On the other hand, in (23), the function U_a satisfies the one-way wave equation in the $-z$ direction, corrected by the term $-\partial_x U_b$, whereas the function U_b satisfies the one-way wave equation in the $+z$ direction, corrected by the term $-\partial_x U_a$. Because we aim to derive ABCs for $z = 0$, we focus afterwards on (23) and the absorption of the component U_a. Hence we further consider the system of equations as follows:

$$(\partial_\tau - \partial_z)U_a - \partial_x U_b = 0 \tag{24}$$

$$-\partial_x U_a + (\partial_\tau + \partial_z)U_b = 0. \tag{25}$$

From this point on, we assume that $U_a = E_x$ and $U_b = E_z$, i.e., we restrict our considerations to the 2-D configuration in Fig. 1. Additional to (24)–(25) in electromagnetics, Gauss's law in free space without charges (i.e., $\partial_x E_x + \partial_z E_z = 0$) remains a constraint for the plane-wave propagation. This enforces the orthogonality of the Fourier-transformed electric-field vector and the wavevector (i.e., $k_x E_x + k_z E_z = 0$). This also means that $\tan\theta_e = E_z/E_x = -k_x/k_z = \tan\theta$ (where $k_z < 0$, see Fig. 1), hence $\theta_e = \theta$. In numerical methods, such as FDTD, the condition $\theta_e = \theta$ is not valid in general [14]–[16]. That is, $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{k}$ in the discrete FDTD domain do not form a mutually orthogonal set. Therefore, we introduce the real parameter $p = \tan\theta_e/\tan\theta$ which describes the relation between the angles θ_e and θ

$$\tan\theta_e = \frac{E_z}{E_x} = -p\frac{k_x}{k_z} = p\tan\theta \tag{26}$$

at the boundary $z = 0$ for a single frequency (note that $k_z < 0$ in Fig. 1). For the normal wave incidence at ABC (i.e., $\theta = 0°$), one obtains $k_x = 0$ and $E_z = 0$ (i.e., $\theta_e = 0°$) independent of the p value. If $\theta_e = 0°$, then $E_z = 0$. This case is satisfied when (i) $p = 0$ or (ii) $p \neq 0$ and $k_x = 0$ (i.e., $\theta = 0°$). Assuming the perfect plane-wave propagation in the continuous domain, one obtains $\theta = \theta_e$ and $p = 1$ due to Gauss's law in free space without charges. Otherwise, the parameter $p \neq 1$ can provide a matching between the dispersion relations valid for ABC and the wave equation. To some extent, (26) implies Gauss's law within ABC in the form $p\partial_x E_x + \partial_z E_z = 0$, which can be written as $\partial_x E_x + \partial_z E_z = (1 - p)\partial_x E_x$ where the right-hand side denotes the charge depending on the derivative $\partial_x E_x$. Based on (26), we propose the additional condition (i.e., assumption)

$$p\,\partial_x U_a + \partial_z U_b = 0 \tag{27}$$

which is a constraint for (24)–(25) in our ABC derivations. In this case, one obtains from (25) that

$$\partial_\tau U_b = (1 + p)\partial_x U_a. \tag{28}$$

Because $\tau = ct$, the constant coefficient $(1 + p)$ before the spatial derivative $\partial_x U_a$ can also be considered as a tuning parameter for the wave-propagation velocity. In numerical simulations such as FDTD, the wave-propagation velocity is different from c and depends on the direction of propagation.

Now we are going to show that certain approximations (i.e., the wave incidence is almost normal to ABC, or the field vector and the wavevector are not orthogonal within ABC) applied to (24)–(25) lead to Trefethen and Halpern ABC.

A. 1st-order ABC

Let us consider an almost normal wave incidence at ABC (i.e., $\theta \approx 0°$). One obtains $U_b = E_z = 0$ which implies $\partial_x U_b = 0$. Then, one obtains from (24) the one-way wave equation for the function U_a

$$G^-|_{z=0}\, U_a = (\partial_z - \partial_\tau)|_{z=0}\, U_a = 0 \tag{29}$$

which is equivalent to (8) proposed by Engquist and Majda. Equation (29) appears in various methods of computational electromagnetics when the plane-wave approximation is applicable. For instance, the one-way wave equation can be demonstrated as an outgoing wave in the discontinuous Galerkin pseudospectral time-domain method, during the Riemann problem solving procedure [17], [18].

B. 2nd-order ABC

Let us assume that $\partial_z U_b = \partial_z E_z \approx 0$, i.e., the component of the electric field normal to ABC does not vary significantly along the direction normal to the boundary around $z = 0$. Thus one obtains from (25) that

$$\partial_\tau U_b = \partial_x U_a \tag{30}$$

which is equivalent to (28) for $p = 0$. Then, we differentiate (24) with respect to the time τ and obtain

$$(\partial_{\tau\tau} - \partial_{\tau z})U_a - \partial_{\tau x}U_b = 0. \quad (31)$$

Because we assumed that the order of partial derivatives can be changed for the solutions of the wave equation (4), one finally obtains ABC at $z = 0$, i.e.,

$$G^-|_{z=0}\, U_a = (\partial_{z\tau} - \partial_{\tau\tau} + \partial_{xx})|_{z=0}\, U_a = 0. \quad (32)$$

It has the form of Trefethen and Halpern ABC (11) with $p_0 = 1$ and $p_2 = -1$. As one can note, it is associated with the Newman-points approximation with the angles of the exact absorption equal to $0°$ and $\pm 90°$ (cf. [4, Table 6.1]).

Let us generalize the result presented above substituting (28) into (31). Hence one obtains ABC at $z = 0$, i.e.,

$$G^-|_{z=0}\, U_a = [\partial_{z\tau} - \partial_{\tau\tau} + (1 + p)\partial_{xx}]|_{z=0}\, U_a = 0. \quad (33)$$

Equation (33) has the general form of Trefethen and Halpern ABC (11) with $p_0 = 1$ and $p_2 = -(1+p)$ as the varying values of the coefficient p allow for changing the angles of the exact absorption in this ABC (cf. [4, Table 6.1]). Moreover, for $p = 0$, one obtains the analysed before case: $\partial_z U_b = \partial_z E_z \approx 0$ represented by ABC (32).

C. 3rd-order ABC

Let us differentiate (31) again with respect to the time τ

$$(\partial_{\tau\tau\tau} - \partial_{\tau\tau z})U_a - \partial_{\tau\tau x}U_b = 0. \quad (34)$$

Then, we differentiate (25) twice with respect to the spatial variable x and the time τ, and obtain

$$\partial_{\tau\tau x}U_b = \partial_{\tau xx}U_a - \partial_{\tau xz}U_b. \quad (35)$$

In the next step, after substituting (35) into (34), one obtains

$$(\partial_{\tau\tau\tau} - \partial_{\tau\tau z} - \partial_{\tau xx})U_a + \partial_{\tau xz}U_b = 0. \quad (36)$$

Let us assume that $\partial_z U_b = \partial_z E_z \approx 0$ as in the first part of Subsection V-B. Then, based on (25), one obtains (30). Hence the following ABC formula is obtained from (36) :

$$G^-|_{z=0}\, U_a = (\partial_{z\tau\tau} - \partial_{\tau\tau\tau} - \partial_{zxx} + \partial_{\tau xx})|_{z=0}\, U_a = 0. \quad (37)$$

This is Trefethen and Halpern ABC (12) with $p_0 = 1$, $p_2 = -1$, $q_0 = 1$ and $q_2 = -1$. Some values of these coefficients are available in [4, Table 6.2] for varying angles of the exact absorption, but those obtained above are not available there.

Let us finally assume the validity of the formula (27). One obtains (28), which is substituted into (36), and then the following formula for ABC is obtained:

$$\begin{aligned} &G^-|_{z=0}\, U_a = \\ &[\partial_{z\tau\tau} - \partial_{\tau\tau\tau} - (1 + p)\partial_{zxx} + \partial_{\tau xx}]|_{z=0}\, U_a = 0. \end{aligned} \quad (38)$$

It has the general form of Trefethen and Halpern ABC (12) with $p_0 = 1$, $p_2 = -1$, $q_0 = 1$, and $q_2 = -(1 + p)$. As one can note in [4, Table 6.2], the coefficient $q_0 = 1$ whereas $p_0 \approx 1$ and $p_2 \approx -1$. This stems from the fact that the first two terms in (38) correspond to the ones in the one-way wave equation (8) twice differentiated with respect to the time τ. Assuming that the wave-propagation velocity is different from c in numerical simulations, one can introduce the variable coefficient $-p_2$ before the derivative $\partial_{\tau xx}$ in (38) to obtain a better level of absorption of incident waves at ABC. Such an equation is a general Trefethen and Halpern ABC (12).

To sum up, we can derive Engquist and Majda ABCs and the generalized ABCs proposed by Trefethen and Halpern in an alternative way. The reflection error is at the level of 0.1-1% for these ABCs, as reported in [4]. Currently, ABCs based on PMLs can provide an arbitrarily small reflection error. Furthermore, PMLs can terminate domains comprising inhomogeneous, dispersive, anisotropic, and nonlinear media. Therefore, the presented results can fill the gap in understanding ABCs, rather than propose analytical ABCs better than the state of the art in 2024.

VI. Verification

Let us consider the incidence of the harmonic plane wave at ABC (33) as in Fig. 1, i.e., $U_a = U_{a0}e^{i(k_x x + k_z z - \omega\tau)}$ and $U_b = U_{b0}e^{i(k_x x + k_z z - \omega\tau)}$ (where $k_z < 0$ and $\omega > 0$). The substitution of these formulas into the wave equation (5) and the system (24)–(25) provides the same dispersion relation

$$\omega^2 = k_x^2 + k_z^2. \quad (39)$$

However, the substitution of the plane-wave formula into (33) provides the following dispersion relation:

$$\omega^2 + k_z\omega - (1 + p)k_x^2 = 0. \quad (40)$$

For the angles of the exact absorption, the wavevector length ω should be the same for free space and ABC, i.e., calculated based on (39) and (40). Hence one obtains from (39) and (40)

$$k_z\sqrt{k_x^2 + k_z^2} + k_z^2 - pk_x^2 = 0. \quad (41)$$

Because $k_z < 0$, one obtains the first solution $k_x = 0$ which implies $\theta = 0°$. The second solution can be derived taking into account the fact that $\tan\theta = -k_x/k_z$. The condition $\omega > 0$ implies that $p < \tan^{-2}\theta$, which allows for selecting the correct solution in terms of p, providing the exact absorption

$$p = -\frac{1}{1 + \sqrt{1 + \tan^2\theta}}. \quad (42)$$

Taking into account the fact that $p_0 = 1$ and $p_2 = -(1 + p)$ in (33), one obtains from (42) the Higdon formula (14) when $\alpha_1 = 0°$ and $\alpha_2 = \theta$. It demonstrates the logical consistency of our ABC derivations.

VII. Conclusion

We demonstrate that the factorization of the wave equation with the use of Pauli matrices algebra allows one to decompose it into two 2-D wave equations with first-order time derivatives. Then, assuming that the wave incidence is almost normal to ABC or the field vector and the wavevector are not orthogonal within ABC, as it can occur in FDTD, we derive in a systematic way a general set of ABCs already proposed by Trefethen and Halpern, which includes Engquist and Majda ABCs as special cases. The presented theory is logically consistent because it allows one to obtain exactly the same absorption angles as in Higdon's ABC theory.

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