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Gauss–Kronrod quadrature formulae for weight functions of Bernstein–Szegö type *

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Abstract: We study the Kronrod extensions of Gaussian quadrature rules whose weight functions on $[-1, 1]$ consist of any one of the four Chebyshev weights divided by an arbitrary quadratic polynomial that remains positive on $[-1, 1]$. We show that in almost all cases these extended "Gauss–Kronrod" quadrature rules have all the desirable properties: Kronrod nodes interlacing with Gauss nodes, all nodes contained in $[-1, 1]$, and all weights positive and representable by semiexplicit formulas. Exceptions to these properties occur only for small values of n (the number of Gauss nodes), namely $n \leqslant 3$, and are carefully identified. The precise degree of exactness of each of these Gauss–Kronrod formulae is determined and shown to grow like $4n$, rather than $3n$, as is normally the case. Our findings are the result of a detailed analysis of the underlying orthogonal polynomials and "Stieltjes polynomials". The paper concludes with a study of the limit case of a linear divisor polynomial in the weight function.

Keywords: Gauss–Kronrod quadrature formulae, weight functions of Bernstein–Szegö type, orthogonal polynomials, Stieltjes polynomials.

1. Introduction

The idea of embedding Gaussian quadrature formulae in higher-order quadrature rules to improve upon their accuracy, or estimating their errors, was advanced in 1964 by Kronrod [8]. Kronrod proposed to insert $n+1$ nodes into an n-point Gauss–Legendre formula and to determine them, and the weights of the resulting $(2n+1)$-point formula, in such a way as to achieve maximum degree of exactness. He showed that the nodes to be inserted are the zeros of a polynomial of degree $n+1$—now called the Stieltjes polynomial—that is orthogonal to all lower-degree polynomials with respect to a sign-changing weight function, the Legendre polynomial of degree n. He computed these zeros, and all weights involved, to 16 decimal digits for $n = 1(1)40$. Mysovskih [13] noted that the same kind of orthogonality has previously been studied by Szegö [14], independently of its application to quadrature. Szegö indeed followed up on an

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idea already expressed in 1894 by Stieltjes in his last letter to Hermite [1, Vol. II, pp. 439–441]. Szegö proved that the zeros in question are all real, are contained in the interval $(-1, 1)$, and interlace with the zeros of the Legendre polynomial. He showed this to be true not only for the Legendre weight (constant weight function), but also for a subclass of Gegenbauer weights. For a further subclass of these, including, however, Legendre's weight function, Monegato [10] in 1978 established positivity of all quadrature weights, a result that was suggested by Kronrod's numerical tables. The interlacing and inclusion properties of the nodes, and positivity of all weights, for Gegenbauer and Jacobi weights are further studied in [3].

During the last ten years, interest in such quadrature rules has intensified, in part because of their potential use in automatic quadrature routines, but also, undoubtedly, because of the intriguing mathematical problems they pose. Recent surveys on the subject can be found in [11] and [2]. Nevertheless, relatively little has been rigorously proved in this area. Apart from the early examples of Gauss–Kronrod quadratures for Chebyshev weights [13] and Gegenbauer weights [14,9,10], only one additional family of weight functions is presently known for which the existence of Gauss–Kronrod quadrature rules with the properties mentioned, and indeed semi-explicit formulae for them, have been established; these are the symmetric weight functions considered in [4] consisting of the Chebyshev weight of the second kind divided by an even quadratic polynomial.

In the following, we substantially enlarge this class of weight functions by considering Chebyshev weight functions of any of the four kinds and dividing them by an arbitrary quadratic polynomial that remains positive on the interval $[-1, 1]$. Such weight functions, even for divisor polynomials of arbitrary degree, have been studied by Bernstein and Szegö (see, e.g., [15, §2.6]). We develop the Gauss–Kronrod formulae in these cases and provide semiexplicit formulae for them analogous to those obtained in [4]. We also prove that the desirable properties of the interlacing of nodes, their containment in the interval $[-1, 1]$, and positivity of all quadrature weights, hold true in almost all cases, exceptions occurring only for small values of n. We begin in Section 2 with identifying explicitly the class of quadratic polynomials that are positive on the interval $[-1, 1]$. We also compute the integrals of the weight functions they generate. In Section 3 we develop the relevant orthogonal polynomials and establish some of their properties. The corresponding Stieltjes polynomials are derived in Section 4. The core of the paper is Section 5 and 6. In Section 5 we study the respective Gauss–Kronrod formulae and establish the interlacing and inclusion properties of the nodes. We also determine the precise polynomial degree of exactness for each one of these quadrature formulae. Section 6 is devoted to explicit formulae for the quadrature weights and their positivity. Finally, in Section 7, we specialize the results to weight functions in which the divisor polynomial is linear, rather than quadratic.

2. The weight functions

We shall be interested in weight functions on $(-1, 1)$ of the form

$$w^{(\pm 1/2)}(t) = (1-t^2)^{\pm 1/2}/\rho(t) \tag{2.1}$$

and

$$w^{(\pm 1/2, \mp 1/2)}(t) = (1-t)^{\pm 1/2}(1+t)^{\mp 1/2}/\rho(t), \tag{2.2}$$

where $\rho(t)$ is a polynomial of exact degree 2 which remains positive on $[-1, 1]$. Our first concern is to find the explicit form of the quadratic polynomial ρ having the stated positivity property.

Proposition 2.1. *A real polynomial* ρ *of exact degree* 2 *satisfies* $\rho(t) > 0$ *for* $-1 \leqslant t \leqslant 1$ *if and only if it has the form*

$$\rho(t) = \rho(t;\ \alpha, \beta, \delta) = \beta(\beta - 2\alpha)t^2 + 2\delta(\beta - \alpha)t + \alpha^2 + \delta^2 \tag{2.3}$$

with

$$0 < \alpha < \beta, \quad \beta \neq 2\alpha, \quad |\delta| < \beta - \alpha. \tag{2.4}$$

Remark. Proposition 2.1 has previously been stated without proof in [12, p. 497].

Proof. Letting

$$\rho(t) = at^2 + bt + c, \quad a, b, c \in \mathbb{R}, \quad a \neq 0, \tag{2.5}$$

we have that $\rho(\cos\theta)$ is a cosine polynomial of degree 2 with real coefficients which is positive for all real values of θ. By [15, Theorem 1.2.2] there then exists a unique polynomial h of exact degree 2, with real coefficients, satisfying

$$h(z) \neq 0 \quad \text{in } |z| \leqslant 1, \quad h(0) > 0, \tag{2.6}$$

and such that

$$\rho(\cos\theta) = |h(e^{i\theta})|^2. \tag{2.7}$$

Writing

$$h(z) = pz^2 + qz + r, \quad p, q, r \in \mathbb{R}, \quad p \neq 0, \tag{2.8}$$

one finds by an elementary computation that

$$|h(e^{i\theta})|^2 = 4pr\cos^2\theta + 2q(p + r)\cos\theta + q^2 + (p - r)^2,$$

hence, by (2.7) and (2.5),

$$a = 4pr, \qquad b = 2q(p + r), \qquad c = q^2 + (p - r)^2. \tag{2.9}$$

On the other hand, all zeros of h are outside the closed unit disc $|z| \leqslant 1$ (the first condition in (2.6)) if and only if

$$(Th)(0) > 0, \qquad (T^2h)(0) > 0, \tag{2.10}$$

where $(Th)(\cdot)$ is the Schur transform of h and $(T^2h)(\cdot)$ its first iterate (see, e.g., [7, Theorem 6.8b]). One easily calculates $(Th)(z) = q(r - p)z + r^2 - p^2$, $(T^2h)(z) = (r - p)^2[(r + p)^2 - q^2]$. Therefore, (2.10), together with the last conditions in (2.6) and (2.8), is equivalent to

$$r > |p|, \quad r + p > |q|, \qquad p \neq 0. \tag{2.11}$$

Letting $\alpha = r - p$, $\beta = 2r$, $\delta = q$, or equivalently,

$$p = \tfrac{1}{2}\beta - \alpha, \qquad q = \delta, \qquad r = \tfrac{1}{2}\beta, \tag{2.12}$$

we obtain from (2.9) and (2.11)

$$a = \beta(\beta - 2\alpha), \qquad b = 2\delta(\beta - \alpha), \qquad c = \alpha^2 + \delta^2$$

with

$$\beta > |\beta - 2\alpha|, \qquad \beta - \alpha > |\delta|, \qquad \beta \neq 2\alpha. \tag{2.13}$$

It remains to observe that (2.13) is equivalent to (2.4). $\square$

We will call the parameters α, β, δ *admissible*, if they satisfy (2.4).

The discriminant of the polynomial (2.3) computes to $4\alpha^2(\delta^2-\beta^2+2\alpha\beta)$, so that ρ has a pair of conjugate complex zeros if $\delta^2<\beta(\beta-2\alpha)$ (which implies $\beta>2\alpha$), and two real zeros if $\delta^2\geqslant\beta(\beta-2\alpha)$. If $\beta-2\alpha>0$, the real zeros are both negative (hence less than -1) if $\delta>0$ and both positive (hence larger than 1) if $\delta<0$ ($\delta=0$ is not possible in this case); if $\beta-2\alpha<0$, they are on opposite sides of $[-1, 1]$.

We note from (2.7) that

$$\rho\left(\tfrac{1}{2}(u+u^{-1})\right)=0,\quad u\in\mathbb{C},\quad |u|>1\quad\text{implies } h(u)=0. \tag{2.14}$$

Indeed, if u_1, u_2 denote the zeros of h (both larger than 1 in modules by (2.6)), then (2.7) can be written as

$$\rho\left(\tfrac{1}{2}(e^{i\theta}+e^{-i\theta})\right)=p^2(e^{i\theta}-u_1)(e^{-i\theta}-\bar{u}_1)(e^{i\theta}-u_2)(e^{-i\theta}-\bar{u}_2),$$

an identity valid for all real θ. By the Identity Theorem for holomorphic functions, the same relation holds for complex θ as well, which, letting $u=e^{i\theta}$ (θ complex), yields (2.14).

The polynomial ρ and the weight functions (2.1), (2.2) become particularly simple when $\delta=0$. In this case we write

$$\alpha/\beta=\tfrac{1}{2}(\gamma+1),\quad -1<\gamma<1, \tag{2.15}$$

and obtain by a simple computation

$$\rho(t;\ \alpha,\ \beta,\ 0)=\alpha^2\left[1-\left(4\gamma/(\gamma+1)^2\right)t^2\right]. \tag{2.16}$$

Thus, apart from a constant factor, we are led to the weight functions

$$w_0^{(\pm 1/2)}(t)=(1-t^2)^{\pm 1/2}/(1-\mu t^2), \tag*{(2.1)^0}$$

$$w_0^{(\pm 1/2,\mp 1/2)}(t)=\frac{(1-t)^{\pm 1/2}(1+t)^{\mp 1/2}}{1-\mu t^2},\quad -\infty<\mu=\frac{4\gamma}{(\gamma+1)^2}<1. \tag*{(2.2)^0}$$

The Gauss–Kronrod quadrature rules for $w_0^{(1/2)}$ have been studied in [4].

It is of interest to compute the integrals of the weight functions (2.1), (2.2). We begin with $w^{(-1/2)}$. Letting $a=\beta(\beta-2\alpha)$ and denoting the zeros of the polynomial ρ in (2.3) by z_1, z_2, we have

$$\beta_0^{(-1/2)}=\int_{-1}^{1}w^{(-1/2)}(t)\,dt=\frac{1}{a(z_1-z_2)}\left\{\int_{-1}^{1}\frac{(1-t^2)^{-1/2}}{t-z_1}\,dt-\int_{-1}^{1}\frac{(1-t^2)^{-1/2}}{t-z_2}\,dt\right\}. \tag{2.17}$$

It is known (see, e.g., Gradshteyn and Ryzhik [6, Eq. 3.613.2]) that

$$\int_{-1}^{1}\frac{T_n(t)}{z-t}(1-t^2)^{-1/2}\,dt=\frac{2\pi}{(u-u^{-1})u^n}, \tag{2.18}$$

where T_n is the Chebyshev polynomial of degree n and

$$z=\tfrac{1}{2}(u+1/u),\quad |u|>1. \tag{2.19}$$

The relationship between u and z represents a well-known conformal map which transforms the exterior of the unit circle, $|u| > 1$, into the whole z-plane cut along $[-1, 1]$, concentric circles going into confocal ellipses. Letting

$$z_i = \tfrac{1}{2}(u_i + 1/u_i), \quad |u_i| > 1, \qquad i = 1, 2, \tag{2.20}$$

one finds from (2.17), (2.18) (with $n = 0$) by a simple computation that

$$\beta_0^{(-1/2)} = \frac{4\pi}{a} \frac{u_1u_2 + 1}{u_1u_2 - 1} \frac{u_1u_2}{(u_1^2 - 1)(u_2^2 - 1)}, \qquad a = \beta(\beta - 2\alpha). \tag{2.21}$$

Since, by (2.20) and (2.14), u_1, u_2 are zeros of h, the symmetric functions of u_1, u_2 in (2.21) can be expressed rationally in terms of the coefficients of h, hence by (2.12) in terms of α, β and δ. One finds

$$\begin{gathered} u_1u_2 = \frac{\beta}{\beta - 2\alpha}, \qquad u_1u_2 + 1 = \frac{2(\beta - \alpha)}{\beta - 2\alpha}, \qquad u_1u_2 - 1 = \frac{2\alpha}{\beta - 2\alpha}, \\ (u_1^2 - 1)(u_2^2 - 1) = (u_1u_2 + 1)^2 - (u_1 + u_2)^2 = \frac{4\left[(\beta - \alpha)^2 - \delta^2\right]}{(\beta - 2\alpha)^2}. \end{gathered} \tag{2.22}$$

Substituted in (2.21), this yields

$$\beta_0^{(-1/2)} = \pi \frac{\beta - \alpha}{\alpha\left[(\beta - \alpha)^2 - \delta^2\right]}. \tag{2.23}$$

We proceed to the weight function $w^{(1/2)}$ and the integral

$$\beta_0^{(1/2)} = \int_{-1}^{1} w^{(1/2)}(t)\, \mathrm{d}t. \tag{2.24}$$

A decomposition analogous to the one in (2.17), and using (cf. [6, Eq. 3.613.3])

$$\int_{-1}^{1} \frac{U_n(t)}{z - t}(1 - t^2)^{1/2}\, \mathrm{d}t = \frac{\pi}{u^{n+1}} \tag{2.25}$$

in place of (2.18), yields

$$\beta_0^{(1/2)} = \frac{2\pi}{a} \frac{1}{u_1u_2 - 1}, \tag{2.26}$$

with u_i defined as in (2.20), hence, by the third relation in (2.22),

$$\beta_0^{(1/2)} = \pi/\alpha\beta. \tag{2.27}$$

Interestingly, the integral in (2.24) does not depend on the parameter δ.

Finally, for the integral

$$\beta_0^{(1/2,-1/2)} = \int_{-1}^{1} w^{(1/2,-1/2)}(t)\, \mathrm{d}t, \tag{2.28}$$

we use (cf. [5, §5.2], where the case $(1 - t)^{-1/2}(1 + t)^{1/2}$ is treated, which is easily transformed to the present case)

$$\int_{-1}^{1} \frac{(1 - t)^{1/2}(1 + t)^{-1/2}}{z - t}\, \mathrm{d}t = \frac{2\pi}{u + 1} \tag{2.29}$$

and find

$$\beta_0^{(1/2,-1/2)} = \frac{4\pi}{a} \frac{u_1 u_2}{(u_1 u_2 - 1)(u_1 + 1)(u_2 + 1)},$$

which by (2.22) and

$$(u_1 + 1)(u_2 + 1) = 2(\beta - \alpha - \delta)/(\beta - 2\alpha)$$

becomes

$$\beta_0^{(1/2,-1/2)} = \pi/\alpha(\beta - \alpha - \delta). \tag{2.30}$$

For the remaining weight function, $w^{(-1/2,1/2)}$, see (3.24) below.

In the following, for ease of readability, we shall often drop the superscripts $\pm 1/2$ in the notation for weight functions and related quantities, when there is no danger of ambiguity.

3. The orthogonal polynomials

In the limiting case $\rho(t) \equiv 1$ (corresponding to $\alpha = 1$, $\beta \to 2$, $\delta = 0$ in (2.3)), the orthogonal polynomials associated with (2.1) are the Chebyshev polynomials $T_n(t)$, $U_n(t)$ of the first and second kind (corresponding, respectively, to the minus and plus sign in (2.1)), whereas those associated with (2.2) are similarly the Chebyshev polynomials $V_n(t)$, $W_n(t)$ of the "third and fourth kind" (corresponding, again, to the minus and plus sign, respectively). These are characterized by the well-known formulae

$$T_n(\cos\theta) = \cos n\theta, \qquad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \tag{3.1}$$

and

$$V_n(\cos\theta) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}, \qquad W_n(\cos\theta) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}. \tag{3.2}$$

They all satisfy the same recurrence relation,

$$y_{k+1} = 2t y_k - y_{k-1}, \quad k = 1, 2, 3, \ldots, \tag{3.3}$$

where

$$\begin{aligned} &y_0 = 1, \quad y_1 = t && \text{for } T_n(t), \\ &y_0 = 1, \quad y_1 = 2t && \text{for } U_n(t), \\ &y_0 = 1, \quad y_1 = 2t - 1 && \text{for } V_n(t), \\ &y_0 = 1, \quad y_1 = 2t + 1 && \text{for } W_n(t). \end{aligned} \tag{3.4}$$

It is natural to expect that the orthogonal polynomials for the more general weight functions (2.1), (2.2), with ρ as given in (2.3), can be expressed in a simple manner in terms of these same Chebyshev polynomials. This is indeed the case, and the respective expressions will be derived in this section.

We will denote the (monic) orthogonal polynomials relative to the weight functions (2.1), (2.2) by

$$\pi_n^{(\pm 1/2)}(t) = \pi_n(t;\, w^{(\pm 1/2)}), \quad \pi_n^{(\pm 1/2,\mp 1/2)}(t) = \pi_n(t;\, w^{(\pm 1/2,\mp 1/2)}),$$
$$n = 0, 1, 2, \ldots, \tag{3.5}$$

and shall drop superscripts when their values are clear from the context. With $h(z)$ the unique polynomial in (2.7), let $h(\mathrm{e}^{\mathrm{i}\theta}) = c(\theta) + \mathrm{i}s(\theta)$ where $c(\theta)$, $s(\theta)$ are real. Then, by [15, Theorem 2.6], we have, up to constant factors,

$$\begin{aligned}
\pi_n^{(-1/2)}(\cos\theta) &= \text{const}\cdot\left[c(\theta)\cos n\theta + s(\theta)\sin n\theta\right],\\
\pi_n^{(1/2)}(\cos\theta) &= \text{const}\cdot\left[c(\theta)\,\frac{\sin(n+1)\theta}{\sin\theta} - s(\theta)\frac{\cos(n+1)\theta}{\sin\theta}\right],\\
\pi_n^{(1/2,-1/2)}(\cos\theta) &= \text{const}\cdot\left[c(\theta)\frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta} - s(\theta)\frac{\cos(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\right].
\end{aligned} \tag{3.6}$$

From (2.8) and (2.12) we find that

$$c(\theta) = (\beta - 2\alpha)\cos^2\theta + \delta\cos\theta + \alpha, \qquad s(\theta) = \sin\theta\left[(\beta - 2\alpha)\cos\theta + \delta\right]. \tag{3.7}$$

Writing $\cos\theta = t$ in (3.7), we get in view of (3.1),

$$\begin{aligned}
\pi_n^{(-1/2)}(t) &= \text{const}\cdot\left\{\left[(\beta-2\alpha)t^2 + \delta t + \alpha\right]T_n(t) + (1-t^2)\left[(\beta-2\alpha)t + \delta\right]U_{n-1}(t)\right\}\\
&= \text{const}\cdot\left\{\left[t(\beta-2\alpha)+\delta\right]\left[tT_n(t) + (1-t^2)U_{n-1}(t)\right] + \alpha T_n(t)\right\}.
\end{aligned}$$

Since $tT_n + (1-t^2)U_{n-1} = T_{n-1}$, as follows easily from (3.1), this yields, if $n \geqslant 2$,

$$\begin{aligned}
\pi_n^{(-1/2)}(t) &= \text{const}\cdot\left[(\beta-2\alpha)tT_{n-1}(t) + \delta T_{n-1}(t) + \alpha T_n(t)\right]\\
&= \text{const}\cdot\left\{(\beta-2\alpha)\cdot\tfrac{1}{2}\left[T_n(t) + T_{n-2}(t)\right] + \delta T_{n-1}(t) + \alpha T_n(t)\right\}\\
&= \text{const}\cdot\left[\tfrac{1}{2}\beta T_n(t) + \delta T_{n-1}(t) + \tfrac{1}{2}(\beta-2\alpha)T_{n-2}(t)\right].
\end{aligned}$$

As the leading coefficient of the expression in brackets is $2^{n-1}\cdot\frac{1}{2}\beta$, we obtain

$$\pi_n^{(-1/2)}(t) = \frac{1}{2^{n-1}}\left[T_n(t) + \frac{2\delta}{\beta}T_{n-1}(t) + \left(1 - \frac{2\alpha}{\beta}\right)T_{n-2}(t)\right], \quad n \geqslant 2. \tag{3.8}$$

When $n = 1$, one finds

$$\pi_1^{(-1/2)}(t) = t + \delta/(\beta - \alpha). \tag{3.8^1}$$

In a similar way one computes

$$\pi_n^{(1/2)}(t) = \frac{1}{2^n}\left[U_n(t) + \frac{2\delta}{\beta}U_{n-1}(t) + \left(1 - \frac{2\alpha}{\beta}\right)U_{n-2}(t)\right], \quad n \geqslant 1, \tag{3.9}$$

where $U_{-1}(t) = 0$ when $n = 1$.

To obtain $\pi_n^{(1/2,-1/2)}$, we use (3.2) and (3.7) in the last equation of (3.6) and find

$$\pi_n^{(1/2,-1/2)}(t) = \text{const}\cdot\left\{\left[(\beta-2\alpha)t + \delta\right]\left[tW_n(t) - (1+t)V_n(t)\right] + \alpha W_n(t)\right\}.$$

Noting that $tW_n - (1+t)V_n = W_{n-1}$ and, for $n \geqslant 2$, by (3.3), $tW_{n-1} = \frac{1}{2}(W_n + W_{n-2})$, we obtain

$$\pi_n^{(1/2,-1/2)}(t) = \text{const}\cdot\left[\tfrac{1}{2}\beta W_n(t) + \delta W_{n-1}(t) + \tfrac{1}{2}(\beta-2\alpha)W_{n-2}(t)\right].$$

Since W_n has leading coefficient 2^n (cf. (3.3), (3.4)), this yields

$$\pi_n^{(1/2,-1/2)}(t) = \frac{1}{2^n}\left[W_n(t) + \frac{2\delta}{\beta}W_{n-1}(t) + \left(1 - \frac{2\alpha}{\beta}\right)W_{n-2}(t)\right], \quad n \geqslant 2. \tag{3.10}$$

For $n = 1$ one finds, by virtue of (3.4),

$$\pi_1^{(1/2,-1/2)}(t) = t + (\alpha + \delta)/\beta. \tag{3.10}$$[1]

Denoting the polynomials (3.5) more precisely by $\pi_n^{(\pm 1/2)}(t;\ \alpha,\ \beta,\ \delta)$, and $\pi_n^{(\pm 1/2,\mp 1/2)}(t;\ \alpha,\ \beta,\ \delta)$, if we want to stress their dependence on the parameters α, β, δ, then a simple argument will show that

$$\pi_n^{(-1/2,1/2)}(t;\ \alpha,\ \beta,\ \delta) = (-1)^n \pi_n^{(1/2,-1/2)}(-t;\ \alpha,\ \beta,\ -\delta). \tag{3.11}$$

There is an alternative derivation of (3.8), (3.9) and (3.10), which explains why the coefficients on the right are the same in all three formulae, and in fact equal to $-(u_1 + u_2)/(u_1u_2)$ and $1/(u_1u_2)$, respectively (cf. (2.20)). We explain the method for the case of (3.8). Expand $\pi_n^{(-1/2)}$ in Chebyshev polynomials of the first kind, $2^{n-1}\pi_n^{(-1/2)}(t) = \sum_{k=0}^{n} c_k T_k(t)$, $c_n = 1$. It is easily seen that $c_k = 0$ for $k < n-2$. To obtain the desired orthogonality

$$\int_{-1}^{1} \pi_n^{(-1/2)}(t) \frac{p(t)}{\rho(t)} (1-t^2)^{-1/2}\ \mathrm{d}t = 0, \quad \text{all } p \in \mathbb{P}_{n-1},$$

write

$$p(t) = \rho(t)q(t) + r(t), \qquad q \in \mathbb{P}_{n-3}, \quad r \in \mathbb{P}_1$$

to see that it suffices to make $\pi_n^{(-1/2)}$ orthogonal (with respect to $w^{(-1/2)}$) to linear functions. Choosing in $\mathbb{P}_1$ the basis functions $z_1 - t$ and $z_2 - t$ (where z_i are the zeros of ρ, assumed distinct), one arrives at the system of equations

$$c_{n-1}\int_{-1}^{1} \frac{T_{n-1}(t)}{z_i - t}(1-t^2)^{-1/2}\ \mathrm{d}t + c_{n-2}\int_{-1}^{1} \frac{T_{n-2}(t)}{z_i - t}(1-t^2)^{-1/2}\ \mathrm{d}t$$
$$= -\int_{-1}^{1} \frac{T_n(t)}{z_i - t}(1-t^2)^{-1/2}\ \mathrm{d}t, \quad i = 1, 2,$$

which, by virtue of (2.18), reduces to

$$u_1c_{n-1} + u_1^2c_{n-2} = -1, \qquad u_2c_{n-1} + u_2^2c_{n-2} = -1,$$

and has the solution $c_{n-1} = -(u_1 + u_2)/(u_1u_2)$, $c_{n-2} = 1/(u_1u_2)$, as claimed. If ρ has a double root, the result follows by continuity. The same argument goes through for the other weight functions.

Proposition 3.1. *We have, for* $n \geqslant 1$,

$$\pi_n^{(1/2)}(t)\pi_{n+1}^{(-1/2)}(t) = \pi_{2n+1}^{(1/2)}(t) + (\delta/\beta)\pi_{2n}^{(1/2)}(t) + \tfrac{1}{4}(1 - 2\alpha/\beta)\pi_{2n-1}^{(1/2)}(t). \tag{3.12}$$

Proof. We first note from (3.1) that

$$T_m(t)U_n(t) = \begin{cases} \frac{1}{2}\left[U_{m+n}(t) + U_{n-m}(t)\right], & m \leqslant n, \\ \frac{1}{2}U_{2n+1}(t), & m = n+1, \\ \frac{1}{2}\left[U_{m+n}(t) - U_{m-n-2}(t)\right], & m > n+1. \end{cases} \tag{3.13}$$

From (3.9) and (3.8) we thus obtain for the product in (3.12), if $n \geqslant 1$,

$$\begin{aligned}
&\frac{1}{2^{2n}}\left[U_n + \frac{2\delta}{\beta}U_{n-1} + \left(1-\frac{2\alpha}{\beta}\right)U_{n-2}\right]\left[T_{n+1} + \frac{2\delta}{\beta}T_n + \left(1-\frac{2\alpha}{\beta}\right)T_{n-1}\right]\\
&\quad= \frac{1}{2^{2n+1}}\left\{U_{2n+1} + \frac{2\delta}{\beta}U_{2n} + \left(1-\frac{2\alpha}{\beta}\right)U_{2n-1}\right.\\
&\qquad + \frac{2\delta}{\beta}\left[U_{2n} + \frac{2\delta}{\beta}U_{2n-1} + \left(1-\frac{2\alpha}{\beta}\right)U_{2n-2}\right]\\
&\qquad \left. + \left(1-\frac{2\alpha}{\beta}\right)\left[U_{2n-1} + \frac{2\delta}{\beta}U_{2n-2} + \left(1-\frac{2\alpha}{\beta}\right)U_{2n-3}\right]\right\},
\end{aligned}$$

which, by (3.9), is precisely the right-hand side of (3.12). □

Proposition 3.2. *We have, for $n \geqslant 2$,*

$$\pi_n^{(1/2,-1/2)}(t)\pi_n^{(-1/2,1/2)}(t) = \pi_{2n}^{(1/2)}(t) + (\delta/\beta)\pi_{2n-1}^{(1/2)}(t) + \tfrac{1}{4}(1-2\alpha/\beta)\pi_{2n-2}^{(1/2)}(t). \tag{3.14}$$

Proof. From (3.2) and elementary trigonometric identities one gets

$$V_m(t)W_n(t) = \begin{cases} U_{m+n}(t) + U_{n-m-1}(t), & m < n,\\ U_{2n}(t), & m = n,\\ U_{m+n}(t) - U_{m-n-1}(t), & m > n. \end{cases} \tag{3.15}$$

Furthermore, replacing θ by $\theta + \pi$ in (3.2) gives

$$W_n(-t) = (-1)^n V_n(t). \tag{3.16}$$

Using (3.10) and (3.11), the product in (3.14) is then computed similarly as in the proof of Proposition 3.1 to be equal to the right-hand side of (3.14). □

The formulae (3.8), (3.9) and (3.10), in conjunction with (3.3), (3.4), immediately yield the recurrence relation

$$\pi_{k+1}(t) = (t-\alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t), \quad k = 0, 1, 2, \ldots, \tag{3.17}$$

for the respective orthogonal polynomials. One finds for $w^{(-1/2)}$:

$$\begin{aligned}
&\alpha_0^{(-1/2)} = -\frac{\delta}{\beta-\alpha}, \qquad \alpha_1^{(-1/2)} = \frac{\alpha\delta}{\beta(\beta-\alpha)}, \qquad \alpha_k^{(-1/2)} = 0 \quad \text{for } k \geqslant 2;\\
&\beta_1^{(-1/2)} = \alpha\frac{(\beta-\alpha)^2-\delta^2}{\beta(\beta-\alpha)^2}, \qquad \beta_2^{(-1/2)} = \frac{\beta-\alpha}{2\beta}, \qquad \beta_k^{(-1/2)} = \tfrac{1}{4} \quad \text{for } k \geqslant 3;
\end{aligned} \tag{3.18}$$

for $w^{(1/2)}$:

$$\begin{aligned}
&\alpha_0^{(1/2)} = -\frac{\delta}{\beta}, \qquad \alpha_k^{(1/2)} = 0 \quad \text{for } k \geqslant 1;\\
&\beta_1^{(1/2)} = \frac{\alpha}{2\beta}, \qquad \beta_k^{(1/2)} = \tfrac{1}{4} \quad \text{for } k \geqslant 2;
\end{aligned} \tag{3.19}$$

and for $w^{(1/2,-1/2)}$:

$$\begin{aligned}&\alpha_0^{(1/2,-1/2)} = -\frac{\alpha+\delta}{\beta}, \qquad \alpha_1^{(1/2,-1/2)} = \frac{2\alpha-\beta}{2\beta}, \qquad \alpha_k^{(1/2,-1/2)} = 0 \quad \text{for } k \geqslant 2;\\ &\beta_1^{(1/2,-1/2)} = \frac{\alpha(\beta-\alpha-\delta)}{\beta^2}, \qquad \beta_k^{(1/2,-1/2)} = \tfrac{1}{4} \quad \text{for } k \geqslant 2.\end{aligned} \tag{3.20}$$

Accordingly, for the norms

$$\|\pi_n\|^2 = \int_{-1}^{1} \pi_n^2(t) w(t)\,\mathrm{d}t = \beta_0\beta_1 \cdots \beta_n,$$

with β_0 given in (2.23), (2.27), and (2.30), one obtains

$$\|\pi_1^{(-1/2)}\|^2 = \frac{\pi}{\beta(\beta-\alpha)}, \qquad \|\pi_n^{(-1/2)}\|^2 = \frac{\pi}{2^{2n-3}\beta^2} \quad \text{for } n \geqslant 2; \tag{3.21}$$

$$\|\pi_n^{(1/2)}\|^2 = \frac{\pi}{2^{2n-1}\beta^2} \quad \text{for } n \geqslant 1; \tag{3.22}$$

$$\|\pi_n^{(1/2,-1/2)}\|^2 = \frac{\pi}{2^{2n-2}\beta^2} \quad \text{for } n \geqslant 1. \tag{3.23}$$

None of these norms, remarkably enough, depends on δ.

The analogous results for the remaining weight function, $w^{(-1/2,1/2)}$, can be obtained from those for $w^{(1/2,-1/2)}$ by means of (3.11), giving

$$\begin{aligned}&\alpha_k^{(-1/2,1/2)}(\alpha, \beta, \delta) = -\alpha_k^{(1/2,-1/2)}(\alpha, \beta, -\delta),\\ &\beta_k^{(-1/2,1/2)}(\alpha, \beta, \delta) = \beta_k^{(1/2,-1/2)}(\alpha, \beta, -\delta), \qquad k = 0, 1, 2, \ldots;\end{aligned} \tag{3.24}$$

$$\|\pi x_n^{(-1/2,1/2)}\|^2 = \|\pi_n^{(1/2,-1/2)}\|^2, \quad n \geqslant 1. \tag{3.25}$$

The recursion coefficients in (3.18)–(3.20) allow us to compute the zeros of the orthogonal polynomials (3.5), as well as the zeros of the corresponding Stieltjes polynomials (except for the first few, cf. (4.2), (4.5) and (4.8)) efficiently as eigenvalues of symmetric tridiagonal matrices.

4. The Stieltjes polynomials

Given an orthogonal polynomial $\pi_n(\cdot) = \pi_n(\cdot;\, w)$ of degree n with respect to a weight function w on $[-1, 1]$, there is associated with it a unique (monic) polynomial $\pi_{n+1}^*(\cdot) = \pi_{n+1}^*(\cdot;\, w)$ of degree $n+1$, called *Stieltjes polynomial*, which satisfies the orthogonality relation (see, e.g., [2, §1])

$$\int_{-1}^{1} \pi_{n+1}^*(t)\pi_n(t)p(t)w(t)\,\mathrm{d}t = 0, \quad \text{all } p \in \mathbb{P}_n. \tag{4.1}$$

In this section we express the Stieltjes polynomials $\pi_{n+1}^{(\pm 1/2)*}(\cdot) = \pi_{n+1}^*(\cdot;\, w^{(\pm 1/2)})$, $\pi_{n+1}^{(\pm 1/2,\mp 1/2)*}(\cdot) = \pi_{n+1}^*(\cdot;\, w^{(\pm 1/2,\mp 1/2)})$ for the weight functions (2.1), (2.2) in terms of the orthogonal polynomials of Section 3 and determine the respective products $\pi_{n+1}^*(\cdot)\pi_n(\cdot)$ appearing in the integrand of (4.1).

Proposition 4.1. *We have*

$$\pi_{n+1}^{(-1/2)*}(t) = (t^2-1)\pi_{n-1}^{(1/2)}(t), \quad n \geqslant 4, \tag{4.2}$$

and

$$\pi_{n+1}^{(-1/2)*}(t)\pi_n^{(-1/2)}(t)$$
$$= (t^2-1)\left[\pi_{2n-1}^{(1/2)}(t) + \frac{\delta}{\beta}\pi_{2n-2}^{(1/2)}(t) + \frac{1}{4}\left(1-\frac{2\alpha}{\beta}\right)\pi_{2n-3}^{(1/2)}(t)\right], \quad n \geqslant 4. \tag{4.3}$$

Furthermore, for $n = 1$, 2, and 3,

$$\pi_2^{(-1/2)*}(t) = t^2 + \frac{\delta}{\beta}t - \frac{1}{2}\left(1+\frac{\alpha}{\beta}\right), \tag*{(4.2)^1}$$

$$\pi_2^{(-1/2)*}(t)\pi_1^{(-1/2)}(t) = \pi_3^{(-1/2)}(t) + \frac{\delta}{\beta-\alpha}\pi_2^{(-1/2)}(t); \tag*{(4.3)^1}$$

$$\pi_3^{(-1/2)*}(t) = t^3 + \frac{\delta}{\beta}t^2 - \frac{1}{4}\left(3+\frac{2\alpha}{\beta}\right)t - \frac{3}{4}\frac{\delta}{\beta}, \tag*{(4.2)^2}$$

$$\pi_3^{(-1/2)*}(t)\pi_2^{(-1/2)}(t) = \pi_5^{(-1/2)}(t) + \frac{\delta}{\beta}\pi_4^{(-1/2)}(t) + \frac{1}{4}\left(1-\frac{4\alpha}{\beta}\right)\pi_3^{(-1/2)}(t); \tag*{(4.3)^2}$$

$$\pi_4^{(-1/2)*}(t) = t^4 + \frac{\delta}{\beta}t^3 - \left(1+\frac{\alpha}{2\beta}\right)t^2 - \frac{\delta}{\beta}t + \frac{1}{16}\left(1+\frac{6\alpha}{\beta}\right), \tag*{(4.2)^3}$$

$$\pi_4^{(-1/2)*}(t)\pi_3^{(-1/2)}(t) = \pi_7^{(-1/2)}(t) + \frac{\delta}{\beta}\pi_6^{(-1/2)}(t) - \frac{\alpha}{2\beta}\pi_5^{(-1/2)}(t) - \frac{\delta}{4\beta}\pi_4^{(-1/2)}(t). \tag*{(4.3)^3}$$

Proof. Define $q_{n+1}(t) = (t^2-1)\pi_{n-1}^{(1/2)}(t)$. Clearly, q_{n+1} is monic of degree $n+1$, and using Proposition 3.1 (with n replaced by $n-1$), we find (for $n \geqslant 2$)

$$q_{n+1}(t)\pi_n^{(-1/2)}(t) = (t^2-1)\pi_{n-1}^{(1/2)}(t)\pi_n^{(-1/2)}(t)$$
$$= (t^2-1)\left[\pi_{2n-1}^{(1/2)}(t) + \frac{\delta}{\beta}\pi_{2n-2}^{(1/2)}(t) + \frac{1}{4}\left(1-\frac{2\alpha}{\beta}\right)\pi_{2n-3}^{(1/2)}(t)\right]. \tag{4.4}$$

Consequently, if $n \geqslant 4$, using $(t^2-1)w^{(-1/2)}(t) = -w^{(1/2)}(t)$, we get

$$\int_{-1}^{1} q_{n+1}(t)\pi_n^{(-1/2)}(t)p(t)w^{(-1/2)}(t)\,\mathrm{d}t$$
$$= -\int_{-1}^{1}\left[\pi_{2n-1}^{(1/2)}(t) + \frac{\delta}{\beta}\pi_{2n-2}^{(1/2)}(t) + \frac{1}{4}\left(1-\frac{2\alpha}{\beta}\right)\pi_{2n-3}^{(1/2)}(t)\right]p(t)w^{(1/2)}(t)\,\mathrm{d}t = 0, \quad \text{all } p \in \mathbb{P}_n,$$

by the orthogonality of the $\pi_m^{(1/2)}$, since $2n-3 > n$ for $n \geqslant 4$. Thus, q_{n+1} has the orthogonality property (4.1) required for $\pi_{n+1}^{(-1/2)*}$, and by uniqueness, $q_{n+1} \equiv \pi_{n+1}^{(-1/2)*}$. This proves (4.2), and (4.3) follows from (4.4).

To prove the special cases $n=1$, 2 and 3, first write the polynomials $\pi_{n+1}^{(-1/2)*}$ in $(4.2)^n$ in terms of the U's and then verify $(4.3)^n$ by expressing the product $\pi_{n+1}^{(-1/2)*}\pi_n^{(-1/2)}$ in terms of the U's, using (3.8) and (3.13), and finally simplify by making use of the relations obtained by multiplying (3.8) by U_0 (thereby expressing $\pi_n^{(-1/2)}$ in terms of the U's). The computations are elementary, but tedious, and will not be reproduced here. □

Proposition 4.2. *We have*

$$\pi_{n+1}^{(1/2)*}(t)=\pi_{n+1}^{(-1/2)}(t), \quad n \geqslant 2, \tag{4.5}$$

and

$$\pi_{n+1}^{(1/2)*}(t)\pi_n^{(1/2)}(t)=\pi_{2n+1}^{(1/2)}(t)+\frac{\delta}{\beta}\pi_{2n}^{(1/2)}(t)+\frac{1}{4}\left(1-\frac{2\alpha}{\beta}\right)\pi_{2n-1}^{(1/2)}(t), \quad n \geqslant 2. \tag{4.6}$$

Furthermore, for $n=1$,

$$\pi_2^{(1/2)*}(t)=t^2+(\delta/\beta)t-\tfrac{1}{4}(1+2\alpha/\beta), \tag*{$(4.5)^1$}$$

$$\pi_2^{(1/2)*}(t)\pi_1^{(1/2)}(t)=\pi_3^{(1/2)}(t)+(\delta/\beta)\pi_2^{(1/2)}(t). \tag*{$(4.6)^1$}$$

Proof. Let $q_{n+1}(t)=\pi_{n+1}^{(-1/2)}(t)$. Then by Proposition 3.1, for $n \geqslant 1$,

$$\begin{aligned} q_{n+1}(t)\pi_n^{(1/2)}(t) &= \pi_{n+1}^{(-1/2)}(t)\pi_n^{(1/2)}(t) \\ &= \pi_{2n+1}^{(1/2)}(t)+(\delta/\beta)\pi_{2n}^{(1/2)}(t)+\tfrac{1}{4}(1-2\alpha/\beta)\pi_{2n-1}^{(1/2)}(t), \end{aligned} \tag{4.7}$$

from which there follows, if $n \geqslant 2$,

$$\int_{-1}^{1} q_{n+1}(t)\pi_n^{(1/2)}(t)p(t)w^{(1/2)}(t)\,\mathrm{d}t=0, \quad \text{all } p \in \mathbb{P}_n,$$

by virtue of $2n-1>n$. Hence, $q_{n+1} \equiv \pi_{n+1}^{(1/2)*}$, which together with (4.7) proves (4.5) and (4.6). The case $n=1$ can be verified directly. □

Proposition 4.3. *We have*

$$\pi_{n+1}^{(1/2,-1/2)*}(t)=(t+1)\pi_n^{(-1/2,1/2)}(t), \quad n \geqslant 3, \tag{4.8}$$

and

$$\begin{aligned} &\pi_{n+1}^{(1/2,-1/2)*}(t)\pi_n^{(1/2,-1/2)}(t) \\ &\quad =(t+1)\left[\pi_{2n}^{(1/2)}(t)+\frac{\delta}{\beta}\pi_{2n-1}^{(1/2)}(t)+\frac{1}{4}\left(1-\frac{2\alpha}{\beta}\right)\pi_{2n-2}^{(1/2)}(t)\right], \quad n \geqslant 3. \end{aligned} \tag{4.9}$$

Furthermore, for $n=1$ and 2,

$$\pi_2^{(1/2,-1/2)*}(t)=t^2+\frac{1}{2}\left(1+\frac{2\delta}{\beta}\right)t-\frac{1}{4}\left(1-\frac{2\delta-2\alpha}{\beta}\right), \tag*{$(4.8)^1$}$$

$$\pi_2^{(1/2,-1/2)*}(t)\pi_1^{(1/2,-1/2)}(t)=\pi_3^{(1/2,-1/2)}(t)+\frac{\alpha+\delta}{\beta}\pi_2^{(1/2,-1/2)}(t); \tag*{$(4.9)^1$}$$

$$\pi_3^{(1/2,-1/2)*}(t)=t^3+\frac{1}{2}\left(1+\frac{2\delta}{\beta}\right)t^2-\frac{1}{2}\left(1-\frac{\delta-\alpha}{\beta}\right)t-\frac{1}{8}\left(1+\frac{2\alpha+4\delta}{\beta}\right), \tag*{$(4.8)^2$}$$

$$\pi_3^{(1/2,-1/2)*}(t)\pi_2^{(1/2,-1/2)}(t) = \pi_5^{(1/2,-1/2)}(t) + \frac{1}{2}\left(1 + \frac{2\delta}{\beta}\right)\pi_4^{(1/2,-1/2)}(t)$$
$$+ \frac{1}{4}\left(1 + \frac{2\delta - 2\alpha}{\beta}\right)\pi_3^{(1/2,-1/2)}(t). \qquad (4.9)^2$$

Proof. Let $q_{n+1}(t) = (t+1)\pi_n^{(-1/2,1/2)}(t)$. Then, by Proposition 3.2, for $n \geqslant 2$,

$$q_{n+1}(t)\pi_n^{(1/2,-1/2)}(t) = (t+1)\pi_n^{(-1/2,1/2)}(t)\pi_n^{(1/2,-1/2)}(t)$$
$$= (t+1)\left[\pi_{2n}^{(1/2)}(t) + \frac{\delta}{\beta}\pi_{2n-1}^{(1/2)}(t) + \frac{1}{4}\left(1 - \frac{2\alpha}{\beta}\right)\pi_{2n-2}^{(1/2)}(t)\right]. \qquad (4.10)$$

Using $(t+1)w^{(1/2,-1/2)}(t) = w^{(1/2)}(t)$, one thus gets, if $n \geqslant 3$,

$$\int_{-1}^{1} q_{n+1}(t)\pi_n^{(1/2,-1/2)}(t)p(t)w^{(1/2,-1/2)}(t)\,\mathrm{d}t$$
$$= \int_{-1}^{1}\left[\pi_{2n}^{(1/2)}(t) + \frac{\delta}{\beta}\pi_{2n-1}^{(1/2)}(t) + \frac{1}{4}\left(1 - \frac{2\alpha}{\beta}\right)\pi_{2n-2}^{(1/2)}(t)\right]p(t)w^{(1/2)}(t)\,\mathrm{d}t = 0,$$

all $p \in \mathbb{P}_n$,

since $2n - 2 > n$. This proves (4.8), and (4.9) follows from (4.10).

To verify the special cases $n = 1$ and 2, it is convenient to first express $\pi_{n+1}^{(1/2,-1/2)*}$ in $(4.8)^n$ in terms of the T's, then compute the products in $(4.9)^n$ as linear combinations of the W's, using

$$T_m(t)W_n(t) = \begin{cases} \frac{1}{2}[W_{m+n}(t) + W_{n-m}(t)], & m < n, \\ \frac{1}{2}[W_{2n}(t) + W_0(t)], & m = n, \\ \frac{1}{2}[W_{m+n}(t) - W_{m-n-1}(t)], & m > n, \end{cases} \qquad (4.11)$$

and finally simplify the results by using (3.10). The details of the computations are left to the reader. □

Proposition 4.4. *We have*

$$\pi_{n+1}^{(-1/2,1/2)*}(t;\ \alpha,\ \beta,\ \delta) = (-1)^{n+1}\pi_{n+1}^{(1/2,-1/2)*}(-t;\ \alpha,\ \beta,\ -\delta), \quad n \geqslant 1. \qquad (4.12)$$

Proof. We verify that the polynomial on the right has the required orthogonality property (4.1). Using $\rho(-t;\ \alpha,\ \beta,\ \delta) = \rho(t;\ \alpha,\ \beta,\ -\delta)$, we find by the substitution of variables $t \mapsto -t$,

$$(-1)^{n+1}\int_{-1}^{1}\pi_{n+1}^{(1/2,-1/2)*}(-t;\ \alpha,\ \beta,\ -\delta)\pi_n^{(-1/2,1/2)}(t;\ \alpha,\ \beta,\ \delta)p(t)$$
$$\times w^{(-1/2,1/2)}(t;\ \alpha,\ \beta,\ \delta)\,\mathrm{d}t$$
$$= (-1)^{n+1}\int_{-1}^{1}\pi_{n+1}^{(1/2,-1/2)*}(t;\ \alpha,\ \beta,\ -\delta)\pi_n^{(-1/2,1/2)}(-t;\ \alpha,\ \beta,\ \delta)p(-t)$$
$$\times w^{(1/2,-1/2)}(t;\ \alpha,\ \beta,\ -\delta)\,\mathrm{d}t,$$

since $w^{(-1/2,1/2)}(-t;\ \alpha,\ \beta,\ \delta) = w^{(1/2,-1/2)}(t;\ \alpha,\ \beta,\ -\delta)$, which by (3.11) is equal to

$$-\int_{-1}^{1} \pi_{n+1}^{(1/2,-1/2)*}(t;\ \alpha,\ \beta,\ -\delta)\pi_n^{(1/2,-1/2)}(t;\ \alpha,\ \beta,\ -\delta)p(-t) \times w^{(1/2,-1/2)}(t;\ \alpha,\ \beta,\ -\delta)\ \mathrm{d}t.$$

This is zero for each $p \in \mathbb{P}_n$ by definition of $\pi_{n+1}^{(1/2,-1/2)*}$. □

Proposition 4.5. *Let τ_ν be the zeros of $\pi_n(\cdot;\ w)$. Then*

$$\pi_{n+1}(\tau_\nu;\ w) = \tfrac{1}{2}\pi_{n+1}^*(\tau_\nu;\ w), \quad \nu = 1, 2, \ldots, n, \tag{4.13}$$

for all $n \geqslant 2$ if $w = w^{(1/2)}$, for all $n \geqslant 4$ if $w = w^{(-1/2)}$, and for all $n \geqslant 3$ if $w = w^{(\pm 1/2,\mp 1/2)}$

Proof. Consider first $w = w^{(1/2)}$. By (3.9) we have

$$\pi_n^{(1/2)}(t) = \frac{1}{2^n}\left[U_n(t) + \frac{2\delta}{\beta}U_{n-1}(t) + \left(1 - \frac{2\alpha}{\beta}\right)U_{n-2}(t)\right]. \tag{4.14}$$

Here, $U_{n-1}(\tau_\nu) \neq 0$. In fact, if we had $U_{n-1}(\tau_\nu) = 0$, then, by (4.14), since τ_ν is a zero of $\pi_n^{(1/2)}$, we would obtain

$$U_n(\tau_\nu) + \left(1 - \frac{2\alpha}{\beta}\right)U_{n-2}(\tau_\nu) = 0,$$

and by the recurrence relation (3.3), $(2\alpha/\beta)U_{n-2}(\tau_\nu) = 0$, i.e., $U_{n-2}(\tau_\nu) = 0$, since $\alpha > 0$. This is impossible, since two consecutive orthogonal polynomials cannot vanish at the same point. We therefore obtain

$$\frac{\delta}{\beta} = \frac{\alpha}{\beta}\frac{U_{n-2}(\tau_\nu)}{U_{n-1}(\tau_\nu)} - \frac{1}{2}\frac{U_n(\tau_\nu) + U_{n-2}(\tau_\nu)}{U_{n-1}(\tau_\nu)}.$$

Letting $\tau_\nu = \cos\theta_\nu$, $0 < \theta_\nu < \pi$, this yields, in view of (3.1),

$$\frac{\delta}{\beta} = \frac{\alpha}{\beta}\frac{\sin(n-1)\theta_\nu}{\sin n\theta_\nu} - \cos\theta_\nu. \tag{4.15}$$

Setting $t = \cos\theta_\nu$ in (4.14), after replacing n by $n+1$, and then substituting for δ/β from (4.15), one obtains, after some elementary computation,

$$\pi_{n+1}^{(1/2)}(\tau_\nu) = -\frac{\alpha}{2^n\beta}\frac{\sin\theta_\nu}{\sin n\theta_\nu}. \tag{4.16}$$

A similar substitution in (4.5), if $n \geqslant 2$, using (3.8) (with n replaced by $n+1$), gives

$$\pi_{n+1}^{(1/2)*}(\tau_\nu) = -\frac{\alpha}{2^{n-1}\beta}\frac{\sin\theta_\nu}{\sin n\theta_\nu}. \tag{4.17}$$

Comparing (4.16) with (4.17) immediately yields (4.13).

For $w = w^{(-1/2)}$ and $n \geqslant 4$, one obtains in the same manner

$$\pi_{n+1}^{(-1/2)}(\tau_\nu) = -\frac{\alpha}{2^{n-1}\beta}\frac{\sin^2\theta_\nu}{\cos(n-1)\theta_\nu} = \tfrac{1}{2}\pi_{n+1}^{(-1/2)*}(\tau_\nu) \qquad \left(\pi_n^{(-1/2)}(\tau_\nu) = 0, \quad \tau_\nu = \cos\theta_\nu\right), \tag{4.18}$$

and for $w = w^{(1/2,-1/2)}$, $n \geqslant 3$,

$$\pi_{n+1}^{(1/2,-1/2)}(\tau_\nu) = -\frac{\alpha}{2^n\beta}\frac{\sin^2\theta_\nu}{\sin\frac{1}{2}\theta_\nu \sin(n-\frac{1}{2})\theta_\nu} = \tfrac{1}{2}\pi_{n+1}^{(1/2,-1/2)*}(\tau_\nu)$$
$$\left(\pi_n^{(1/2,-1/2)}(\tau_\nu) = 0, \quad \tau_\nu = \cos\theta_\nu\right). \tag{4.19}$$

The analogous result for $w^{(-1/2,1/2()}$ follows easily from (3.11) and (4.12). □

5. Interlacing, inclusion, and exactness properties

If $\tau_\nu = \tau_\nu^{(n)}$ denote the zeros of the orthogonal polynomial $\pi_n(\cdot) = \pi_n(\cdot;\ w)$, and $\tau_\mu^* = \tau_\mu^{(n)*}$ those of the polynomial $\pi_{n+1}^*(\cdot) = \pi_{n+1}^*(\cdot;\ w)$ satisfying (4.1), then, if they are distinct among themselves and between one another, one calls the (interpolatory) quadrature rule

$$\int_{-1}^{1} f(t)w(t)\,\mathrm{d}t = \sum_{\nu=1}^{n}\sigma_\nu f(\tau_\nu) + \sum_{\mu=1}^{n+1}\sigma_\mu^* f(\tau_\mu^*) + R_n(f) \tag{5.1}$$

the $(2n+1)$-point *Gauss–Kronrod quadrature rule*, or the *Kronrod extension* of the n-point Gauss rule, relative to the weight function w. The formula (5.1) is known to have precise degree of exactness $d = 2n + k$, where k is the unique integer satisfying

$$\int_{-1}^{1}\pi_{n+1}^*(t)\pi_n(t)p(t)w(t)\,\mathrm{d}t \begin{cases} = 0 & \text{for all } p \in \mathbb{P}_{k-1}, \\ \neq 0 & \text{for some } p \in \mathbb{P}_k \end{cases} \tag{5.2}$$

(see, e.g., [2, §2]). By (4.1) clearly $k \geqslant n+1$, hence $d \geqslant 3n+1$.

We say that (5.1) has the *interlacing* property if all nodes τ_ν, τ_μ^* are real and satisfy, when ordered decreasingly,

$$\tau_{n+1}^* < \tau_n < \tau_n^* < \cdots < \tau_2^* < \tau_1 < \tau_1^*. \tag{5.3}$$

We say that (5.1) has the *inclusion* property if all nodes τ_ν, τ_μ^* are contained in the closed interval $[-1, 1]$, i.e.,

$$-1 \leqslant \tau_{n+1}^* \quad \text{and} \quad \tau_1^* \leqslant 1, \tag{5.4}$$

if (5.3) holds.

In this section we show that, with a few exceptions for small n, the Gauss–Kronrod formula (5.1) enjoys the interlacing property when $w(t) = w^{(\pm 1/2)}(t;\ \alpha,\ \beta,\ \delta)$ and $w(t) = w^{(\pm 1/2,\mp 1/2)}(t;\ \alpha,\ \beta,\ \delta)$ and the parameters α, β, δ are admissible. In addition, we state conditions under which the inclusion property holds, and we determine the precise degree of exactness for all Gauss–Kronrod quadrature formulae in this class. As mentioned at the end of Section 2, superscripts $\pm\frac{1}{2}$ will be deleted when there is no danger of confusion.

Theorem 5.1. *Consider the weight function $w(t) = w^{(-1/2)}(t;\ \alpha,\ \beta,\ \delta)$, with α, β, δ admissible.*

(a) *The Gauss–Kronrod rule* (5.1) *has the interlacing property for all $n \geqslant 1$, except when $n = 3$ and $\beta > 2\alpha$, in which case* (5.3) *holds if*

$$\delta^2 < \delta_0^2, \qquad \delta_0^2 = \frac{1}{32}\frac{(3\beta - 2\alpha)^2(\beta + 6\alpha)}{\beta + 2\alpha}. \tag{5.5}$$

(b) *The inclusion property holds for all* $n \geqslant 4$. *For* $n = 3$ (*assuming interlacing*; cf. (a)), *it holds precisely if* $\beta > 2\alpha$; *for* $n = 2$ *precisely if* $\beta > 2\alpha$ *and* $|\delta| \leqslant \beta - 2\alpha$; *and for* $n = 1$ *precisely if* $|\delta| \leqslant \frac{1}{2}(\beta - \alpha)$.

(c) *The precise degree of exactness* d *of* (5.1) *is equal to* $4n - 3$ *if* $n \geqslant 4$. *If* $n = 3$, *then* $d = 10$ *unless* $\delta = 0$, *in which case* $d = 11$. *If* $n = 2$, *then* $d = 7$ *if* $\beta \neq 4\alpha$, $d = 8$ *if* $\beta = 4\alpha$ *and* $\delta \neq 0$, *and* $d = 9$ *if* $\beta = 4\alpha$ *and* $\delta = 0$. *If* $n = 1$, *then* $d = 4$ *if* $\delta \neq 0$ *and* 5 *otherwise*.

Proof. (a) If $n = 1$, we have by $(3.8)^1$ that $\tau_1 = -\delta/(\beta - \alpha)$, hence by $(4.2)^1$,

$$\begin{aligned}\pi_2^*(\tau_1) &= \frac{1}{2\beta(\beta-\alpha)^2}\left[2\alpha\delta^2 - (\alpha+\beta)(\beta-\alpha)^2\right]\\ &< \frac{\alpha}{\beta} - \frac{\alpha+\beta}{2\beta} = \frac{1}{2}\left(\frac{\alpha}{\beta} - 1\right) < 0,\end{aligned}$$

by virtue of (2.4). This proves (5.3) for $n = 1$.

If $n = 2$, we have by (3.8) and $(4.2)^2$

$$\begin{aligned}\pi_2(t) &= t^2 + \frac{\delta}{\beta}t - \frac{\alpha}{\beta},\\ \pi_3^*(t) &= t^3 + \frac{\delta}{\beta}t^2 - \frac{1}{4} + \left(3 + \frac{2\alpha}{\beta}\right)t - \frac{3}{4}\frac{\delta}{\beta}.\end{aligned} \tag{5.6}$$

The interlacing property holds when $\delta = 0$, since then $\tau_{1,2}^2 = \alpha/\beta$ and π_3^* has a zero at $t = 0$ and two other zeros $\tau_{1,3}^*$ with $\tau_{1,3}^{*2} = \frac{1}{4}(3 + 2\alpha/\beta) > \alpha/\beta$. Upon varying $|\delta|$ from 0 to $\beta - \alpha$, interlacing can only break down when π_2 and π_3^* have a common zero, t_0, for some δ in that range. If that is the case, then by (5.6)

$$t_0^2 + \frac{\delta}{\beta}t_0 = \frac{\alpha}{\beta}, \qquad t_0 \cdot \frac{\alpha}{\beta} - \frac{1}{4}\left(3 + \frac{2\alpha}{\beta}\right)t_0 - \frac{3}{4}\frac{\delta}{\beta} = 0,$$

which, upon eliminating t_0, yields

$$\delta^2 = \tfrac{1}{6}(3\beta - 2\alpha)^2.$$

It is elementary to see that the right-hand side is larger than $(\beta - \alpha)^2$ for admissible α, β. Consequently, the interlacing property holds for all admissible α, β, δ.

We now discuss the case $n = 3$. One shows, similarly as in the case $n = 2$, that (5.3) is true when $\delta = 0$. As in the previous case, we determine the first value of $|\delta|$ for which the polynomials

$$\begin{aligned}\pi_3(t) &= t^3 + \frac{\delta}{\beta}t^2 - \frac{1}{2}\left(1 + \frac{\alpha}{\beta}\right)t - \frac{1}{2}\frac{\delta}{\beta},\\ \pi_4^*(t) &= t^4 + \frac{\delta}{\beta}t^3 - \left(1 + \frac{\alpha}{2\beta}\right)t^2 - \frac{\delta}{\beta}t + \frac{1}{16}\left(1 + \frac{6\alpha}{\beta}\right)\end{aligned} \tag{5.7}$$

(cf. (3.8) and $(4.2)^3$) have a common zero t_0. Writing

$$\pi_4^*(t) = t\left[t^3 + \frac{\delta}{\beta}t^2 - \frac{1}{2}\left(1 + \frac{\alpha}{\beta}\right)t\right] - \tfrac{1}{2}t^2 - \frac{\delta}{\beta}t + \frac{1}{16}\left(1 + \frac{6\alpha}{\beta}\right)$$

and using

$$\pi_3(t_0)=0, \tag{5.8}$$

the fact that $\pi_4^*(t_0)=0$ yields

$$t_0^2+\frac{\delta}{\beta}t_0-\frac{1}{8}\left(1+\frac{6\alpha}{\beta}\right)=0,$$

which, upon reusing (5.8), simplifies to

$$\frac{1}{8}\left(1+\frac{6\alpha}{\beta}\right)t_0-\frac{1}{2}\left(1+\frac{\alpha}{\beta}\right)t_0-\frac{1}{2}\frac{\delta}{\beta}=0,$$

that is, to

$$t_0=4\delta/(2\alpha-3\beta).$$

Inserting this into (5.8) one finds, after a little computation, $\delta^2=\delta_0^2$ with δ_0^2 as given in (5.5). Hence, for all values of $|\delta|$ smaller than $|\delta_0|$ we are assured of the interlacing property. An elementary computation shows that

$$\delta_0^2 \gtrless (\beta-\alpha)^2 \quad \text{iff } 40r^3-28r^2-42r+23 \lesseqgtr 0, \quad r=\alpha/\beta. \tag{5.9}$$

Since $0<r<1$ by (2.4), and the cubic in (5.9) has a zero at $r=\frac{1}{2}$ and two other zeros outside, and on either side, of [0, 1], we have $|\delta_0| \geqslant \beta-\alpha$ (for admissible α, β) precisely if $\frac{1}{2} \leqslant r < 1$, i.e., $\beta \leqslant 2\alpha$, where equality is excluded by (2.4). Therefore, if $\beta < 2\alpha$, the interlacing property holds; if $\beta > 2\alpha$, it holds when $\delta^2 < \delta_0^2$ as claimed.

Finally, for $n \geqslant 4$, the assertion follows immediately from $\pi_n^{(-1/2)}(t)=\pi_n^{(1/2)*}(t)$ (cf. (4.5)) and $\pi_{n+1}^{(-1/2)*}(t)=(t^2-1)\pi_{n-1}^{(1/2)}(t)$ (cf. (4.2)), and the fact that the interlacing property holds for the weight function $w^{(1/2)}$ (cf. Theorem 5.2(a) below).

(b) For $n \geqslant 4$, the inclusion property (5.4) follows immediately from (4.2). For $n=3$, assuming interlacing, (5.4) is equivalent to $\pi_4^*(\pm 1) \geqslant 0$, hence, by $(4.2)^3$, to $\beta \geqslant 2\alpha$. Since $\beta \neq 2\alpha$, we have inclusion precisely for $\beta > 2\alpha$. If $n=2$, we have (5.4) if and only if $\pi_3^*(1) \geqslant 0$ and $\pi_3^*(-1) \leqslant 0$. By $(4.2)^2$, this is the same as $\delta \geqslant 2\alpha-\beta$ and $\delta \leqslant \beta-2\alpha$. These conditions are incompatible when $\beta < 2\alpha$, and equivalent to $|\delta| \leqslant \beta-2\alpha$ when $\beta > 2\alpha$. Finally, when $n=1$, using $(4.2)^1$, we have by a similar argument as before that (5.4) holds precisely when $\delta \geqslant -\frac{1}{2}(\beta-\alpha)$ and $\delta \leqslant \frac{1}{2}(\beta-\alpha)$, i.e., $|\delta| \leqslant \frac{1}{2}(\beta-\alpha)$.

(c) We have $d=2n+k$, with k determined as in (5.2). If $n \geqslant 4$, then (4.3), in view of $(1-t^2)w^{(-1/2)}(t)=w^{(1/2)}(t)$, yields $k=2n-3$, since $\beta \neq 2\alpha$, hence $d=4n-3$. The remaining assertions follow similarly from $(4.3)^n$, $n=3$, 2, and 1. □

Theorem 5.2. *Consider the weight function $w(t)=w^{(1/2)}(t;\ \alpha,\ \beta,\ \delta)$, with α, β, δ admissible.*

(a) *The Gauss–Kronrod rule* (5.1) *has the interlacing property for all $n \geqslant 1$.*

(b) *The inclusion property holds for all $n \geqslant 1$, except when $n=1$ and $\beta > 2\alpha$, in which case* (5.4) *holds precisely when $|\delta| \leqslant \frac{1}{4}(3\beta-2\alpha)$.*

(c) *The precise degree of exactness d of* (5.1) *is equal to $4n-1$ if $n \geqslant 2$; if $n=1$, then $d=4$ if $\delta \neq 0$ and $d=5$ if $\delta=0$.*

Proof. (a) We first note that interlacing holds when $\delta=0$. Indeed, we are then in the case of the weight function $w_0^{(1/2)}$ of $(2.1)^0$ for which the interlacing property (5.3), including (5.4), is known from the work in [4]. Moving δ away from 0, either to the left or to the right, within the allowable range $|\delta| < \beta - \alpha$, the interlacing property ceases to hold only if for some δ_0 in this range the polynomials π_n and π_{n+1}^* have a common zero, t_0. Proposition 4.5, if $n \geqslant 2$, would then imply $\pi_{n+1}(t_0) = \pi_n(t_0) = 0$, which is impossible, and (3.9), $(4.5)^1$, if $n = 1$, would imply $\beta + 2\alpha = 0$, contradicting (2.4). Hence, interlacing prevails for all admissible, α, β, δ.

(b) The inclusion property (5.4) follows immediately from (4.5) when $n \geqslant 2$. If $n = 1$, we have (5.4) precisely when $\pi_2^*(\pm 1) \geqslant 0$, which, on using $(4.5)^1$ is equivalent to $|\delta| \leqslant \frac{1}{4}(3\beta - 2\alpha)$. Here, the bound is larger than $\beta - \alpha$ if $\beta < 2\alpha$, so that the constraint is active only if $\beta > 2\alpha$. (The case $\beta = 2\alpha$ is excluded by (2.4).)

(c) If $n \geqslant 2$, then (4.6) shows, since $\beta \neq 2\alpha$, that (5.2) holds with $k = 2n - 1$, so that $d = 2n + k = 4n - 1$. The assertion for $n = 1$ follows from $(4.6)^1$ by a similar argument. □

Theorem 5.3. *Consider the weight function* $w(t) = w^{(1/2,-1/2)}(t;\ \alpha, \beta, \delta)$, *with* α, β, δ *admissible.*

(a) *The Gauss–Kronrod rule* (5.1) *has the interlacing property for all* $n \geqslant 1$, *except when* $n = 2$ *and* $\beta > 2\alpha$, *in which case* (5.3) *holds if*

$$-(\beta - \alpha) < \delta < \delta_1, \quad \delta_1 = \frac{\beta^2 + 8\alpha\beta - 4\alpha^2}{8\alpha + 4\beta}. \tag{5.10}$$

(b) *The inclusion property holds for all* $n \geqslant 3$. *For* $n = 2$ (*assuming interlacing*; cf. (a)), *it holds precisely if*

$$\beta > 2\alpha \quad \text{and} \quad -\frac{7\beta - 6\alpha}{8} \leqslant \delta < \beta - \alpha, \tag{5.11}$$

and for $n = 1$ *if both inequalities*

$$6\delta + 5\beta - 2\alpha \geqslant 0 \quad \text{and} \quad 2\delta + 2\alpha - \beta \leqslant 0 \tag{5.12}$$

are satisfied.

(c) *The precise degree of exactness* d *of* (5.1) *is equal to* $4n - 2$ *if* $n \geqslant 3$. *If* $n = 2$, *then* $d = 7$, *unless* $2\delta - 2\alpha + \beta = 0$, *in which case* $d = 8$. *If* $n = 1$, *then* $d = 4$ *if* $\alpha + \delta \neq 0$ *and* $d = 5$ *otherwise.*

Proof. (a) It is elementary to show, using $(3.10)^1$ and $(4.8)^1$, that $\pi_2^*(\tau_1) < 0$, which implies (5.3) for $n = 1$. For $n = 2$, an argument similar to the one in the proof of Theorem 5.1(a) will show that the interlacing property holds if $\beta < 2\alpha$, and if $\beta > 2\alpha$ provided that (5.10) is satisfied. If $n \geqslant 3$, the assertion follows from Proposition 4.5, which, since the zeros of π_n interlace with those of π_{n+1}, implies that sign $\pi_{n+1}(\tau_\nu;\ w) = (-1)^\nu =$ sign $\pi_{n+1}^*(\tau_\nu;\ w)$, $\nu = 1, 2, \ldots, n$, which in turn implies (5.3).

(b) For $n \geqslant 3$, the inclusion property follows trivially from (4.8). For $n = 2$ and $n = 1$, the conditions stated in (5.11) and (5.12) express $\pi_3^*(1) \geqslant 0$, $\pi_3^*(-1) \leqslant 0$ and $\pi_2^*(\pm 1) \geqslant 0$, respectively.

(c) The formulae for the precise degree of exactness follow in the usual way from (4.9), when $n \geqslant 3$, and from $(4.9)^2$, $(4.9)^1$ when $n = 2$ and 1, noting, in the case of $(4.9)^2$, that the coefficients of π_3 and π_4 in $(4.9)^2$ cannot vanish simultaneously because of $\alpha > 0$. □

The discussion for the remaining weight function $w^{(-1/2,1/2)}$ can be reduced, with the help of (3.11) and (4.12), to the one just completed for $w^{(1/2,-1/2)}$.

6. Quadrature weights, positivity, and explicit formulae

In terms of the polynomials $\pi_n(\cdot) = \pi_n(\cdot;\ w)$ and $\pi_{n+1}^*(\cdot) = \pi_{n+1}^*(\cdot;\ w)$, the weights $\sigma_\nu = \sigma_\nu^{(n)}$ and $\sigma_\mu^* = \sigma_\mu^{(n)*}$ in the Gauss–Kronrod formula (5.1) admit the following representations (Monegato [9]),

$$\sigma_\nu = \lambda_\nu + \frac{\|\pi_n\|^2}{\pi_{n+1}^*(\tau_\nu)\pi_n'(\tau_\nu)}, \quad \nu = 1, 2, \dots, n, \tag{6.1}$$

$$\sigma_\mu^* = \frac{\|\pi_n\|^2}{\pi_n(\tau_\mu^*)\pi_{n+1}^{*\prime}(\tau_\mu^*)}, \quad \mu = 1, 2, \dots, n+1, \tag{6.2}$$

where $\lambda_\nu = \lambda_\nu^{(n)}$ are the Christoffel numbers—the weights of the Gaussian quadrature rule—relative to the weight function w. These in turn can be represented in the form (see, e.g., [15, Eq. (3.4.7)])

$$\lambda_\nu = -\frac{\|\pi_n\|^2}{\pi_{n+1}(\tau_\nu)\pi_n'(\tau_\nu)}, \quad \nu = 1, 2, \dots, n, \tag{6.3}$$

and are known to be positive. We show in this section that also the weights $\sigma_\nu^{(n)}$ and $\sigma_\mu^{(n)*}$, with a few exceptions for small n, are all positive when $w(t) = w^{(\pm 1/2)}(t;\ \alpha, \beta, \delta)$ and $w(t) = w^{(\pm 1/2, \mp 1/2)}(t;\ \alpha, \beta, \delta)$, for arbitrary admissible parameters α, β, δ. The positivity of σ_μ^* actually follows from the interlacing property (Monegato [9]). Moreover, we give explicit formulae for the σ_ν and σ_μ^*.

If the interlacing property holds, the second term in (6.1) is always negative, whereas the first is positive. Thus, in view of (6.3), we have $\sigma_\nu > 0$ if and only if $|\pi_{n+1}^*(\tau_\nu)| > |\pi_{n+1}(\tau_\nu)|$, or, since the zeros of π_n interlace with those of π_{n+1},

$$\sigma_\nu > 0 \quad \text{iff } \pi_{n+1}^*(\tau_\nu) \lessgtr \pi_{n+1}(\tau_\nu) \quad \text{for } \nu = \begin{cases} \text{odd}, \\ \text{even} \end{cases} \tag{6.4}$$

(assuming the interlacing property).

We begin with the case $w = w^{(1/2)}$, which is simpler and appears to be more fundamental.

Theorem 6.1. *Consider the weight function $w(t) = w^{(1/2)}(t;\ \alpha, \beta, \delta)$, with α, β, δ admissible. Then all weights $\sigma_\nu = \sigma_\nu^{(n)}$, $\sigma_\mu^* = \sigma_\mu^{(n)*}$ in* (5.1) *are positive for each $n \geqslant 1$. Specifically, when $n = 1$,*

$$\sigma_1^{(1)} = \frac{\pi}{\alpha(2\alpha + \beta)}, \qquad \sigma_1^{(1)*} = \frac{\pi}{\omega(\omega + \delta)}, \qquad \sigma_2^{(1)*} = \frac{\pi}{\omega(\omega - \delta)}, \tag{6.5}$$

where $\omega = \sqrt{\delta^2 + \beta(2\alpha + \beta)}$. For $n \geqslant 2$, letting $\tau_\nu^{(n)} = \cos\,\theta_\nu$ and $\tau_\mu^{(n)*} = \cos\,\theta_\mu^*$, one has for $\nu = 1, 2, \dots, n$,

$$\sigma_\nu^{(n)} = \frac{\pi}{2\alpha^2} \frac{\sin^2 n\theta_\nu}{n - \dfrac{\sin\,n\theta_\nu \cos(n-1)\theta_\nu}{\sin\,\theta_\nu} + \dfrac{\beta}{\alpha}\sin^2 n\theta_\nu}, \tag{6.6}$$

and for $\mu = 1, 2, \dots, n+1$,

$$\sigma_\mu^{(n)*} = \frac{\pi}{2\alpha^2} \frac{\cos^2 n\theta_\mu^*}{n + \dfrac{\sin(n-1)\theta_\mu^* \cos\,n\theta_\mu^*}{\sin\,\theta_\mu^*} + \dfrac{\beta}{\alpha}\cos^2 n\theta_\mu^*}. \tag{6.7}$$

Remark. Alternative forms of (6.6), (6.7), assuming $\theta_\nu \neq \frac{1}{2}\pi$, $\theta_\mu^* \neq \frac{1}{2}\pi$, are

$$\sigma_\nu^{(n)} = \frac{\pi}{2\alpha^2} \frac{\sin^2 n\theta_\nu}{n - \left\{\sin 2n\theta_\nu + \frac{2\delta}{\beta}\sin n\theta_\nu \cos(n-1)\theta_\nu\right\} / \left\{\sin 2\theta_\nu + \frac{2\delta}{\beta}\sin\theta_\nu\right\}}, \tag{6.6$'$}$$

$$\sigma_\mu^{(n)*} = \frac{\pi}{2\alpha^2} \frac{\cos^2 n\theta_\mu^*}{n + \left\{\sin 2n\theta_\mu^* + \frac{2\delta}{\beta}\sin(n-1)\theta_\mu^* \cos n\theta_\mu^*\right\} / \left\{\sin 2\theta_\mu^* + \frac{2\delta}{\beta}\sin\theta_\mu^*\right\}}. \tag{6.7$'$}$$

These, as $\delta \to 0$, reduce in view of (2.16) to the formulae (3.9), (3.11) obtained in [4]. We note, however, that (6.6) remains valid as $\delta \to 0$ also in the case n (odd) $\geqslant 3$, $\nu = \frac{1}{2}(n+1)$, and reduces to [4, Eq. (3.9$'$)]. Likewise, (6.7) in the limit $\delta = 0$ reduces to [4, Eq. (3.11$'$)] when n (even) $\geqslant 2$ and $\mu = \frac{1}{2}(n+2)$.

Proof. The formulae (6.5) for $n = 1$, which clearly imply the positivity of all weights, are easily obtained from (6.1), (6.2), and (6.3), using the expressions (3.9) for π_1 and π_2, (4.5)[1] for π_2^*, and (3.22) for $\| \pi_1 \|^2$.

Thus assume $n \geqslant 2$. Substituting $\pi_{n+1}^*(\tau_\nu)$ from (4.13) into (6.1), we get $\sigma_\nu = \frac{1}{2}\lambda_\nu$, which proves positivity of σ_ν. Positivity of σ_μ^*, as already mentioned, follows from the interlacing property, which holds according to Theorem 5.2 (a).

We proceed to derive the explicit expressions in (6.6) and (6.7). First note that, by what has just been shown,

$$\sigma_\nu = \tfrac{1}{2}\lambda_\nu = -\| \pi_n \|^2 / \pi_{n+1}^*(\tau_\nu)\pi_n'(\tau_\nu). \tag{6.8}$$

Putting $t = \cos\theta$ in (3.9) and differentiating gives

$$\begin{aligned} &2^n\left[\cos\theta \cdot \pi_n(\cos\theta) - \sin^2\theta \cdot \pi_n'(\cos\theta)\right] \\ &\quad = (n+1)\cos(n+1)\theta + \frac{2\delta}{\beta} n \cos n\theta + \left(1 - \frac{2\alpha}{\beta}\right)(n-1)\cos(n-1)\theta. \end{aligned}$$

Now put $\theta = \theta_\nu$ and substitute δ/β from (4.15); then, after a short calculation, one finds

$$\pi_n'(\tau_\nu) = \frac{\alpha}{2^{n-1}\beta} \frac{1}{\sin\theta_\nu \sin n\theta_\nu}\left\{n - \frac{\sin n\theta_\nu \cos(n-1)\theta_\nu}{\sin\theta_\nu} + \frac{\beta}{\alpha}\sin^2 n\theta_\nu\right\}.$$

Multiplying this by $\pi_{n+1}^*(\tau_\nu)$ from (4.17), inserting the result in (6.8), and recalling (3.22) immediately yields the desired formula (6.6).

To derive (6.7), we let $\tau_\mu^* = \cos\theta_\mu^*$ (which is possible by virtue of (4.5)) and find from (4.5) and (3.8), similarly as in the proof of Proposition 4.5 (cf. (4.15)), that

$$\frac{\delta}{\beta} = \frac{\alpha}{\beta}\frac{\cos(n-1)\theta_\mu^*}{\cos n\theta_\mu^*} - \cos\theta_\mu^*, \tag{6.9}$$

and from (3.9) that

$$\pi_n(\tau_\mu^*) = \frac{\alpha}{2^{n-1}\beta}\frac{1}{\cos n\theta_\mu^*}.$$

Furthermore,

$$\pi_{n+1}^{*\prime}(\tau_\mu^*) = \frac{\alpha}{2^{n-1}\beta}\frac{1}{\cos n\theta_\mu^*}\left\{n + \frac{\sin(n-1)\theta_\mu^* \cos n\theta_\mu^*}{\sin\theta_\mu^*} + \frac{\beta}{\alpha}\cos^2 n\theta_\mu^*\right\},$$

from which as before, using (6.2) and (3.22), one obtains (6.7).

The alternative formulae (6.6′) and (6.7′) follow from those just derived by expressing β/α (under the assumptions made on θ_ν and θ_μ^*) in terms of δ/β by means of (4.15) and (6.9), respectively, and using elementary trigonometric identities. □

Theorem 6.2. *Consider the weight function* $w(t) = w^{(-1/2)}(t;\ \alpha, \beta, \delta)$, *with* α, β, δ *admissible. Then all weights* $\sigma_\nu = \sigma_\nu^{(n)}$, $\sigma_\mu^* = \sigma_\mu^{(n)*}$ *in* (5.1) *are positive for each* $n \geqslant 1$, *except when* $n = 3$ *and* $\beta > 2\alpha$, *in which case positivity holds if* $\delta^2 < \beta(\beta + 2\alpha)/8$.

If $\{\bar{\tau}_\mu\}_{\mu=1}^{n-1}$, $\{\bar{\tau}_\nu^*\}_{\nu=1}^{n}$ *denote the Gauss and Kronrod nodes, respectively, of the* $(2n-1)$*-point Gauss–Kronrod formula for the weight function* $w = w^{(1/2)}$, *and* $\bar{\sigma}_\mu$, $\bar{\sigma}_\nu^*$ *the respective weights, then for* $n \geqslant 4$ *one has*

$$\sigma_\nu^{(n)} = \frac{\bar{\sigma}_\nu^*}{1-\tau_\nu^2},\quad \nu = 1, 2, \ldots, n;\qquad \sigma_\mu^{(n)*} = \frac{\bar{\sigma}_{\mu-1}}{1-\tau_\mu^{*2}},\quad \mu = 2, 3, \ldots, n, \tag{6.10}$$

while

$$\begin{aligned}\sigma_1^{(n)*} &= \frac{\pi}{4(\beta-\alpha+\delta)[(\beta-\alpha+\delta)n + 2\alpha-\beta-\delta]},\\ \sigma_{n+1}^{(n)*} &= \frac{\pi}{4(\beta-\alpha-\delta)[(\beta-\alpha-\delta)n + 2\alpha-\beta+\delta]}.\end{aligned} \tag{6.11}$$

Remark. Explicit formulae could easily be obtained for $\sigma_\nu^{(1)}$, $\sigma_\nu^{(2)}$ and $\sigma_\mu^{(1)*}$, but we refrain from writing them down here.

Proof. We begin with the cases $n = 1, 2, 3$, which require special treatment. We verify the conditions in (6.4).

For $n = 1$, the condition in (6.4) is immediate from $(4.2)^1$ and (5.6), since $\alpha < \beta$. For $n = 2$, the two inequalities in (6.4), by $(4.2)^2$ and (5.7), amount to $\tau_1 > -\delta/\beta$, $\tau_2 < -\delta/\beta$, which are true since $\pi_2(-\delta/\beta) = -\alpha/\beta < 0$ by (5.6). When $n = 3$, the three inequalities in (6.4), by (5.7) and (3.8), are equivalent to

$$\tau_\nu^2 + \frac{\delta}{\beta}\tau_\nu \gtrless \frac{1}{4}\left(1 + \frac{2\alpha}{\beta}\right),\qquad \nu = \begin{cases}1 \text{ and } 3,\\ 2,\end{cases} \tag{6.12}$$

where τ_ν as a zero of π_3 satisfies (cf. (5.7))

$$\tau_\nu\left(\tau_\nu^2 + \frac{\delta}{\beta}\tau_\nu\right) - \frac{1}{2}\left(1 + \frac{\alpha}{\beta}\right)\tau_\nu - \frac{1}{2}\frac{\delta}{\beta} = 0. \tag{6.13}$$

Consider first the case $\delta = 0$. Then, by symmetry, $\tau_3 < \tau_2 = 0 < \tau_1$, and (6.12) is trivially true for $\nu = 2$ and easily seen true for $\nu = 1, 3$ in view of (6.13) (for $\delta = 0$). Now moving $|\delta|$ continuously away from zero, positivity ceases to hold the first time we have equality in (6.12) for some ν. Combining this equality with (6.13) then yields $\tau_\nu = -2\delta/\beta$ for some ν, which, reinserted in

(6.13), yields $\delta^2 = \beta(\beta + 2\alpha)/8 =: \delta_2^2$. Thus, positivity holds for all δ with $\delta^2 < \delta_2^2$. Since $\delta_2^2 \geqslant (\beta - \alpha)^2$ precisely for $\beta \leqslant 2\alpha$ (assuming α, β admissible), positivity holds for all admissible α, β, δ when $\beta < 2\alpha$, since then the interlacing property holds by Theorem 5.1(a). In the remaining case, $\beta > 2\alpha$, we have positivity if $\delta^2 < \delta_2^2$, which can be verified to be a subregion of the region $\delta^2 < \delta_0^2$ (cf. (5.5)) in which interlacing holds.

Assume now $n \geqslant 4$. It follows from (4.2) and Theorem 5.1(c) that the Gauss–Kronrod formula under study has the form

$$\int_{-1}^{1} f(t) w^{(-1/2)}(t)\, \mathrm{d}t = \sum_{\nu=1}^{n} \sigma_\nu f(\tau_\nu) + \sigma_1^* f(1) + \sum_{\mu=2}^{n} \sigma_\mu^* f(\tau_\mu^*) + \sigma_{n+1}^* f(-1),$$

all $f \in \mathbb{P}_{4n-3}$.

Putting here $f(t) = (1 - t^2) g(t)$, and taking note of (4.2) and (4.5), this yields

$$\begin{aligned}\int_{-1}^{1} g(t) w^{(1/2)}(t)\, \mathrm{d}t &= \sum_{\nu=1}^{n} \sigma_\nu (1 - \tau_\nu^2) g(\tau_\nu) + \sum_{\mu=2}^{n} \sigma_\mu^* (1 - \tau_\mu^{*2}) g(\tau_\mu^*) \\ &= \sum_{\mu=1}^{n-1} \sigma_{\mu+1}^* (1 - \tau_{\mu+1}^{*2}) g(\bar{\tau}_\mu) + \sum_{\nu=1}^{n} \sigma_\nu (1 - \tau_\nu^2) g(\bar{\tau}_\nu^*),\end{aligned}$$

all $g \in \mathbb{P}_{4n-5}$,

which is precisely the $(2n-1)$-point Gauss–Kronrod formula for $w = w^{(1/2)}$. By uniqueness, (6.10) follows immediately.

To prove (6.11), it suffices to apply (6.2) for $\mu = 1$ and $\mu = n + 1$, noting that $\tau_1^* = 1$, $\tau_{n+1}^* = -1$, making use of (3.8) to evaluate $\pi_n(\pm 1)$, and of (4.2) to obtain $\pi_{n+1}^{*\prime}(\pm 1) = \pm 2\pi_{n-1}^{(1/2)}(\pm 1)$, and finally using (3.9) to evaluate $\pi_{n-1}^{(1/2)}(\pm 1)$ and recalling (3.21). The fact that both σ_1^* and σ_{n+1}^* are positive follows from $|\delta| < \beta - \alpha$ and from

$$\begin{aligned} n(\beta - \alpha \pm \delta) + 2\alpha - \beta \mp \delta &\geqslant 4(\beta - \alpha \pm \delta) + 2\alpha - \beta \mp \delta \\ &= 3\beta - 2\alpha \pm 3\delta > 3(\beta - \alpha \pm \delta) > 0. \qquad \square \end{aligned}$$

Theorem 6.3. *Consider the weight function* $w(t) = w^{(1/2,-1/2)}(t;\ \alpha, \beta, \delta)$, *with* α, β, δ *admissible. Then all weights* $\sigma_\nu = \sigma_\nu^{(n)}$, $\sigma_\mu^* = \sigma_\mu^{(n)*}$ *in* (5.1) *are positive for each* $n \geqslant 1$, *except when* $n = 2$ *and* $\beta > 2\alpha$, *in which case they are positive if* $-(\beta - \alpha) < \delta < \alpha$.

Specifically, for $n \geqslant 3$, *letting* $\tau_\nu^{(n)} = \cos\theta_\nu$ *and* $\tau_\mu^{(n)*} = \cos\theta_\mu^*$, *one has for* $\nu = 1, 2, \ldots, n$,

$$\sigma_\nu^{(n)} = \frac{\pi}{2\alpha^2} \frac{\sin^2(n - \frac{1}{2})\theta_\nu}{(1 + \cos\theta_\nu)\left[n - \dfrac{1}{2} - \dfrac{\sin(n - \frac{1}{2})\theta_\nu \cos(n - \frac{3}{2})\theta_\nu}{\sin\theta_\nu} + \dfrac{\beta}{\alpha}\sin^2(n - \frac{1}{2})\theta_\nu\right]}, \tag{6.14}$$

for $\mu = 1, 2, \ldots, n$,

$$\sigma_\mu^{(n)*} = \frac{\pi}{2\alpha^2} \frac{\cos^2(n - \frac{1}{2})\theta_\mu^*}{(1 + \cos\theta_\mu^*)\left[n - \dfrac{1}{2} + \dfrac{\sin(n - \frac{3}{2})\theta_\mu^* \cos(n - \frac{1}{2})\theta_\mu^*}{\sin\theta_\mu^*} + \dfrac{\beta}{\alpha}\cos^2(n - \frac{1}{2})\theta_\mu^*\right]}, \tag{6.15}$$

and for $\mu = n+1$,

$$\sigma_{n+1}^{(n)*} = \frac{\pi}{(\beta-\alpha-\delta)[2n(\beta-\alpha-\delta)+3\alpha-\beta+\delta]}. \tag{6.16}$$

Remark. It is possible to obtain explicit formulae for $\sigma_\nu^{(1)}$, $\sigma_\nu^{(2)}$, and $\sigma_\mu^{(1)*}$, but we will not bother writing them down here.

Proof. For $n = 1, 2$, we verify the conditions in (6.4). When $n = 1$, the inequality in (6.4) follows readily from $(4.8)^1$ and (3.10). When $n = 2$, we must discuss

$$\pi_3^*(\tau_\nu) \lesseqgtr \pi_3(\tau_\nu), \quad \nu = \begin{cases} 1, \\ 2, \end{cases} \tag{6.17}$$

where τ_ν satisfies $\pi_2(\tau_\nu) = 0$ with (cf. (3.10))

$$\pi_2(t) = t^2 + \tfrac{1}{2}(1+2\delta/\beta)t + \tfrac{1}{2}(\delta-\alpha)/\beta. \tag{6.18}$$

The top inequality in (6.17), by $(4.8)^2$ and (3.10), turns out to be equivalent to

$$\tau_1 > -\tfrac{1}{2}(1+2\delta/\beta), \tag{6.19}$$

which, by virtue of

$$\pi_2(-\tfrac{1}{2}(1+2\delta/\beta)) = (\delta-\alpha)/2\beta, \tag{6.20}$$

is certainly true if $\delta = 0$. Assume first $\beta < 2\alpha$, in which case interlacing holds by Theorem 5.3(a). Since the value of π_2 in (6.20) is negative when $\delta < \alpha$, and $\alpha > \beta - \alpha$, we have (6.19), hence $\sigma_1^{(2)} > 0$, for all admissible α, β, δ. The discussion of the lower inequality in (6.17) (for $\nu = 2$) is analogous and leads to the same conclusion. If $\beta > 2\alpha$, one needs to distinguish the cases $\delta > \alpha$ and $\delta < \alpha$. In the former case, since $\delta > 0$, both zeros τ_1 and τ_2 of π_2 are negative and sum up to $-\tfrac{1}{2}(1+2\delta/\beta)$, by (6.18). Therefore, (6.19) holds for both τ_1 and τ_2, hence $\sigma_1^{(2)} > 0$, but $\sigma_2^{(2)} < 0$. If $\delta < \alpha$, then (6.20) implies as before that $\sigma_1^{(2)} > 0$, $\sigma_2^{(2)} > 0$. Thus we have positivity of both weights if $-(\beta-\alpha) < \delta < \alpha$, which is easily seen to be a subinterval of the interval in (5.10) in which the interlacing property holds.

Positivity for $n \geqslant 3$ follows as in the proof of Theorem 6.1, and the explicit formulae (6.14), (6.15) are obtained by a procedure entirely analogous to the one used to derive (6.6), (6.7), using the appropriate polynomials π_{n+1}^*, π_n in (4.8) and (3.10), respectively, and such properties as (3.11), (3.2) and (3.16). The expression (6.16) for σ_{n+1}^*, and its positivity, follow similarly as in the proof of (6.11). □

7. Linear divisors

Up until now, we assumed that the divisor ρ in (2.1), (2.2) is a polynomial of *exact* degree 2. We now relax this condition and allow the case of *linear* divisors. Formally, this case is obtained in the limit as $\beta \to 2\alpha$, which yields

$$\rho(t) = \rho(t;\, \alpha, 2\alpha, \delta) = \alpha^2(2\mu t + \mu^2 + 1),$$
$$\mu = \delta/\alpha, \quad \alpha > 0, \quad |\mu| < 1. \tag{7.1}$$

Table 1
Gauss–Kronrod quadrature for the weight functions (7.2).

w	$2^n\pi_n$	π^*_{n+1}	Interlacing	Inclusion	Degree of exactness	
					$\mu \neq 0$	$\mu = 0$
$v^{(-1/2)}$	$2(T_n + \mu T_{n-1})$	$(t^2-1)\pi_{n-1}^{(1/2)}$ $(n \geqslant 3)$	$n \geqslant 1$	$n \geqslant 3$	$4n-2$ $(n \geqslant 3)$	$4n-1$ $(n \geqslant 2)$
		$t(t^2-1)+\frac{1}{2}\mu(t^2-\frac{3}{4})$ $(n=2)$		$n=2$ and $\mu=0$	7 $(n=2)$	
		$t^2-\frac{3}{4}+\frac{1}{2}\mu t$ $(n=1)$		$n=1$ and $\lvert\mu\rvert \leqslant \frac{1}{2}$	4 $(n=1)$	5 $(n=1)$
$v^{(1/2)}$	$U_n + \mu U_{n-1}$	$\pi_{n+1}^{(-1/2)}$	$n \geqslant 1$	$n \geqslant 1$	$4n$	$4n+1$
$v^{(1/2,-1/2)}$	$W_n + \mu W_{n-1}$	$(t+1)\pi_n^{(-1/2,1/2)}$ $(n \geqslant 2)$	$n \geqslant 1$	$n \geqslant 2$	$4n-1$ $(n \geqslant 2)$	$4n$ $(n \geqslant 2)$
		$(t+1)(t-\frac{1}{2})+\frac{1}{2}\mu(t+\frac{1}{2})$ $(n=1)$		$n=1$ and $\mu \leqslant 0$	4 $(n=1)$	4 $(n=1)$
$v^{(-1/2,1/2)}$	$V_n + \mu V_{n-1}$	$(t-1)\pi_n^{(1/2,-1/2)}$ $(n \geqslant 2)$	$n \geqslant 1$	$n \geqslant 2$	$4n-1$ $(n \geqslant 2)$	$4n$ $(n \geqslant 2)$
		$(t-1)(t+\frac{1}{2})+\frac{1}{2}\mu(t-\frac{1}{2})$ $(n=1)$		$n=1$ and $\mu \geqslant 0$	4 $(n=1)$	4 $(n=1)$

Apart from a constant factor, we are thus led to the weight functions

$$
\begin{aligned}
v^{(\pm 1/2)}(t) &= v^{(\pm 1/2)}(t;\,\mu) = \frac{(1-t^2)^{\pm 1/2}}{2\mu t + \mu^2 + 1},\\
v^{(\pm 1/2, \mp 1/2)}(t) &= v^{(\pm 1/2, \mp 1/2)}(t;\,\mu) = \frac{(1-t)^{\pm 1/2}(1+t)^{\mp 1/2}}{2\mu t + \mu^2 + 1}.
\end{aligned}
\tag{7.2}
$$

The case $\mu = 0$ corresponds to the classical Chebyshev weight functions.

The results of the previous sections, and their proofs, are easily specialized to the case $\beta = 2\alpha$. The resulting orthogonal polynomials, Stieltjes polynomials, interlacing and inclusion properties, and (sharp) degrees of exactness are summarized in Table 1, in this order.

All weights $\sigma_\mu^{(n)*}$ are positive, without exceptions, because of the interlacing property holding for all $n \geqslant 1$. The same turns out to be true for the weights $\sigma_\nu^{(n)}$. The explicit formulae given in Section 6 simplify somewhat (note that they are to be multiplied by α^2 on account of (7.1)); for $w = v^{(1/2)}$ and $n \geqslant 2$ one obtains

$$
\sigma_\nu^{(n)} = \frac{\pi}{2}\,\frac{\sin^2 n\theta_\nu}{n - \dfrac{\sin n\theta_\nu}{\sin\theta_\nu}\cos(n+1)\theta_\nu}, \quad \nu = 1, 2, \ldots, n, \tag{7.3}
$$

$$
\sigma_\mu^{(n)*} = \frac{\pi}{2}\,\frac{\cos^2 n\theta_\mu^*}{n + \dfrac{\sin(n+1)\theta_\mu^*}{\sin\theta_\mu^*}\cos n\theta_\mu^*}, \quad \mu = 1, 2, \ldots, n+1, \tag{7.4}
$$

whereas for $w = v^{(1/2,-1/2)}$ and $n \geqslant 3$,

$$
\sigma_\nu^{(n)} = \frac{\pi}{2}\,\frac{\sin^2(n-\frac{1}{2})\theta_\nu}{(1+\cos\theta_\nu)\left[n - \dfrac{1}{2} - \dfrac{\sin(n-\frac{1}{2})\theta_\nu}{\sin\theta_\nu}\cos(n+\frac{1}{2})\theta_\nu\right]}, \quad \nu = 1, 2, \ldots, n, \tag{7.5}
$$

$$
\sigma_\mu^{(n)*} = \frac{\pi}{2}\,\frac{\cos^2(n-\frac{1}{2})\theta_\mu^*}{(1+\cos\theta_\mu^*)\left[n - \dfrac{1}{2} + \dfrac{\sin(n+\frac{1}{2})\theta_\mu^*}{\sin\theta_\mu^*}\cos(n-\frac{1}{2})\theta_\mu^*\right]}, \quad \mu = 1, 2, \ldots, n,
$$

$$
\sigma_{n+1}^{(n)*} = \frac{\pi}{(1-\mu)[2(1-\mu)n + 1 + \mu]}, \tag{7.6}
$$

where θ_ν, θ_μ^* are as defined in Theorems 6.1 and 6.3, respectively. For $w = v^{(-1/2)}$ and $n \geqslant 4$, the weights are obtained as in Theorem 6.2 in terms of the weights $\sigma_\nu^{(n-1)}$, $\sigma_\mu^{(n-1)*}$ in (7.3), (7.4).

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Erratum

To: Gauss–Kronrod quadrature formulae for weight functions of Bernstein–Szegö type

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The publishers sincerely apologize for the typographical errors, which appear in the above-mentioned paper.

The first name of the second author is "Sotirios", not "Sotorios".

On p.202, line 9, read "modulus", not "modules".

On p.208, delete "x" in (3.25).

On p.213, line 4, read "$w^{(-1/2, 1/2)}$".

On p.214, in the second relation of (5.6), the second "+" sign, between "1/4" and "(", should be deleted.

On p.216, line 5, no comma after "admissible".

On p.218, line 7, read "$\mu = \frac{1}{2}(n+1)$" instead of "$\mu = \frac{1}{2}(n+2)$".