LX. Systemic Geopolitical Modeling. Part 1: Prediction of Geopolitical Events

[& N.J. Darras]

[Published first in: *GeoJournal, Spatially Integrated Social Sciences and Humanities*, vol. 79, no 4, 2014, [ISSN 0343-2521, GeoJournal, DOI 10.1007/s10708-014-9569-3]

Abstract: We give two general mathematical models predicting geopolitical events into a geopolitical system according to Mazis' lakatosian formulation methodology for a Systemic Geopolitical Analysis. To this end, we consider weighted geopolitical indices and their measurements. When the weighted geopolitical indices, as well as the related geopolitical measurements take values in different times and different geographical points, then they form two sets in the four-dimensional Euclidean space. The distance between these sets can be considered as a measure for assessing the occurrence or not of a geopolitical event. To this direction, we give general frameworks of two algorithms for determining the time moments and geographical points at which is expected the appearance of peculiar geopolitical events.

Keywords: systemic geopolitical analysis, universality of weighted geopolitical indices, parameterized surface, section of geopolitical measurement, interpolation, non-linear optimization

GeoJournal
DOI 10.1007/s10708-014-9554-x

Systemic Geopolitical Modeling
Part 1. Prediction of Geopolitical Events

Nicholas J. Daras • John Th. Mazis

Abstract We give two general mathematical models predicting geopolitical events into a geopolitical system according to Mazis' lakatosian formulation methodology for a Systemic Geopolitical Analysis. To this end, we consider weighted geopolitical indices and their measurements. When the weighted geopolitical indices, as well as the related geopolitical measurements take values in different times and different geographical points, then they form two sets in the four-dimensional Euclidean space. The distance between these sets can be considered as a measure for assessing the occurrence or not of a geopolitical event. To this direction, we give general frameworks of two algorithms for determining the time moments and geographical points at which is expected the appearance of peculiar geopolitical events.

Keywords systemic geopolitical analysis, universality of weighted geopolitical indices, parameterized surface, section of geopolitical measurement, interpolation, non-linear optimization

I Introduction

In many modern scientific studies, quantifying assumptions, data and variables can contribute to the accurate description of the phenomena through appropriate mathematical models.

So, in many disciplines, the analysts resort to a mathematical foundation of the concepts, in order to create a solid base for the theoretical formulation and solving all relevant problems. As classic examples of such an integrated mathematization, we can mention Mechanics, Physics, Biology, Earth Science, Meteorology, Medicine, Statistics, Operations Research and other branches of Theoretical Engineering disciplines such as Theoretical Computer Science, Network Security, Electronics, and Artificial Intelligence etc. In recent years, it has begun an effort to mathematical modeling of the social sciences, such as Economics, Psychology, Sociology, Political Science and Geopolitics.

Especially in the case of Geopolitics, we want mention the indicative papers (Cederman 2002; Cederman 2003; Cederman 2004; Cederman and L. Girardin 2005; Gawlik 2010; Hoff and Ward 2004) and the cited therein references. In all these contributions, Geopolitics is explored mainly by means of an agent-based modelling theory and with an analytic hierarchy process which is applied for assessment of individual and collective utility of various indices describing the actual economic situation and short–term prospects of enterprises operating locally and internationally. Also, in 2010, Kuperman presented a model based on a competitive dynamics that intends to imitate the processes leading to some characteristics of the geopolitical division (Kuperman 2010). However, although these presentations are innovative and promising, it seems that they lack a holistic view of the geopolitical environment. Moreover, there is no predictability of geopolitical events.

The aim of the present paper is to document this holistic geopolitical systemic modeling. To this end, we will give two general mathematical models

N. J. Daras ( )

Department of Mathematics and Engineering Sciences, Hellenic Military Academy, 166 73, Vari Attikis, Greece
E-mails: ndaras@sse.gr , njdaras@gmail.com

J. Th. Mazis

Faculty of Economic and Political Sciences
National & Kapodistrian University of Athens
26, Kaplanon str., 106 80, Athens, Greece
E-mails: mazis@her.forthnet.gr ,
yianmazis@turkmas.uoa.gr

predicting geopolitical events in a geopolitical system. The starting point is to consider weighted geopolitical indices and their measurements. A weighted geopolitical index is a quantity which refers exclusively to a geopolitical entity at any point of the space-time, endowed with an associated threshold above and below which it is marked a geopolitical change in the conduct of the geopolitical system (e.g. distinct spatiotemporal historical phases). A geopolitical measurement gives the value of a geopolitical entity measured at some discrete time moments, in the same, homogeneous spatiotemporal historical phase, and some geographic location points. If we limit ourselves within a given region of space-time, then the corresponding set of weighted geopolitical indices over this region is said to be a universality of weighted geopolitical indices. The magnitude of the (Euclidean or not) distance between such a universality of weighted geopolitical indices and a parameterized surface which interpolates the discrete points representing the values of a geopolitical measurement can be considered as a measure for assessing the occurrence or not of a geopolitical event. To this direction, we will demonstrate and give two general frameworks for determining the time moments and geographical location points at which is expected the appearance of peculiar geopolitical events. The corresponding algorithmic formulations show that the prediction problem is reduced to two respective classical nonlinear optimization problems.

Two basic and reasonable questions arise immediately and may be constitute the central subject of discussion in subsequent additional scientific studies. The first question relates to the *subjectivity of geopolitical choices and priorities*: given that it is very doubtful whether the considered set of weighted geopolitical indices could be considered as exhaustive, one wonders if the above prediction is ultimately reliable. Equivalently, if a geopolitical analyst considers a set of weighted geopolitical indices and if another geopolitical analyst considers a different set of weighted geopolitical indices, then how much the two predictions will differ or diverge? The second question concerns *the reliability of geopolitical measurements*: given that many geopolitical measurements are based on qualitative data, hierarchical structures and design practices, statistical methodologies, and information files, how much the reliability of the result of geopolitical measurements could affect the validity of a prediction? In other words, any divergences in the measurement values how much affect the accuracy of the time moments and of the locations where we expect appearance of a geopolitical event?

Without loss of generality and in order to simplify the overall formulation of the model, in what follows, we will assume continuously that *there is a complete objectivity in all geopolitical options and priorities*, in the sense that all geopolitical analysts have agreed for the finalized selection of all weighted geopolitical indices. For the case in which it arises question concerning subjectivity of geopolitical analysts' preference priorities, the interested readers are referred to the forthcoming article by Daras and Mazis (in preparation). Moreover, for the same reasons, we will assume also regularly that *all geopolitical measurements were carried out with sufficient reliability* to such an extent as to preclude any discrepancy in the estimates of the predictions.

In the sections 2 and 3, we will provide basic definitions of the geopolitical concepts which will be used subsequently, such as the weighted index geopolitical and the geopolitical measurement. The whole content of the second section, as well as the definition of the weighted geopolitical index, are derived entirely from (Mazis 2002; Mazis 2008). Moreover, the development we adopt in the third section for the concept of geopolitics measurement is fully aligned to a summarized description of the classic analysis of measurement systems (Levin 2006; Robertson 1993). In the fourth section, we will give algebraic formulations that refer to the geopolitical space of weighted geopolitical indices over a given system. The algebraic approach will give the possibility of introducing new concepts, such as the concept of the geopolitical fiber at a time and in one geographical location and the concept of the geopolitical affinity between two geopolitical systems. This approach within the fourth section will come with the consideration of a fiber product between two spaces consisting of weighted geopolitical indices. In the fifth section we will introduce and examine the structure of the, so-called, universalities of weighted geopolitical indices. We will distinguish between two cases: the case in which such a given universality forms a parameterized surface into the geopolitical space of weighted geopolitical indices over a given system and the case where the universality exhibits discontinuities. In the same section, we will describe the aspect of a geopolitical measurement at some discrete time moments and some geographic location points, and then we will discuss the concept of deviation which can have such a geopolitical measurement from a given universality of weighted geopolitical indices. Based on this background, in the sixth, and final, section we will move on to considering a distance between a given universality of weighted geopolitical indices and the parameterized surface which interpolates the points that represent the values of the corresponding geopolitical measurement. Notice that *the choice of an appropriate distance is non-unique, and may be determined according to the formulation of each problem*. This approach allows predicting of time moments and of geographical location points at which is expected to happen a geopolitical event. Indeed, *if at some point of space-time, the distance between the two surfaces exceeds a given critical value, then it means that at the prescribed time moment is expected a geopolitical event in this location point*. This prediction will be described in two cases. Firstly, in we will consider the case where the measurements are conducted at discrete time moments and the geographical location remains constant. In such a case, the parameterized surface which interpolates the points of the measurement is defined by means of the Lagrange unique polynomial (Gasca Sauer 2000). Secondly, we will consider the case where the measurements are conducted at discrete time moments and (generally) over different location points. In such a case, the parameterized surface which interpolates the points of the geopolitical measurement is defined by means of the Kergin unique polynomial (Kergin 1980). In both cases, the methods, which lead to the prediction of a geopolitical event, are given by two corresponding algorithms, resulting in two constrained nonlinear optimization problems which can be solved by one of the beautiful methods of the relevant literature (Avriel 2003; Bazaraa & Shetty 1979; Bertsekas 1999; Bonnans et all. 2006; Luenberger & Ye 2008; Nocedal and Wright 2006; Parsopoulos & Vrahatis 2002; Parsopoulos & Vrahatis 2010; Ruszczyński 2006). Notice that the choice of the interpolation method giving the parameterized surface is not binding. For example, one can use spline functions, instead of interpolation polynomials, but the central idea of the method remains unchanged.

II Geopolitical analysis. Aspects and methodology

According to I. Th. Mazis' definition, the ***geopolitical analysis*** of a geographical system is characterized by an uneven distribution of power and is "*the geographical method that studies, describes and predicts the attitudes and the consequences ensuing from relations between the opposing and distinct political practices for the redistribution of power as well as their ideological metaphysics, within the framework of the geographical complexes where these practices apply*" (Mazis 2002; Mazis 2008; Mazis 2013; Mazis 2014; Mazis in preparation).

2.1 Designing the methodological proposal of a geopolitical analysis

Mazis' Systemic Geopolitical Analysis consists of the following six general stages (Mazis 2002; Mazis 2008; Mazis 2013; Mazis 2014; Mazis in preparation).

1st Stage: Decoding the title of the topic

The title of the topic of a study of geopolitical analysis (should) define(s) the facts and the objectives of our problem. In particular it defines:

1) The *boundaries of the geographical complex* which constitutes the geographical area to be analyzed.

2) The (internal or external) *space of the complex* under study as a field of distribution or redistribution of power due to the activity of a specific geopolitical factor.

3) The above-mentioned *geopolitical factor*, the impact of which may affect the distribution of power, within or outside this geographical complex.

Example 2.1. Analysis of the title of the topic: *the geopolitics of the Islamic movement in the Greater Middle East*.

1) *Identification of the boundaries of the geographical complex:* The boundaries of the geographical Complex are defined by the term "Greater Middle East".

2) *Precise identification of the space under study:* The space under study of this specific complex is the "interior" space of the geographical complex of the Greater Middle East and this is evident by the use of "in", i.e. "in the inside of...", "within the boundaries of...".

3) *Identification of the geopolitical factor:* The designated geopolitical factor is the "Islamist movement". ■

2nd Stage: Identifying the boundaries of the Geopolitical Systems under study

At this stage, we identify the boundaries of the geopolitical systems within which we are going to study the activity (or activities) of the geopolitical factor defined in the title.

There are three levels of systems defined according to the extent of the geographic area they refer to:

(i). *Sub-systems* that are subsets of the systems.

(ii). *The system* that is the geographical complex under investigation.

(iii). *Supra-systems*, containing the main system under study -as a subset- along with other Systems that may not concern the current analysis.

Remark 2.1. In general, in order to define the system/geographical complex in question in terms of geographical extent, a qualitative trait is also required, one that will identify -with its very presence, its forms and its level of influences- the extent of the geographical areas of the above-mentioned systemic levels/scales. Without this qualitative trait and its particular characteristics, the definition of the three above-mentioned levels of systems would not only be impossible, but also meaningless. ■

Example 2.2 (continuation of Example 2.1) In the above-mentioned topic the boundaries of the Systemic levels are defined as follows.

(i). *System:* The geographical complex of the Greater Middle East, not only because it is stated in the title, which already consists a fundamental criterion, but also because of the fact that the "geopolitical factor", i.e., the "Islamist movement", exists, acts, and affects the whole geographical area of the complex.

(ii). *Sub-systems:*

- The "Islamist movement in Maghreb" constitutes a sub-system due to its peculiarities that relate to the cultural, economic, political and organizational character of Islam in this geographical area.
- The "Islamist movement in the Middle East" for the same reasons.
- The "Afghan-Pakistani and the Iranian Islamic movement".

(iii). *Supra-system:* We can define as supra-system the entity with the following characteristics:

- State power poles;
- International collective Security systems (e.g., NATO);
- Supranational collective systems in general (e.g., EU, UN);
- International multinational financial or operational power poles which influence the "geopolitical factor" acting, however, from the External space of the geographical complex. ■

3rd Stage: Defining the fields of influence of a geopolitical factor

Once we have defined the three levels of systems, we should identify the fields of geopolitical influence of the geopolitical factor under study. In other words, we should determine which combination of the four fields or ***geopolitical pillars*** of the given geopolitical factor (GF) we are going to investigate, always within the framework of the chosen systemic scale (e.g. on the level of system or on the level of sub-systems).

In order to follow a rational order in the examination of the influences of the GF, we should start the investigation from the "supra-systems" level and continue with the system level. Such a sequential order should prove that, in most cases, if the analysis of the influences of the GF on the level of the sub-systems is completed, and if sub-systems have been correctly identified, the respective analysis on the level of the whole system is also completed.

The four main geopolitical pillars are as follows:

1) Defense / Security 2) Economy
3) Politics and 4)Culture and Information

4th Stage: Identifying the function of a geopolitical factor for the specific pillars of influence

At this stage we are going to identify the geopolitical trends-dynamics for each designated subsystem. These trends may answer the following questions:

– *The pillars (defense, economy, politics, culture) where the GF under study prevails* (in our case the GF "Islamist movement"), *and* by consequence *already determines or may determine their attitude within the framework of each sub-system*. This type of conclusion is defined as ***positive sub-systemic component of the trend power*** of the GF in the interior of the system.

– *Which pillars absorb the influence of the GF*, *and* by consequence, *it does not influence the whole attitude of the sub-system*. This form of conclusion is defined as ***zero sub-systemic component power trend*** of the GF in the interior of the system.

5th Stage: Synthesis

The term ***synthesis*** refers to the procedure through which we can detect the so-called Resultant Power Trend (Mazis 2002; Mazis 2008; Mazis in preparation) of the given GF on whichever final systemic scale (e.g. sub-system, system or supra-system level). We may distinguish between two cases.

– 1st case: If we have detected and defined the particular power components (of the GF at hand) on the sub-system level, and our objective is the component of the system on the systemic level, then the stage of synthesis begins from the level of the system.

– 2nd case: If the component in question is on the level of the supra-system, then the stage of synthesis starts after the conclusion of the analysis of the components of the individual systems. This means that the synthesis should start from the level of sub-systems, and we should then shape the image of the components on the level of systems, and finally conclude with the identification of the component on the level of supra-system.

6th Stage: Conclusions

The last stage of the geopolitical analysis is that of conclusions. At this stage we must describe the geopolitical dynamics, to which the *component of power* of the GF under study, subjects the attitude of the system examined, in the context of the supra-system.

We must stress that: at this stage of the study, as in any other stage of the aforementioned geopolitical analysis, we make no proposals. Moreover, at this stage

(i). we discover structures, actions, functions, influences, forms and dynamics of the geopolitical factor and we describe them and

(ii). we describe how they affect the attitude of the system.

Proposals do not form part of a geopolitical analysis. They are part of the geo-strategic approach which may be carried out, only if asked and by exploiting the results of the geopolitical analysis preceding.

2.2. The Lakatosian structure of the systemic geopolitical analysis contents

Now, we must present four main fields of our methodological construction*:* the definition of the fundamental axiomatic conditions (:*assumptions 1-3*) of the hard core of the geopolitical research program; the definition of the auxiliary hypotheses (: *assumptions 4-9*) of the protective belt of the geopolitical research project; the issue of the positive heuristics of the geopolitical research program; and the assumptions of the positive heuristics of the geopolitical research program.

2.2.1. Fundamental axiomatic conditions of the hard core of a geopolitical research program

According to the Lakatosian meta-theoretical approach, as it is encoded by C. Elman and F. Elman (2003), the *hard core* (fundamental assumptions) constitutes the basic premise of a research program.

The hard core is protected by negative heuristics, in short, by the rule that prohibits researchers to contradict the fundamental ideas of a given research program, i.e., with the hard core of the program (as an attempt to address new empirical data which tend to invalidate the theory). Any change to the hard core would mean the creation of a new Research Program, since it is clear that the hard core is the presupposition that determines the character of a Program. It is therefore obvious, from a Lakatosian point of view, that if the core changes, the Research Program also changes. That being said we must introduce the following hypotheses (Mazis 2002; Mazis 2008; Mazis 2013; Mazis in preparation).

Assumption 1. The first fundamental axiomatic condition, which constitutes the centre of the hard core of the geopolitical research program, is that ***all the characteristics of the above-mentioned sub-spaces of the geographical complex are countable or can be counted***, *through the countable results which they produce*, e.g., the concept of "democraticity" of a polity (according to western standards, since there are no other). ■

This is a concept identified as a "***geopolitical index***" within the framework of the secondary causative "Political Space", as defined earlier, and can be countable by means of a multitude of specific results, which it produces in the society where this form of political governance is applied. Such are, for example, *the number of printed and electronic media in the specific society*, *the number of political prisoners or their absence*, *the level of protection of children of single-parent families*, *the number of reception areas for immigrants and density of the latter per square meter*, etc. These figures are classified, systematized and evaluated according to their specific ***weight*** concerning the function of the figure to be quantified, and constitute the *geopolitical indices* that we are going to present and examine in detail below.

For obvious reasons, and to avoid confounding effects, in what follows, we will assume continuously the ideal situation:

Assumption 2.*There is a complete objectivity in all geopolitical options and priorities*, in the sense that *the geopolitical analysts, who study the given geopolitical system, have agreed for the finalized selection of all geopolitical indices governing the geopolitical system behavior.* ■

Assumption 3. The second fundamental axiomatic assumption of the hard core of the systemic geopolitical program is that *within the framework of the geographical area under study, there exist more than two consistent and homogeneous geopolitical poles which are also*:

a) *self-determined* (as to "what" they consider "gain" and "loss" for themselves), and also in relation to their international environment, and
b) *hetero-determined, uniformly and identically to their international environment which is determined by the international actors that well within them and their common systemic relation is their characteristic*. ■

2.2.2. Auxiliary conditions of the protective belt of a geopolitical research project

Consequently, following the Lakatosian meta-theoretical paradigm, the protective belt of the geopolitical research program should be defined, complemented with the following auxiliary hypotheses (Mazis 2002; Mazis 2008).

Assumption 4.*The size of the power is analyzed in four fundamental entities (Defense, Economy, Politics, Culture/Information), which in turn are analyzed in a number of geopolitical indices*. ***These geopolitical indices, as already mentioned, are countable or can be counted and they are detected and counted in the internal structures of those geopolitical poles that constitute the sub-systems of the geographical complexes under geopolitical analysis***.■

Assumption 5.*The geopolitical poles constitute fundamental structural components of an international, and ever-changing, unstable system*. ■

Assumption 6.*These geopolitical poles express social volitions or volitions of the deciding factors that characterize the international attitude of the pole*. Consequently, these poles can be national states, collective international institutions (e.g., international collective security systems, international development institutions, and international cultural institutions), economic organizations of an international scope (i.e., multinational companies, bank consortia) or combinations of the above which, however, present uniformity of action within the international framework concerning their systemic functioning. ■

Assumption 7. The fourth auxiliary hypothesis of the protective belt of the geopolitical research program consists of the above-mentioned ***"causal*** *and causative" notions of the "primary", "secondary" and "tertiary space", as well as their combinations ("complete" and "special composite spaces")*. ■

Assumption 8.*The international system has a completely unsure, unstable and changing structure.*■

Assumption 9.*Systemic geopolitical analysis aims to conclusions of "practicology", shortly, of some "theory of practice"* (Aron 1967)*, i.e., to the construction of a predictive model of the trends of power redistribution and in no case to "guidelines for action under some specific national or "polarized" perspective.* The latter is nothing but the "geo-strategic biased synthesis", not a "geopolitical analysis". This equals the use of the results (of the model of power redistribution) of the geopolitical analysis and follows the stage of geopolitical analysis.■

We must note that the "historicity" of these assumptions of the research program is represented by the cultural formations developing in the context of the fourth geopolitical pillar. Thus, their countability is possible in the same way as is for the rest of the geopolitical pillars that have a "qualitative nature", by means of the geopolitical indices of the cultural pillar.

2.2.3. The issue of positive heuristics of the geopolitical research program

We should not forget that replacing a set of auxiliary assumptions by another set is an intra-program problem shift, since only the protective belt and not the hard core is altered. The intra-program problem shifts should be made in accordance with the positive heuristics of the problem, which is with a set of suggestions or advices that function as guidelines for the development of particular theories within the program.

We should also emphasize that, a key concern of the geopolitical research program is to describe the suggestions to the researcher that will determine the content of the positive heuristics of the program in question. Without them, it is impossible to assess the progressivism of the geopolitical analysis according to the necessary "novel empirical content" expected in our analytical spatial paradigm (model).

Given these necessary clarifications concerning the assumptions of the positive heuristics of the geopolitical research program, we define the following:

1) *The methodology of each theoretical approach should remain stable until a possible detection of continuous degeneration*.

2) *The requirement of predictive ability and the expansion of the empirical basis of the theoretical approach should be maintained*.

3) *The empirical facts should constitute the final measure for assessing competitive theoretical approaches of the same set [research program]*.

4) *The facts that have been used to test a theoretical approach should not be the only ones used for verifying this approach but, with the progress of time of research, the testing of the theoretical approach should be re-fed also with facts that derive from the expansion of the empirical basis of the given approach* (Mazis 2002; Mazis 2008; Mazis in preparation).

III Weighted geopolitical indices and geopolitical measurements

It is well known that any quantification requires a corresponding assumption in the measurement mode of quantities. But, even if it looks easy in mathematical sciences, in the social sciences, quantitative methods need to be improved and tested yet, creating scope for debate and reflection. Therefore, our first concern should be directed towards this just field. For this purpose, we define as ***geopolitical index***, each quotient of the following form

$$g_S^{(j)}(t,x,y,z)/\mathfrak{D}_S^{(j)}(t,x,y,z)$$

where

—

- $g_S^{(j)}(t,x,y,z)$ is the measured value of the geopolitical characteristic j at date $t \in \mathbb{R}$ and location $(x,y,z) \in \mathbb{R}^3$, into the geopolitical system (or geopolitical complex) S
- $\mathfrak{D}_S^{(j)}(t,x,y,z)$ is the ***weighted geopolitical index*** of the geopolitical characteristic j at date $t \in \mathbb{R}$ and location $(x,y,z) \in \mathbb{R}^3$, into the geopolitical system (or geopolitical complex) S.

Definition 3.1.*A **weighted geopolitical index***

$$\mathfrak{D}_S^{(j)}(t,x,y,z)$$

is a specified reference value (or weighted threshold / limit) of the geopolitical index $g_S^{(j)}$ for the geopolitical characteristic j depending on the date $t \in \mathbb{R}$ and location $(x,y,z) \in \mathbb{R}^3$, above and below which, there is a change in the behavior of any active within the geographical complex, affecting both the other geopolitical indices, as well as the power and influence of others geopolitical characteristics acting in the geographical complex. ■

In other words, weighted geopolitical indices can constitute the basis of indices, which are defined as a function of international Treaties dictated by supra-systems. As an example, we can mention the Treaty of Maastricht, which dictated all economic conditions, and not only for the enlargement of the European Union supra-systems.

As an immediate consequence of *Assumption 2*, we will assume that

Assumption 2'.*The geopolitical analysts, who study the given geopolitical system, have agreed for the finalized selection of all weighted geopolitical indices governing the geopolitical system behavior.* ■

On the other hand, apart from the values of the weighted geopolitical indices $\mathfrak{D}_S^{(j)}(t,x,y,z)$, we are also interested in the numerical values $g_S^{(j)}(t,x,y,z)$ of the relevant geopolitical characteristics. For this purpose, we must measure the actual numerical values which take the selected geopolitical characteristics at any given time and any geographical location. However, the ***geopolitical measurements***, taken by a single person or instrument on the same item and under the same conditions, may contain errors due to various causes (rounding of measurements, erroneous information, limited databases etc.). Therefore, there should be a good analysis, the results of which if implemented, all attempted geopolitical measurements will be as reliable as possible.

In general, such an analysis uses scientific tools to determine the amount of total variation from the geopolitical measurement. An obvious method to assess the validity of a geopolitical measurement is to minimize the factors that could excessively contribute to the variation in the data. Towards this direction, the objective of the analysis is to confirm that the geopolitical measurement used to collect the data is valid. So, the main aim is to quantify the equipment/process variation and appraiser variation and the total measurement system variation. The following areas components of geopolitical measurement error need to be studied and quantified in order to ensure the quality of any geopolitical measurement, before establishing capability of a process making decisions from the data:

1) *resolution*,
2) *accuracy (bias)*,
3) *linearity*,
4) *repeatability*
5) *reproducibility* and
6) *stability*.

Definitions 3.2.i. ***Resolution*** *is the incremental ability of a geopolitical measurement to discriminate between geopolitical measurement values.* The geopolitical measurement should have a minimum of 20 geopolitical measurement increments within the product tolerance (e.g. for a full tolerance of 1, minimum resolution is 0.05).

ii. ***Accuracy*** —or ***bias*** —*is a measure of the distance between the average value of the geopolitical measurement of a part and the True, certified, or assigned value of this part.*

iii. ***Linearity*** *is the consistency of geopolitical accuracy (bias) over the range of geopolitical measurement; a slope of one (unity) between measured and true value is perfect.*

iv. ***Repeatability*** *is the consistency of a single appraiser to measure the same part multiple times with the same measurement system; it is related to the standard deviation of the measured values.*

v. ***Reproducibility*** *is the consistency of different appraisers in measuring the same part with the same geopolitical measurement; it is related to standard deviation of the distribution of appraiser averages.*

vi. ***Stability*** *is the ability of a geopolitical measurement to produce the same values over time when measuring the same sample As with statistical process control charts, stability means the absence of "Special Cause Variation" which is indicated by an "in control" condition, leaving only "Common Cause" or random variation.* ■

Substantially, the accuracy is the difference from the true value and the value from the geopolitical measurement. In other words, it represents the closeness degree of the geopolitical measurement value of a geopolitical characteristic to its actual (true) value. In conceptual pertinence to the concept of accuracy, we have the concept of precision that is slightly different and is covered under the geopolitical repeatability. More specifically, the *precision* of a geopolitical measurement is the degree to which repeated geopolitical measurements under unchanged conditions show the same results. Although the two words precision and accuracy can be synonymous in colloquial use, they are deliberately contrasted in the context of the geopolitical method.

For best accuracy of the geopolitical data, we can follow the following general rules.

1) Accept all geopolitical data as it is collected. Assigning special cause and scrutinizing the data can come later.

2) Record the geopolitical data at the time it occurs.

3) Avoid rounding off the geopolitical data, record it as it is.

4) On the geopolitical data collection plan, record as many details around the geopolitical data such as the exact source, machine, operator, conditions, collector's name, material, gage, and time. Record legibly and carefully. The geopolitical data should be screened for misplaced decimal points, duplicate data entries by mistake or improper recording procedure, missing date points if frequency is important and other obvious non-representative geopolitical data.

5) Verify the gage is accurate. If using a weigh scale, verify it with a known and calibrated weight. Use gage blocks for calipers or micrometers. Use hardness blocks to verify hardness testers.

The goal of the resolution is to have at least 5 distinct values or categories of readings. Adhere to the 10-bucket rule. If the geopolitical measurement requires geopolitical measurements to the hundredths (x.xx), then divide that by 10. Collect and record the geopolitical data to the nearest thousandths (x.xxx). The geopolitical measurement shall be sensitive to change and capable of detecting change. The lack of resolution will not allow a geopolitical measurement detect change. If the measurement of downtime and use of geopolitical measurement to the nearest hour and most downtime is less than an hour then most of the reading will either be a 0 (for 0 hours) or a 1 (for 1 hour). However, using a stop watch and recording geopolitical data to the nearest minute will provide 60 × more resolution and allow better distribution of geopolitical data points, more variety of geopolitical data, with fewer repeat geopolitical measurements. You could have 60 different readings. Actually recording the nearest 6 minutes would satisfy the 10-bucket rule, but it is a guide to help ensure resolution in the geopolitical measurement.

When gathering geopolitical data only collect with the acceptable limits where there is proven linearity. This is a test to examine the performance of the geopolitical measurement throughout the range of geopolitical measurements.

Stability is analyzed using control charts. Ensuring the geopolitical measurements taken by appraiser(s) for the process is stable and consistent over time. SPC Charts use a variety of tests to determine stability. Many software programs will have these as options to include when analyzing geopolitical data and will even indicate the point(s) and test that each failed. Some of the corrective measures once again include Standard Operating Procedures. Each geopolitical appraiser should measure the same way every time over a long period of time and each geopolitical appraiser should measure the same way as all the others. Recall that special causes can also occur with the process control limits and these must be given corrective action before proceeding to validate the geopolitical measurement.

Repeatability or test–retest reliability is the variation in geopolitical measurements. A less-than-perfect test–retest reliability causes test–retest variability. Such variability can be caused by, for example, intra-individual variability and intra-observer variability. Reproducibility is the degree of agreement between measurements or observations conducted on replicate specimens in different locations by different people, as part of the precision of a test method[1].

Gage Repeatability and Reproducibility (in brief, ***gage R&R***), which stands for gage repeatability and reproducibility, is a statistical tool that measures the amount of variation in the geopolitical measurement arising from the measurement device and the geopolitical analysts taking the measurement. It is the amount of geopolitical measurement variation introduced by a geopolitical measurement, which consists of the measuring instrument itself and the geopolitical analysts using the instrument. A gage R&R study quantifies three things:

1. Repeatability (: variation from the measurement instrument)
2. Reproducibility (: variation from the geopolitical analysts using the measurement instrument)
3. Overall gage R&R, which is the combined effect of (1) and (2).

The overall gage R&R is normally expressed as a percentage of the tolerance for the ***CTQ*** (***Critical-to-Quality Characteristic***[2]) being studied, and a value of 20% gage R&R or less may be considered acceptable in most cases.

Remark 3.1. In a variable gage R&R there are generally two to three geopolitical operator appraisers with 5-10 process outputs measured by each geopolitical appraiser. Each process output is measured 2-3 times by each operator. Depending on the cost and time involved one can add more geopolitical appraisers and geopolitical measurements and replications. When performing the replicated appraisals it is critical that the geopolitical measurement is randomized so that no patterns or predictability can be entered in by the geopolitical appraiser. This geopolitical bias will mislead the team and create a useless gage R&R. For example, a geopolitical appraiser may remember the 7th part that was measured was borderline and made a decision to give it one geopolitical measurement. He/she may have spend a lot of time of that part and if the 2nd round of geopolitical measurements are not randomized, that person will remember the geopolitical measurement (appraisal) they gave it on the first round. So, move the parts around each repeat set of geopolitical measurements. However, the parts must be indentified so the person entering the data into the statistical software enters the reading under the correct part. ■

Generally, precision is the principle concern; inaccuracy due to linearity or constant bias can typically be corrected through calibration.

The measurement error is the statistical summing of the error generated by repeatability (the variation within an appraiser) and reproducibility (the variation between appraisers) is given by

$$\sigma_{error} = \sqrt{\left(\sigma_{repeatability}\right)^2 + \left(\sigma_{reproducibility}\right)^2}.$$

The total geopolitical measurement error spans the interval that contains 99% of probable geopolitical measurement values from a geopolitical measurement, using a single part:

$$total\ geopolitical\ measurement\ error = 5.15 * \sigma_{error}.$$

[1] Reproducibility also refers to the ability of an entire experiment or study to be reproduced, either by the researcher or by someone else working independently. It is one of the main principles of the scientific method and relies on ceteribus paribus. The result values are said to be *commensurate* if they are obtained (in distinct experimental trials) according to the same reproducible experimental description and procedure. The basic idea can be seen in Aristotle's dictum that there is no scientific knowledge of the individual, where the word used for *individual* in Greek had the connotation of the *idiosyncratic*, or wholly isolated occurrence. Thus all knowledge, all science, necessarily involves the formation of general concepts and the invocation of their corresponding symbols in language (http://en.wikipedia.org/wiki/Reproducibility).

[2] CTQ is a characteristic of a product or service which fulfills a critical customer requirement. CTQ's are the basic elements to be used in driving process measurement, improvement, and control.

Geopolitical measurement precision is defined by the "precision/ tolerance ratio", the ratio between total geopolitical measurement error and the part tolerance

$$\frac{P}{T} = \frac{5.15 * \sigma_{error}}{Upper\ Spec\ Limit - Lower\ Spec\ Limit}.$$

Remark 3.2.*Error independence* is defined by the lack of a relationship between geopolitical measurement error and the geopolitical measurement value; error generated by the geopolitical measurement process should be independent of the measured value. In this sense, stability can be defined by the randomness of the measurement error; purely random measurement error is evidence of good stability. Also, linearity can be defined by the slope of measured value vs. true value; a slope of 1 (a 1:1 relationship) is perfect. Bias Offset is defined by the average difference between the measured value and the true value at the specification target; a value of zero is perfect. The combination of bias offset and linearity define the amount of ***systematic geopolitical measurement error*** across the entire geopolitical measurement range; they are typically corrected through calibration. ■

We will complete this section, by giving some sufficient technical requirements relating to the practical acceptance or rejection of geopolitical measurements.

Geopolitical Measurement Parameter	Requirement
Precision/ Tolerance Ratio	$\begin{cases} \frac{GP}{GT} < 10\% \Rightarrow \boldsymbol{Accept} \\ 10\% < \frac{GP}{GT} < 30\% \Rightarrow \boldsymbol{Marginal\ Acce} \\ 30\% < \frac{GP}{GT} \Rightarrow \boldsymbol{Fall} \end{cases}$
Error independence	***Pass*** *the hypothesis test that geopolitical error is independent of measured value*
Stability	*Geopolitical Measurement Error is* ***in control****, when plotted on a control Chart*
Bias	***Pass*** *the hypothesis test that no offset exists between true and measured value at the spec target*
Linearity	***Pass*** *the hypothesis test that slope between the true and measured values is equal to one*

For obvious reasons, and to avoid confounding effects, in what follows, we will assume continuously the ideal situation:

Assumption 10.*All geopolitical measurements are carried out with sufficient reliability* to such an extent as to preclude any discrepancy in the estimates of the predictions. ■

IV Basic algebraic considerations

4.1. The space of weighted geopolitical indices over a geopolitical system

Let S be a geopolitical system (or geopolitical complex). A ***weighted geopolitical index*** *of the system S at date t and location (x, y, z)*

$$\mathfrak{D}_S^{(j)} = \mathfrak{D}_S^{(j)}\left(P_{1/S}^{(j)}, \dots, P_{N_j/S}^{(j)}; t, x, y, z\right)$$

is a numerical function of the values of its N_j intrinsic properties (physical characteristics) $\left(P_{1/S}^{(j)}, \dots, P_{N_j/S}^{(j)}\right)$ into the system S, the date $t \in \mathbb{R}$ and the location $(x, y, z) \in \mathbb{R}^3$ at which it is studied.

It is assumed that *there are a finite number of distinguishable weighted geopolitical indices of the system S, say $\mathfrak{D}_S^{(1)}$, $\mathfrak{D}_S^{(2)}$,..., $\mathfrak{D}_S^{(\ell+1)}$ for any date t and any location (x, y, z)*. Further, to simplify our study, *any geopolitical index of S at date t and location (x, y, z) is supposed to be continuous in t and (x, y, z)*.

Definition 4.1.*If every unit vector*

$$\mathbb{I}^{(j)} = \underbrace{(0, \dots 0,1,0, \dots ,0)}_{j}$$

of the vector space $\mathbb{R}^{\ell+1}$ is identified with one unit of the geopolitical index $D^{(j)}$ of the system S at date t and location (x, y, z), $(j = 1,2, \dots, \ell + 1)$, then the linear space

$$\mathbb{G}_{t,x,y,z}(S) \coloneqq \left\{\mathfrak{D}_S = c_1\mathfrak{D}_S^{(1)} + \dots + c_\ell \mathfrak{D}_S^{(\ell+1)} : \; : c_1, c_2, \dots, c_{\ell+1} \in \mathbb{R}\right\} \equiv \mathbb{R}^{\ell+1}$$

*is called a **momentary local weighted geopolitical index space** of the system S at date t and location (x, y, z).*■

Proposition 4.1. *$\mathbb{G}_{t,x,y,z}(S)$ is a linear topological space with respect to the usual Euclidean distance in $\mathbb{R}^{\ell+1}$.*■

Let us now consider the union

$$\mathbb{G}(S) := \bigcup_{t\in\mathbb{R},(x,y,z)\in\mathbb{R}^3} \mathbb{G}_{t,x,y,z}(S)$$

$$\equiv \mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^{\ell+1} = \mathbb{R}^{\ell+5}$$

Definition 4.2. *The linear space $\mathbb{G}(S)$, endowed with the usual Euclidean topology in $\mathbb{R}^{\ell+5}$, is called the* ***space of geopolitical indices*** *over the system S. The elements of $\mathbb{G}(S)$ are called the* ***weighted geopolitical indices*** *of the system S.* ■

With this notation, we have the following.

Proposition 4.2. *The geopolitical index space $\mathbb{G}(S)$ of the system S is a connected space over the topological space $\mathbb{B} = \mathbb{R}\times\mathbb{R}^3$, endowed with a continuous projection $\pi_S\colon \mathbb{G}(S)\to\mathbb{B}$.* ■

It is now immediately seen that

Proposition 4.3. *For each point $(t,x,y,z)\in\mathbb{B}$, the momentary local geopolitical index space $\mathbb{G}_{t,x,y,z}(S)$ of the system S at date t and location (x,y,z) coincides with the* ***geopolitical fiber***

$$\pi_S^{-1}(t,x,y,z)$$

of $\mathbb{G}(S)$ at the point (t,x,y,z). ■

Remark 4.1. For any $\mathfrak{D}_S$ and $\mathfrak{B}_S$ in the geopolitical fiber $\pi_S^{-1}(t,x,y,z)\equiv\mathbb{G}_{t,x,y,z}(S)$ *of $\mathbb{G}(S)$ at the point (t,x,y,z),* there are two disjoint neighborhoods of $\mathfrak{D}_S$ and $\mathfrak{B}_S$ (: we say that $\mathbb{G}(S)$ *is* ***separable***.) ■

Since the geopolitical index space $\mathbb{G}(S)$ of the system S is separated and connected, we can answer the question how many momentary local geopolitical indices of the system S exist.

Lemma 4.1 (The Cardinality Lemma) *Whenever $(t,x,y,z)\in\mathbb{B}$, the cardinality $Card\left(\mathbb{G}_{t,x,y,z}(S)\right)$ of the corresponding geopolitical fiber $\pi_S^{-1}(t,x,y,z)\equiv\mathbb{G}_{t,x,y,z}(S)$ does not exceed the infinite cardinality of any basis of open sets in $\mathbb{B}$.* ■

Further, since

$$\mathbb{G}(S) = \mathbb{R}^{\ell+5} = \mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^{\ell+1} = \mathbb{B}\times \underbrace{\mathbb{R}^{\ell+1}}_{\mathbb{G}_{t,x,y,z}(S)},$$

the geopolitical index space $\mathbb{G}(S)$ is a ***trivial bundle of discrete fibres*** $\mathbb{G}_{t,x,y,z}(S) = \mathbb{R}^{\ell+1}$. It follows the next result which we present for later use.

Proposition 4.4. *The geopolitical index space $\mathbb{G}(S)$, endowed with the continuous projection $\pi_S\colon \mathbb{G}(S)\to\mathbb{B}$, is (also) a* ***geopolitical covering space*** *of $\mathbb{B}$.* ■

It is important to see that the inverse image $\pi_S^{-1}(K)$ of any compact set K in $\mathbb{B}=\mathbb{R}^4$ is compact in $\mathbb{G}(S)=\mathbb{R}^{\ell+5}$. Thus, the geopolitical index space $\mathbb{G}(S)$ is a ***quasi-compact space*** in the following sense.

Proposition 4.5. *For any $(t,x,y,z)\in\mathbb{B}$, and for any family $(U_i)_{i\in I}$ of open subsets of $\mathbb{G}(S)$ such that*

$$\bigcup_{i\in I} U_i \supset \pi_S^{-1}(t,x,y,z)\equiv\mathbb{G}_{t,x,y,z}(S)$$

there exists a finite part J of I and an open neighbourhood V of (t,x,y,z) such that

$$\bigcup_{i\in J} U_i \supset \pi_S^{-1}(V). \blacksquare$$

In particular, we have the following.

Corollary 4.1. *The geopolitical index space $\mathbb{G}(S)$ is a* ***proper space*** *over $\mathbb{B}$.* ■

4.2. Affinities between geopolitical systems

Let S and T be two geopolitical systems. Let us consider the corresponding weighted geopolitical index spaces $\mathbb{G}(S)$ and $\mathbb{G}(T)$, with projections π_S and π_T respectively.

Definition 4.3. *A continuous mapping $F\colon\mathbb{G}(S)\to\mathbb{G}(T)$ is said to be a* ***geopolitical affinity*** *between the systems S and T if the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{G}(S) & \overset{F}{\rightrightarrows} & \mathbb{G}(T) \\ \pi_S \searrow & & \swarrow \pi_T. \\ & \mathbb{B} & \end{array} \blacksquare +$$

Proposition 4.6. *If $F\colon\mathbb{G}(S)\to\mathbb{G}(T)$ is a geopolitical affinity between the systems S and T, then, for any $(t,x,y,z)\in\mathbb{B}$, F induces a mapping*

$$F_{t,x,y,z}\colon \mathbb{G}_{t,x,y,z}(S)\to\mathbb{G}_{t,x,y,z}(T)$$

of the momentary local geopolitical index space of the system S at date t and location (x,y,z) into the momentary local geopolitical index space of the system T at date t and location (x,y,z). ■

Proposition 4.7. *Any geopolitical affinity $F\colon\mathbb{G}(S)\to\mathbb{G}(T)$ between the systems S and T is onto the geopolitical index space $\mathbb{G}(T)$.* ■

Proposition 4.8. *Let $F: \mathbb{G}(S) \to \mathbb{G}(T)$ be a geopolitical affinity between the systems S and T. Assume that there exists a point $(t,x,y,z) \in \mathbb{B}$ such that the induced mapping $F_{t,x,y,z}: \mathbb{G}_{t,x,y,z}(S) \to \mathbb{G}_{t,x,y,z}(T)$ is one-to one. Then the geopolitical affinity between the systems S and T is an isomorphism.* ■

Proposition 4.9. *Let F and G be two geopolitical affinities between the systems S and T. The set of all $D \in \mathbb{G}(T)$ such that $F(D) = G(D)$ is open and closed in $\mathbb{G}(T)$.* ■

In particular, since the geopolitical index space $\mathbb{G}(S)$ $(= \mathbb{R}^{\ell+5})$ is connected, we have the following.

Corollary 4.2. *Let F and G be two geopolitical affinities between the systems S and T. If there exists a geopolitical index $g \in \mathbb{G}(T)$ such that $F(D) = G(D)$, the geopolitical affinities F and G coincide.* ■

Corollary 4.3. *Let F and G be two geopolitical affinities between the systems S and T. If there exists a $(t,x,y,z) \in \mathbb{B}$ such that $F_{t,x,y,z} = G_{t,x,y,z}$, the geopolitical affinities F and G coincide.* ■

Definition 4.4. *The category with objects the geopolitical index spaces and morphisms the geopolitical affinities between two systems is called the **category of geopolitical systems**.* It will be denoted by $\mathbb{B} - \mathcal{T}op$. ■

Definition 3.5. *The **sum of** $\mathbb{G}(\boldsymbol{S})$ **and** $\mathbb{G}(\boldsymbol{T})$ into the category $\mathbb{B} - \mathcal{T}op$ of geopolitical systems is the disjoint union $\mathbb{G}(S) \sqcup \mathbb{G}(T)$ endowed with the projection inducing π_S onto $\mathbb{G}(S)$ and π_T onto $\mathbb{G}(T)$.* ■

Proposition 4.10. *It holds*

$$\left(\mathbb{G}(S) \sqcup \mathbb{G}(T)\right)_{t,x,y,z} = \mathbb{G}_{t,x,y,z}(S) \sqcup \mathbb{G}_{t,x,y,z}(T). \blacksquare$$

4.3. The fiber product of two geopolitical index systems

Let S and T be two geopolitical systems, with corresponding geopolitical index spaces $\mathbb{G}(S)$ and $\mathbb{G}(T)$, and projections π_S and π_T respectively.

Definition 4.5. The ***fiber product of the weighted geopolitical index spaces*** *$\mathbb{G}(S)$ and $\mathbb{G}(T)$ is the subspace of the topological space $\mathbb{G}(S) \times \mathbb{G}(T)$ consisting in all pairs $(\mathfrak{D}_S, \mathfrak{D}_T)$ satisfying $\pi_S(\mathfrak{D}_S) = \pi_T(\mathfrak{D}_T)$. The fiber product of $\mathbb{G}(S)$ and $\mathbb{G}(T)$ will be denoted by $\mathbb{G}(S) \times_{\mathbb{B}} \mathbb{G}(T)$.* ■

Proposition 4.11. *The space $\mathbb{G}(S) \times_{\mathbb{B}} \mathbb{G}(T)$ endowed with the mapping $(\mathfrak{D}_S, \mathfrak{D}_T) \to \pi_S(\mathfrak{D}_S)$ is **the product of the geopolitical index spaces $\mathbb{G}(S)$ and $\mathbb{G}(T)$ into the category of geopolitical systems**. It is clear that*

$$\left(\mathbb{G}(S) \times_{\mathbb{B}} \mathbb{G}(T)\right)(t,x,y,z) = \mathbb{G}_{t,x,y,z}(S) \times \mathbb{G}_{t,x,y,z}(T),$$

whenever $(t,x,y,z) \in \mathbb{B}$ ■

Let now

$$h: \mathbb{B} \equiv \mathbb{R} \times \mathbb{R}^3 \to \mathbb{B} \equiv \mathbb{R} \times \mathbb{R}^3$$

be a continuous mapping.

Definition 4.6. The *topological space $\mathbb{G}^*(S) := h(\mathbb{B}) \times_{\mathbb{B}} \mathbb{G}(S)$ endowed with the first projection $\mathbb{G}^*(S) \to h(\mathbb{B})$ is a **space over** the topological space $h(\mathbb{B})$. It is called **the space above $h(\mathbb{B})$ obtained from $\mathbb{G}(S)$ by base change from $\mathbb{B}$ to $h(\mathbb{B})$**. The fibre of $\mathbb{G}^*(S)$ at a point b' of $h(\mathbb{B})$ is identified with the fibre of $\mathbb{G}(S)$ at $h(b')$.* ■

V Geometric foundations of geopolitics

Let S a be any geopolitical system/complex with corresponding geopolitical index space

$$\mathbb{G}(S) := \bigcup_{t\in\mathbb{R},(x,y,z)\in\mathbb{R}^3} \mathbb{G}_{t,x,y,z}(S) \equiv \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{\ell+1} = \mathbb{R}^{\ell+5}.$$

In any geopolitical index $g_S^{(j)} = g_S^{(j)}\left(P_{1/S}^{(j)}, \ldots, P_{N_j/S}^{(j)}; t,x,y,z\right)$ there corresponds a unique ***weighted geopolitical index***

$$\mathfrak{D}_S^{(j)} = \mathfrak{D}_S^{(j)}\left(P_{1/S}^{(j)}, \ldots, P_{N_j/S}^{(j)}; t,x,y,z\right).$$[3]

The weighted geopolitical index $\mathfrak{D}_S^{(j)}$ is a *specified* reference value of the corresponding geopolitical index $g_S^{(j)}$, above or below which, there is a change in the behavior of any active within the geographical complex, affecting both the other geopolitical indices, as well as the behavior of others geopolitical characteristics acting in the geographical complex. For the purposes of this

[3] As usually, $\left(P_{1/S}^{(j)}, \ldots, P_{N_j/S}^{(j)}\right)$ represents N_j intrinsic properties (physical characteristics) into the system.

paper, we will assume that *the fixed values of the weighted geopolitical indices are always given at any date $t \in \mathbb{R}$ and location $(x, y, z) \in \mathbb{R}^3$*.

A tool that would allow us a thorough study of the measurements carried out in the weighted geopolitical space is to attach geopolitical vector field measurements on all points of the space of weighted geopolitical indices.

The measurements made are usually statistical and must be accurate and reliable. The issue of the accuracy and reliability of measurements is large and should be considered outside the limits of this work. For now, we will always assume that *the values obtained from the measurements are reliable and accurate and will compare them with respect to the given and fixed values of the weighted geopolitical indices*.

5.1. Universalities of weighted geopolitical indices

Let U be a non-empty open subset of $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ representing a ***spatiotemporal historical phase***, i.e. *an* ***open*** *set in space-time, identified as a combination of information, criteria and historical facts, which can be derived from the fields of Science/Technology, Defense/Security, Politics/Economy and Culture/Art* (Mazis in preparation).

Definition 5.1.i. *The map*

$$\mathfrak{D}: U \to \mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}: (t, x, y, z) \mapsto$$
$$(t, x, y, z; \mathfrak{D}_S^{(1)}\left(P_{\frac{1}{S}}^{(1)}, \dots, P_{\frac{N_1}{S}}^{(1)}; t, x, y, z\right), \dots$$
$$\dots, \mathfrak{D}_S^{(\ell+1)}\left(P_{1/S}^{(\ell+1)}, \dots, P_{N_{\ell+1}/S}^{(\ell+1)}; t, x, y, z\right))$$

is called a ***universality of weighted geopolitical indices for the system S over the spatiotemporal historical phase U***.

ii. *If the map $\mathfrak{D}$ is smooth and regular, i.e. its differential $d\mathfrak{D}_{(t,x,y,z)}$ is non-singular (has rank 4) for each $(t, x, y, z) \in \mathbb{R} \times \mathbb{R}^3$, then $\mathfrak{D}$ is a parameterized surface of dimension 4 in the geopolitical index space $\mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}$. In such a case, we say that*

$$\mathfrak{S}_{\mathfrak{D}} = \mathfrak{D}(U) \text{ or simply } \mathfrak{D}$$

is the ***parameterized surface of the weighted geopolitical indices for the system S over U.*** ∎

5.1.1 Smooth parameterized surfaces of weighted geopolitical indices

We will first assume that the map $\mathfrak{D}$ is smooth and regular. The ***differential*** of the smooth map $\mathfrak{D}: U \to \mathbb{R}^{\ell+5}$ is the smooth map

$$d\mathfrak{D}: U \times \mathbb{R}^4 \to \mathbb{R}^{\ell+5} \times \mathbb{R}^{\ell+5}$$

defined as follows. A point $v \in U \times \mathbb{R}^4$ is a vector $v = \big((t, x, y, z), u\big)$ at a point $(t, x, y, z) \in U$. Let $\alpha: I \to U$ be any parameterized curve in U with $\dot{\alpha}(t_0) = v$. Then $d\mathfrak{D}(v)$ is the vector at

$$\mathfrak{D}(t, x, y, z)\big(d\mathfrak{D}(v) \in \mathbb{R}^{\ell+5}_{\mathfrak{D}(t,x,y,z)} \subset \mathbb{R}^{\ell+5} \times \mathbb{R}^{\ell+5}\big)$$

defined by $d\mathfrak{D}(v) = \mathfrak{D} \,\dot{\circ}\, \alpha(t_0)$. Note that the value of $d\mathfrak{D}(v)$ does not depend on the choice of parameterized curve α, because

$$\mathfrak{D} \,\dot{\circ}\, \alpha(t_0) =$$
$$\left(\mathfrak{D} \circ \alpha(t_0), \left(\mathfrak{D}_S^{(1)} \circ \alpha\right)'(t_0), \dots, \left(\mathfrak{D}_S^{(\ell+1)} \circ \alpha\right)'(t_0)\right) =$$
$$\Big(\mathfrak{D}(t, x, y, z), \nabla\mathfrak{D}_S^{(1)}\big(\alpha(t_0)\big) \cdot \dot{\alpha}(t_0), \dots$$
$$\dots, \nabla\mathfrak{D}_S^{(\ell+1)}\big(\alpha(t_0)\big) \cdot \dot{\alpha}(t_0)\Big) =$$
$$\Big(\mathfrak{D}(t, x, y, z), \nabla\mathfrak{D}_S^{(1)}(t, x, y, z) \cdot v, \dots$$
$$\dots, \nabla\mathfrak{D}_S^{(\ell+1)}(t, x, y, z) \cdot v\Big),$$

so

$$d\mathfrak{D}(v) = \Big(\mathfrak{D}(t, x, y, z), \nabla_v\mathfrak{D}_S^{(1)}, \dots, \nabla_v\mathfrak{D}_S^{(\ell+1)}\Big).$$

It follows immediately from the above formula that the restriction $d\mathfrak{D}_{(t,x,y,z)}$ of $d\mathfrak{D}$ to $\mathbb{R}^4_{(t,x,y,z)}$ (: the vectors at (t, x, y, z)) is a linear map

$$d\mathfrak{D}_{(t,x,y,z)}: \mathbb{R}^4_{(t,x,y,z)} \to \mathbb{R}^{\ell+5}_{\mathfrak{D}(t,x,y,z)}.$$

Its matrix relative to the standard bases for $\mathbb{R}^4_{(t,x,y,z)}$ and $\mathbb{R}^{\ell+5}_{\mathfrak{D}(t,x,y,z)}$ is just the Jacobian matrix of $\mathfrak{D}$ at (t, x, y, z). ∎

The regularity condition on $\mathfrak{D}$ guarantees that

Proposition 5.1.i. *The image $d\mathfrak{D}_{(t,x,y,z)}\big(\mathbb{R}^4_{(t,x,y,z)}\big)$ of $d\mathfrak{D}_{(t,x,y,z)}$ is a 4-dimensional subspace of $\mathbb{R}^{\ell+5}_{\mathfrak{D}(t,x,y,z)}$ for each $(t, x, y, z) \in U$.*

ii. *Further, the image $d\mathfrak{D}_{(t,x,y,z)}\big(\mathbb{R}^4_{(t,x,y,z)}\big)$ of $d\mathfrak{D}_{(t,x,y,z)}$ is the tangent space to the parameterized hypersurface $\mathfrak{D}$ of dimension 4 in the geopolitical index space $\mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}$ corresponding to the point $(t, x, y, z) \in U$.* ∎

Remark 5.1. Note that the parameterized surface $\mathfrak{D}$ of dimension 4 in the geopolitical index space $\mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}$ need not to be one-to-one, and that $\mathfrak{D}(t,x,y,z) = \mathfrak{D}(t',x',y',z')$ for $(t,x,y,z) \neq (t',x',y',z')$ **does not** necessarily imply that the image $d\mathfrak{D}_{(t,x,y,z)}\left(\mathbb{R}^4_{(t,x,y,z)}\right)$ of $d\mathfrak{D}_{(t,x,y,z)}$ is equal to the image $d\mathfrak{D}_{(t\prime,x\prime,y\prime,z\prime)}\left(\mathbb{R}^4_{(t\prime,x\prime,y\prime,z\prime)}\right)$ of $d\mathfrak{D}_{(t\prime,x\prime,y\prime,z\prime)}$ (i.e. $Image\left[d\mathfrak{D}_{(t,x,y,z)}\right] \neq Imaged\left[\mathfrak{D}_{(t',x',y',z')}\right]$). ■

Definition 5.2. *A **geopolitical vector field along the parameterized surface $\mathfrak{S}_{\mathfrak{D}}$ of the weighted geopolitical indices for the system S over U** is a map $\mathfrak{d}$ which assigns to each point $p = (t,x,y,z) \in U$ a vector $\mathfrak{d}(p) \in \mathbb{R}^{\ell+5}_{\mathfrak{D}(t,x,y,z)}$.* ■

The study of the geopolitical vector fields along the parameterized surface $\mathfrak{S}_{\mathfrak{D}}$ requires consideration of some additional concepts.

Definition 5.3. *Let*

$$\mathfrak{d}: U \to \mathbb{R}^{\ell+5}: p = (t,x,y,z) \longmapsto \mathfrak{d}(p) = (\mathfrak{D}(p); \mathfrak{d}_1, \dots, \mathfrak{d}_{\ell+5}) \in \mathbb{R}^{\ell+5}_{\mathfrak{D}(t,x,y,z)}$$

be a geopolitical vector field along the parameterized surface $\mathfrak{S}_{\mathfrak{D}}$.

i. *We say that $\mathfrak{d} = (\mathfrak{D}; \mathfrak{d}_1, \dots, \mathfrak{d}_{\ell+5})$ is **smooth** if each coordinate $\mathfrak{d}_j: U \to \mathbb{R}$ is smooth $(j = 1,2,\dots,n)$.*

ii. *We say that $\mathfrak{d} = (\mathfrak{D}; \mathfrak{d}_1, \dots, \mathfrak{d}_{\ell+5})$ is **tangent to the parameterized surface $\mathfrak{S}_{\mathfrak{D}}$ of the weighted geopolitical indices for the system S over U** if $\mathfrak{d}$ is of the form*

$$\mathfrak{d}(p) = d\mathfrak{D}_{(t,x,y,z)}(\boldsymbol{y}(p))$$

for some vector field $\boldsymbol{y}$ on U.

iii. *We say that $\mathfrak{d} = (\mathfrak{D}; \mathfrak{d}_1, \dots, \mathfrak{d}_{\ell+5})$ is **normal to the parameterized surface $\mathfrak{S}_{\mathfrak{D}}$ of the weighted geopolitical indices for the system S over U** if*

$$\mathfrak{d}(p) \perp Image\left[d\mathfrak{D}_{(t,x,y,z)}\right] \text{ for all } (t,x,y,z) \in U. \ ■$$

Let us now give a generalization of the concept of the *velocity field* in the case of a geopolitical vector field along the parameterized surface $\mathfrak{S}_{\mathfrak{D}}$ of the weighted geopolitical indices for the system S over U. Let

$$\boldsymbol{\mathcal{E}}^{(1)}, \boldsymbol{\mathcal{E}}^{(2)}, \boldsymbol{\mathcal{E}}^{(3)} \text{ and } \boldsymbol{\mathcal{E}}^{(4)}$$

denote the tangent vector fields along the parameterized surface $\mathfrak{S}_{\mathfrak{D}}$ defined by

$$\boldsymbol{\mathcal{E}}^{(i)}(t,x,y,z) = d\mathfrak{D}_{(t,x,y,z)}\big((t,x,y,z); 0, \dots, 0,1,0 \dots, 0\big);$$

where the 1 is in the $(i+1)^{th}$ spot (i spots after the $(t,x,y,z) \in U$).

Proposition 5.2. *The components of $\boldsymbol{\mathcal{E}}^{(i)}$ are just the entries in the i^{th} column of the Jacobian matrix for $\mathfrak{D}$ at $(t,x,y,z) \in U$:*

$$\boldsymbol{\mathcal{E}}^{(1)}(t,x,y,z) = \left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{D}}{\partial t}(t,x,y,z)\right) = \left(\mathfrak{D}; \frac{\partial t}{\partial t}, \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}, \frac{\partial \mathfrak{D}_1}{\partial t}, \dots, \frac{\partial \mathfrak{D}_{\ell+1}}{\partial t}\right)(t,x,y,z),$$

$$\boldsymbol{\mathcal{E}}^{(2)}(t,x,y,z) = \left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{D}}{\partial x}(t,x,y,z)\right) = \left(\mathfrak{D}; \frac{\partial t}{\partial x}, \frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}, \frac{\partial \mathfrak{D}_1}{\partial x}, \dots, \frac{\partial \mathfrak{D}_{\ell+1}}{\partial x}\right)(t,x,y,z),$$

$$\boldsymbol{\mathcal{E}}^{(3)}(t,x,y,z) = \left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{D}}{\partial y}(t,x,y,z)\right) = \left(\mathfrak{D}; \frac{\partial t}{\partial y}, \frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y}, \frac{\partial \mathfrak{D}_1}{\partial y}, \dots, \frac{\partial \mathfrak{D}_{\ell+1}}{\partial y}\right)(t,x,y,z),$$

$$\boldsymbol{\mathcal{E}}^{(4)}(t,x,y,z) = \left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{D}}{\partial z}(t,x,y,z)\right) = \left(\mathfrak{D}; \frac{\partial t}{\partial z}, \frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z}, \frac{\partial \mathfrak{D}_1}{\partial z}, \dots, \frac{\partial \mathfrak{D}_{\ell+1}}{\partial z}\right)(t,x,y,z),$$

where

$$\mathfrak{D}(t,x,y,z) = \left(t,x,y,z,; \mathfrak{D}_S^{(1)}(t,x,y,z), \dots, \mathfrak{D}_S^{(\ell+1)}(t,x,y,z)\right). \ ■$$

Note that $\boldsymbol{\mathcal{E}}^{(i)}(t,x,y,z)$ is simply the ***velocity at $(t,x,y,z) \in U$*** of the coordinate curve

$$u_i \longmapsto \mathfrak{D}(u_1,u_2,u_3,u_4)$$

(all u_j held constant except u_i)passing through $\mathfrak{D}(t,x,y,z)$. (Here $u_1 = t$, $u_2 = x$, $u_3 = y$, $u_4 = z$. Since $d\mathfrak{D}_{(t,x,y,z)}$ is non-singular, we infer

Proposition 5.3.i. *The tangent vector fields $\boldsymbol{\mathcal{E}}^{(1)}$, $\boldsymbol{\mathcal{E}}^{(2)}$, $\boldsymbol{\mathcal{E}}^{(3)}$ and $\boldsymbol{\mathcal{E}}^{(4)}$ are linearly independent at each point $(t,x,y,z) \in U$.*

ii. *For each point $(t,x,y,z) \in U$, the tangent vector fields $\boldsymbol{\mathcal{E}}^{(1)}$, $\boldsymbol{\mathcal{E}}^{(2)}$, $\boldsymbol{\mathcal{E}}^{(3)}$ and $\boldsymbol{\mathcal{E}}^{(4)}$ form a basis for the tangent $Image\left[d\mathfrak{D}_{(t,x,y,z)}\right]$.* ■

Definition 5.4. *For any smooth geopolitical vector field $\mathfrak{d}: U \to \mathbb{R}^{\ell+5}$ (U open in $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$) along the parameterized surface $\mathfrak{S}_{\mathfrak{D}}$ of the weighted geopolitical indices for the system S, **the derivative***

$$\nabla_u \mathfrak{d} \in \mathbb{R}^{\ell+5}_{\mathfrak{D}(t,x,y,z)}$$

of $\mathfrak{d}$ with respect to $u \in \mathbb{R}^4_{(t,x,y,z)}$ ($(t,x,y,z) \in U$) is defined by

$$\nabla_u\, \mathfrak{d} = \left(\mathfrak{D}(t,x,y,z), \frac{d}{d\tau}\Big|_{\tau_0} (\mathfrak{d}\circ\alpha)\right)$$
$$= (\mathfrak{D}(t,x,y,z), \nabla_u\, \mathfrak{d}_1, \dots, \nabla_u\, \mathfrak{d}_{\ell+5})$$

where

- $\mathfrak{d} = (\mathfrak{d}_1, \dots, \mathfrak{d}_{\ell+5})$ *is the vector part of* $\mathfrak{d}$ ($\mathfrak{d}(q) = (\mathfrak{D}(q); \mathfrak{d}_1(q), \dots, \mathfrak{d}_{\ell+5}(q))$ for $q \in U$) *and*
- α *is any parameterized curve in* U *with* $\dot{\alpha}(\tau_0) = u$. ■

Remark 5.2. Note that, when
$u \in \{e_1 = ((t,x,y,z); 1,0,0,0)$,
$e_2 = ((t,x,y,z); 0,1,0,0), e_3 = ((t,x,y,z); 0,0,1,0)$,
$e_4 = ((t,x,y,z); 0,0,0,1)\}$,
we have

$$\nabla_{e_1}\, \mathfrak{d} = \left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{d}}{\partial t}(t,x,y,z)\right) =$$
$$\left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{d}_1}{\partial t}(t,x,y,z), \dots, \frac{\partial \mathfrak{d}_{\ell+5}}{\partial t}(t,x,y,z)\right),$$
$$\nabla_{e_2}\, \mathfrak{d} = \left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{d}}{\partial x}(t,x,y,z)\right) =$$
$$\left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{d}_1}{\partial x}(t,x,y,z), \dots, \frac{\partial \mathfrak{d}_{\ell+5}}{\partial x}(t,x,y,z)\right),$$
$$\nabla_{e_3}\, \mathfrak{d} = \left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{d}}{\partial y}(t,x,y,z)\right) =$$
$$\left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{d}_1}{\partial y}(t,x,y,z), \dots, \frac{\partial \mathfrak{d}_{\ell+5}}{\partial y}(t,x,y,z)\right),$$
$$\nabla_{e_4}\, \mathfrak{d} = \left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{d}}{\partial z}(t,x,y,z)\right) =$$
$$\left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{d}_1}{\partial z}(t,x,y,z), \dots, \frac{\partial \mathfrak{d}_{\ell+5}}{\partial z}(t,x,y,z)\right). ■$$

5.1.2. Discontinuous Universalities of Weighted Geopolitical Indices

It is quite reasonable to assume that all the components $\mathfrak{D}_S^{(1)}\left(P_{1/S}^{(1)}, \dots, P_{N_1/S}^{(1)}; t,x,y,z\right), \dots, \mathfrak{D}_S^{(\ell+1)}\left(P_{1/S}^{(\ell+1)}, \dots, P_{N_{\ell+1}/S}^{(\ell+1)}; t,x,y,z\right)$ of a universality of weighted geopolitical indices for the system S remain constant for long or short periods and for large or small geographical areas. In other words, we can assume that U is partitioned in different regions, each associated to a different constant expression of the geopolitical indices. In other words, in what follow, we will assume that

there are

- *a finite partition*
  $\{\bar{U}_i\colon U_i$ *is a non void open subset of* U *and* $i = 1,2,\dots,I\}$ *of* U,
  such that
  $U_i \cap U_{i'} = \emptyset$ *whenever* $i \neq i'$, *and*
- *a finite set of constant vectors*
  $c^{(i)} = \left(c_1^{(i)}, \dots, c_{\ell+1}^{(i)}\right)$ in $\mathbb{R}^{\ell+1}$,
  $i = 1,2,\dots,I$,

such that

$$\left(\mathfrak{D}_S^{(1)}\left(P_{1/S}^{(1)}, \dots, P_{N_1/S}^{(1)}; t,x,y,z\right), \dots \right.$$
$$\left. \dots, \mathfrak{D}_S^{(\ell+1)}\left(P_{1/S}^{(\ell+1)}, \dots, P_{N_{\ell+1}/S}^{(\ell+1)}; t,x,y,z\right)\right) =$$
$$\left(c_1^{(i)}, \dots, c_{\ell+1}^{(i)}\right),$$

for any $(t,x,y,z) \in U_i$.

The intersection $\Upsilon_{i,i'} \coloneqq \bar{U}_i \cap \bar{U}_{i'}$ *between the closure (: set plus its boundary) of the sets* U_i *and* $U_{i'}$ *is either an* $\mathbb{R}^3$ *–dimensional manifold included in the boundaries* ∂U_i *and* $\partial U_{i'}$, *or is the empty set. A set* $\Upsilon_{i,i'}$ *is termed a* ***geopolitical border*** or ***geopolitical discontinuous boundary***.

5.2. Geopolitical Measurements

Let U be any non-empty subset of $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ representing a spatiotemporal historical phase.

Definition 5.5. *Select any finite set* E_{K+1} *of points* (t_k, x_k, y_k, z_k) *in* U. *A* ***geopolitical measurement*** $\mathfrak{M}_{\mathfrak{F}}$ *of size* $K+1$ *in* U *is a process*

$$\mathfrak{F}$$

by which each geopolitical index $g_S^{(j)} = g_S^{(j)}\left(P_{1/S}^{(j)}, \dots, P_{N_j/S}^{(j)}; t,x,y,z\right)$ *is assigned to a real number*

$$\mathfrak{F}\left(g_S^{(j)}\left(P_{1/S}^{(j)}, \dots, P_{N_j/S}^{(j)}; t_k, x_k, y_k, z_k\right)\right)$$

for any $(t_k, x_k, y_k, z_k) \in E$ *and* $k \in \{1,2,\dots,K+1\}$. ■

Assume that the Euclidean space $\mathbb{R}^{\ell+5}$ is endowed with the metric

$$\mathbb{d}.$$

Let also

$$\mathfrak{D}\colon U \to \mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}\colon$$
$$(t,x,y,z) \mapsto$$
$$\left(t,x,y,z, \mathfrak{D}_S^{(1)}\left(P_{1/S}^{(1)}, \dots, P_{N_1/S}^{(1)}, t,x,y,z\right), \dots \right.$$
$$\left. \dots, \mathfrak{D}_S^{(\ell+1)}\left(P_{1/S}^{(\ell+1)}, \dots, P_{N_{\ell+1}/S}^{(\ell+1)}, t,x,y,z\right)\right)$$

be a universality of weighted geopolitical indices for a system S in U.

Suppose a geopolitical measurement $\mathfrak{M}_{\mathfrak{F}}$ *of size* $K+1$ in U is given. This means that it has selected a ***finite*** set $E_{K+1} = \{(t_k, x_k, y_k, z_k) \in U : k = 1,2,\dots,K+1\}$ and a process $\mathfrak{F}$ which, for every $(t_k, x_k, y_k, z_k) \in E_{K+1}$, assigns a real number $f_j(t_k, x_k, y_k, z_k)$ to each geopolitical index $g_S^{(j)} = g_S^{(j)}\left(P_{1/S}^{(j)}, \dots, P_{N_j/S}^{(j)}; t_k, x_k, y_k, z_k\right)$. In this way, the following mapping is formed:

$$\mathfrak{F}: E_{K+1} \to \mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}: (t_k, x_k, y_k, z_k) \mapsto.$$
$$\left(t_k, x_k, y_k, z_k, \mathfrak{F}\left(g_S^{(1)}\left(P_{1/S}^{(1)}, \dots, P_{N_j/S}^{(1)}; t_k, x_k, y_k, z_k\right)\right), \dots \right.$$
$$\left. \dots, \mathfrak{F}\left(g_S^{(\ell+1)}\left(P_{1/S}^{(\ell+1)}, \dots, P_{N_j/S}^{(\ell+1)}; t_k, x_k, y_k, z_k\right)\right)\right).$$

Definition 5.6. *The function*

$$\boldsymbol{\mathcal{V}}: E_{K+1} \to \mathbb{R}: (t_k, x_k, y_k, z_k) \mapsto \boldsymbol{\mathcal{V}}_{\boldsymbol{K+1}}(t_k, x_k, y_k, z_k)$$
$$\coloneqq \mathbb{d}(\mathfrak{D}(t_k, x_k, y_k, z_k), \mathfrak{F}(t_k, x_k, y_k, z_k))$$

*is the **deviation of the geopolitical measurement at the points of*** $\boldsymbol{E_{K+1}}$ *from the weighted geopolitical indices over the system S.* ∎

It is clear that U is a *separable* topological space, so it is always possible to choose a sequence

$$\dots \subsetneq E_K \subsetneq E_{K+1} \subsetneq E_{K+2} \subsetneq \cdots.$$

of *finite* sets of points of U, such that

- their union $E = \bigcup_{K=1}^{\infty} E_{K+1}$ is dense in U and
- E_{K+1} contains only one element more that E_K, say $(t_{K+1}, x_{K+1}, y_{K+1}, z_{K+1})$.

Hence, for any $(t, x, y, z) \in U$, there exists a well defined sequence

$$\left((t_{K+1}, x_{K+1}, y_{K+1}, z_{K+1}) \in E_{K+1}\right)_{K=1,2,\dots}$$

such that

$$(t, x, y, z) = lim_{K+1\to\infty}(t_{K+1}, x_{K+1}, y_{K+1}, z_{K+1}).$$

Define

$$\bar{\bar{\mathfrak{F}}}(t, x, y, z) := lim_{K+1\to\infty}\, \mathfrak{F}(t_{K+1}, x_{K+1}, y_{K+1}, z_{K+1})$$

and

$$\bar{\boldsymbol{\mathcal{V}}}(t, x, y, z) := lim_{K+1\to\infty}\boldsymbol{\mathcal{V}}(t_{K+1}, x_{K+1}, y_{K+1}, z_{K+1}).$$

Now, it is straightforward to see that $\bar{\bar{\mathfrak{F}}}$ is a process by which each geopolitical index $g_S^{(j)} = g_S^{(j)}\left(P_{1/S}^{(j)}, \dots, P_{N_j/S}^{(j)}; t, x, y, z\right)$ corresponds to a real number

$$\bar{\bar{\mathfrak{F}}}\left(g_S^{(j)}\left(P_{1/S}^{(j)}, \dots, P_{N_j/S}^{(j)}; t, x, y, z\right)\right).$$

Similarly, $\bar{\boldsymbol{\mathcal{V}}}$ can be considered as a function which maps the distance between the vectors $\mathfrak{D}(t, x, y, z)$ and $\bar{\bar{\mathfrak{F}}}(t, x, y, z)$ at every point (t, x, y, z) of U. Thus, we are led reasonably to the next definition.

Definition 5.7.i. *The map*

$$\bar{\bar{\mathfrak{F}}}: U \to \mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}: (t, x, y, z) \mapsto$$
$$\left(t, x, y, z;\ \mathfrak{F}\left(g_S^{(1)}\left(P_{1/S}^{(1)}, \dots, P_{N_1/S}^{(1)}; t, x, y, z\right)\right), \dots \right.$$
$$\left. \dots, \mathfrak{F}\left(g_S^{(\ell+1)}\left(P_{1/S}^{(\ell+1)}, \dots, P_{N_{\ell+1}/S}^{(\ell+1)}; t, x, y, z\right)\right)\right)$$

*is called a **section of the geopolitical measurement*** $\boldsymbol{\mathfrak{M}_{\mathfrak{F}}}$ ***for the system S over U**.*

ii. *If the set U is open in* $\mathbb{R}^4$ *and if the map* $\bar{\bar{\mathfrak{F}}}$ *is smooth and regular in U, i.e. its differential* $d\bar{\bar{\mathfrak{F}}}_{(t,x,y,z)}$ *is non-singular (has rank 4) for each* $(t, x, y, z) \in U$, *then*

$$\mathfrak{S}_{\mathfrak{F}} = \bar{\bar{\mathfrak{F}}}(U) \text{ or simply } \bar{\bar{\mathfrak{F}}}$$

*is said to be a **geopolitical measurements' parameterized surface*** $\boldsymbol{\mathfrak{M}_{\mathfrak{F}}}$ ***over U for the system S**.*

iii. *The function*

$$\bar{\boldsymbol{\mathcal{V}}}: U \to \mathbb{R}: (t, x, y, z) \mapsto \bar{\boldsymbol{\mathcal{V}}}(t, x, y, z)$$
$$\coloneqq \mathbb{d}(\mathfrak{D}(t, x, y, z), \bar{\bar{\mathfrak{F}}}(t, x, y, z))$$

*is the **deviation of the geopolitical measurement at the points of U** from the weighted geopolitical indices over the system S.* ∎

We can immediately make some useful general observations.

Remark 5.3. If U is a non-empty open subset of $\mathbb{R}^4$ and if the map $\bar{\bar{\mathfrak{F}}}: U \to \mathbb{R}^{\ell+5}$ is smooth and regular, its ***differential*** is the smooth map

$$d\bar{\bar{\mathfrak{F}}}: U \times \mathbb{R}^4 \to \mathbb{R}^{\ell+5} \times \mathbb{R}^{\ell+5}$$

defined as follows. A point $\mathscr{v} \in U \times \mathbb{R}^4$ is a vector $\mathscr{v} = ((t, x, y, z), u)$ at a point $(t, x, y, z) \in U$. Let $\alpha: I \to U$ be any parameterized curve in U with $\dot{\alpha}(t_0) = \mathscr{v}$. Then $d\bar{\bar{\mathfrak{F}}}(\mathscr{v})$ is the vector at

$$\bar{\bar{\mathfrak{F}}}(t, x, y, z)\left(d\bar{\bar{\mathfrak{F}}}(\mathscr{v}) \in \mathbb{R}^{\ell+5}_{\mathfrak{D}(t,x,y,z)} \subset \mathbb{R}^{\ell+5} \times \mathbb{R}^{\ell+5}\right)$$

defined by

$$d\bar{\bar{\mathfrak{F}}}(\mathscr{v}) = \bar{\bar{\mathfrak{F}}} \dot{\circ}\ \alpha(t_0).$$

Note that the value of $d\bar{\bar{\mathfrak{F}}}(\mathscr{v})$ does not depend on the choice of parameterized curve α, because

$$\bar{\bar{\mathfrak{F}}} \dot{\circ}\ \alpha(t_0) =$$
$$\left(\bar{\bar{\mathfrak{F}}} \circ \alpha(t_0), \left(\bar{\bar{\mathfrak{F}}}_S^{(1)} \circ \alpha\right)'(t_0), \dots, \left(\bar{\bar{\mathfrak{F}}}_S^{(\ell+1)} \circ \alpha\right)'(t_0)\right) =$$

$$\Big(\overline{\mathfrak{F}}(t,x,y,z), \nabla\overline{\mathfrak{F}}_S^{(1)}\big(\alpha(t_0)\big)\cdot\dot{\alpha}(t_0), \ldots$$
$$\ldots, \nabla\overline{\mathfrak{F}}_S^{(\ell+1)}\big(\alpha(t_0)\big)\cdot\dot{\alpha}(t_0)\Big) =$$
$$\Big(\overline{\mathfrak{F}}(t,x,y,z), \nabla\overline{\mathfrak{F}}_S^{(1)}(t,x,y,z)\cdot\boldsymbol{v}, \ldots$$
$$\ldots, \nabla\overline{\mathfrak{F}}_S^{(\ell+1)}(t,x,y,z)\cdot\boldsymbol{v}\Big),$$

so

$$d\overline{\mathfrak{F}}(\boldsymbol{v}) = \Big(\overline{\mathfrak{F}}(t,x,y,z), \nabla_{\boldsymbol{v}}\overline{\mathfrak{F}}_S^{(1)}, \ldots, \nabla_{\boldsymbol{v}}\overline{\mathfrak{F}}_S^{(\ell+1)}\Big).$$

It follows immediately from the above formula that the restriction $d\overline{\mathfrak{F}}_{(t,x,y,z)}$ of $d\overline{\mathfrak{F}}$ to $\mathbb{R}^4_{(t,x,y,z)}$ (: the vectors at (t,x,y,z)) is a linear map

$$d\overline{\mathfrak{F}}_{(t,x,y,z)}: \mathbb{R}^4_{(t,x,y,z)} \to \mathbb{R}^{\ell+5}_{\overline{\mathfrak{F}}(t,x,y,z)}.$$

Its matrix relative to the standard bases for $\mathbb{R}^4_{(t,x,y,z)}$ and $\mathbb{R}^{\ell+1}_{\mathfrak{D}(t,x,y,z)}$ is just the Jacobian matrix of $\overline{\mathfrak{F}}$ at (t,x,y,z). ■

The regularity condition on $\overline{\mathfrak{F}}$ guarantees that

Proposition 5.4.i. *The image $d\overline{\mathfrak{F}}_{(t,x,y,z)}\big(\mathbb{R}^4_{(t,x,y,z)}\big)$ of $d\overline{\mathfrak{F}}_{(t,x,y,z)}$ is a 4-dimensional subspace of $\mathbb{R}^{\ell+5}_{\overline{\mathfrak{F}}(t,x,y,z)}$ for each $(t,x,y,z) \in U$.*

ii. *Further, the image $d\overline{\mathfrak{F}}_{(t,x,y,z)}\big(\mathbb{R}^4_{(t,x,y,z)}\big)$ of $d\overline{\mathfrak{F}}_{(t,x,y,z)}$ is the tangent space to the parameterized surface $\overline{\mathfrak{F}}$ of dimension 4 in the geopolitical index space $\mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}$ corresponding to the point $(t,x,y,z) \in U$.* ■

Remark 5.4. Note that a parameterized surface $\overline{\mathfrak{F}}$ of dimension 4 in the geopolitical index space $\mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}$ need not to be one-to-one, and that $\overline{\mathfrak{F}}(t,x,y,z) = \overline{\mathfrak{F}}(t',x',y',z')$ for $(t,x,y,z) \neq (t',x',y',z')$ **does not** necessarily imply that the image $d\overline{\mathfrak{F}}_{(t,x,y,z)}\big(\mathbb{R}^4_{(t,x,y,z)}\big)$ of $d\overline{\mathfrak{F}}_{(t,x,y,z)}$ is equal to the image $d\overline{\mathfrak{F}}_{(t\prime,x\prime,y\prime,z\prime)}\big(\mathbb{R}^4_{(t\prime,x\prime,y\prime,z\prime)}\big)$ of $d\overline{\mathfrak{F}}_{(t\prime,x\prime,y\prime,z\prime)}$ (i.e.$Image\big[d\overline{\mathfrak{F}}_{(t,x,y,z)}\big] \neq Image\ d\Big[\overline{\mathfrak{F}}_{(t',x',y',z')}\Big]$). ■

Definition 5.8. *A **geopolitical vector field along a parameterized surface $\mathfrak{S}_{\mathfrak{F}}$ of geopolitical measurement $\mathfrak{M}_{\mathfrak{F}}$ for the system S over U** is a map $\boldsymbol{f}$ which assigns to each point $p = (t,x,y,z) \in U$ a vector $(p) \in \mathbb{R}^{\ell+5}_{\overline{\mathfrak{F}}(t,x,y,z)}$.* ■

The study of the geopolitical vector fields along a parameterized surface $\mathbb{S}_{\mathfrak{D}}$ requires consideration of some additional concepts.

Definition 5.9. *Let*

$$\boldsymbol{f}: U \to \mathbb{R}^{\ell+5}: p = (t,x,y,z) \mapsto$$
$$\boldsymbol{f}(p) = (\overline{\mathfrak{F}}(p); \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{\ell+5}) \in \mathbb{R}^{\ell+5}_{\overline{\mathfrak{F}}(t,x,y,z)}$$

be a geopolitical vector field along a parameterized surface $\mathfrak{S}_{\mathfrak{F}}$.

i. *We say that $\boldsymbol{f} = (\overline{\mathfrak{F}}; \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{\ell+5})$ is **smooth** if each coordinate $\boldsymbol{f}_j: U \to \mathbb{R}$ is smooth ($j = 1,2,\ldots,\ell+5$).*

ii. *We say that $\boldsymbol{f} = (\overline{\mathfrak{F}}; \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{\ell+5})$ is **tangent to the parameterized surface $\mathbb{S}_{\mathfrak{D}}$ of the weighted geopolitical indices for the system S over U** if $\boldsymbol{f}$ is of the form*

$$\boldsymbol{f}(p) = d\overline{\mathfrak{F}}_{(t,x,y,z)}\big(\boldsymbol{y}(p)\big)$$

for some vector field $\boldsymbol{y}$ on U.

iii. *We say that $\boldsymbol{f} = (\overline{\mathfrak{F}}; \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{\ell+5})$ is **normal to the parameterized surface $\mathbb{S}_{\mathfrak{D}}$ of the geopolitical measurement $\mathfrak{M}_{\mathfrak{F}}$ for the system S over U** if*

$$\boldsymbol{f}(p) \perp Image\,\big[d\overline{\mathfrak{F}}_{(t,x,y,z)}\big] \text{ for all } (t,x,y,z) \in U. \ ■$$

Let us now give a generalization of the concept of the *velocity field* in the case of a geopolitical vector field along a parameterized surface $\mathfrak{S}_{\mathfrak{F}}$ of a geopolitical measurement $\mathfrak{M}_{\mathfrak{F}}$ for the system S over U. Let

$$\boldsymbol{G}^{(1)}, \boldsymbol{G}^{(2)}, \boldsymbol{G}^{(3)} \text{ and } \boldsymbol{G}^{(4)}$$

denote the tangent vector fields along the parameterized surface $\mathfrak{S}_{\mathfrak{F}}$ defined by

$$\boldsymbol{G}^{(i)}(t,x,y,z) =$$
$$d\overline{\mathfrak{F}}_{(t,x,y,z)}\big((t,x,y,z); 0,\ldots,0,1,0\ldots,0\big);$$

where the 1 is in the $(i+1)^{th}$ spot (i spots after the $(t,x,y,z) \in U$).

Proposition 5.5. *The components of $\boldsymbol{G}^{(i)}$ are just the entries in the i^{th} column of the Jacobian matrix for $\overline{\mathfrak{F}}$ at $(t,x,y,z) \in U$:*

$$\boldsymbol{G}^{(1)}(t,x,y,z) = \left(\overline{\mathfrak{F}}(t,x,y,z); \frac{\partial\overline{\mathfrak{F}}}{\partial t}(t,x,y,z)\right) =$$
$$\left(\overline{\mathfrak{F}}; \frac{\partial t}{\partial t}, \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}, \frac{\partial\overline{\mathfrak{F}}_1}{\partial t}, \ldots, \frac{\partial\overline{\mathfrak{F}}_{\ell+1}}{\partial t}\right)(t,x,y,z),$$
$$\boldsymbol{G}^{(2)}(t,x,y,z) = \left(\overline{\mathfrak{F}}(t,x,y,z); \frac{\partial\overline{\mathfrak{F}}}{\partial x}(t,x,y,z)\right) =$$
$$\left(\overline{\mathfrak{F}}; \frac{\partial t}{\partial x}, \frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}, \frac{\partial\overline{\mathfrak{F}}_1}{\partial x}, \ldots, \frac{\partial\overline{\mathfrak{F}}_{\ell+1}}{\partial x}\right)(t,x,y,z),$$
$$\boldsymbol{G}^{(3)}(t,x,y,z) = \left(\overline{\mathfrak{F}}(t,x,y,z); \frac{\partial\overline{\mathfrak{F}}}{\partial y}(t,x,y,z)\right) =$$
$$\left(\overline{\mathfrak{F}}; \frac{\partial t}{\partial y}, \frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y}, \frac{\partial\overline{\mathfrak{F}}_1}{\partial y}, \ldots, \frac{\partial\overline{\mathfrak{F}}_{\ell+1}}{\partial y}\right)(t,x,y,z),$$
$$\boldsymbol{G}^{(4)}(t,x,y,z) = \left(\overline{\mathfrak{F}}(t,x,y,z); \frac{\partial\overline{\mathfrak{F}}}{\partial z}(t,x,y,z)\right) =$$
$$\left(\overline{\mathfrak{F}}; \frac{\partial t}{\partial z}, \frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z}, \frac{\partial\overline{\mathfrak{F}}_1}{\partial z}, \ldots, \frac{\partial\overline{\mathfrak{F}}_{\ell+1}}{\partial z}\right)(t,x,y,z),$$

where

$$\overline{\overline{\mathfrak{F}}}(t,x,y,z) = \left(t,x,y,z,\overline{\overline{\mathfrak{F}}}_1(t,x,y,z),\ldots,\overline{\overline{\mathfrak{F}}}_{\ell+1}(t,x,y,z)\right). \blacksquare$$

Note that $\mathcal{G}^{(i)}(t,x,y,z)$ is simply the ***velocity*** *at* $(t,x,y,z) \in U$ of the coordinate curve

$$u_i \longmapsto \overline{\overline{\mathfrak{F}}}(u_1,u_2,u_3,u_4)$$

(all u_j held constant except u_i) passing through $\overline{\overline{\mathfrak{F}}}(t,x,y,z)$. Here $u_1 = t$, $u_2 = x$, $u_3 = y$, $u_4 = z$. Further, since $d\overline{\overline{\mathfrak{F}}}_{(t,x,y,z)}$ is non-singular, we infer

Proposition 5.6.i. *The tangent vector fields* $\mathcal{G}^{(1)}$, $\mathcal{G}^{(2)}$, $\mathcal{G}^{(3)}$ *and* $\mathcal{G}^{(4)}$ *are linearly independent at each point* $(t,x,y,z) \in U$.

ii. *For each point* $(t,x,y,z) \in U$, *the tangent vector fields* $\mathcal{G}^{(1)}$,$\mathcal{G}^{(2)}$, $\mathcal{G}^{(3)}$ *and* $\mathcal{G}^{(4)}$ *form a basis for the tangent Image* $\left[d\overline{\overline{\mathfrak{F}}}_{(t,x,y,z)}\right]$. ■

Definition 5.10. *For any smooth geopolitical vector field* $\boldsymbol{f}: U \to \mathbb{R}^{\ell+5}$ *(*U *open in* $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$*) along the parameterized surface* $\mathfrak{S}_{\mathfrak{F}}$ *of the geopolitical measurement* $\mathfrak{M}_{\mathfrak{F}}$ *for the system S,* ***the derivative***

$$\nabla_u \boldsymbol{f} \in \mathbb{R}^{\ell+5}_{\overline{\overline{\mathfrak{F}}}(t,x,y,z)}$$

of $\boldsymbol{f}$ *with respect to* $u \in \mathbb{R}^4_{(t,x,y,z)}$ *(*$(t,x,y,z) \in U$*) is defined by*

$$\nabla_u \boldsymbol{f} = \left(\overline{\overline{\mathfrak{F}}}(t,x,y,z), \frac{d}{d\tau}\Big|_{\tau_0} (\boldsymbol{f} \circ \alpha)\right) = (\overline{\overline{\mathfrak{F}}}(t,x,y,z), \nabla_u f_1, \ldots, \nabla_u f_{\ell+5})$$

where

- $f = (f_1, \ldots, f_{\ell+5})$ *is the vector part of* $\boldsymbol{f}$ *(*$\boldsymbol{f}(q) = \left(\overline{\overline{\mathfrak{F}}}(q); f_1(q), \ldots, f_{\ell+5}(q)\right)$ for $q \in U$*) and*
- α *is any parameterized curve in* U *with* $\dot{\alpha}(\tau_0) = u$. ■

Remark 5.5. Note that, when

$$u \in \{e_1 = ((t,x,y,z); 1,0,0,0),\ e_2 = ((t,x,y,z); 0,1,0,0), e_3 = ((t,x,y,z); 0,0,1,0),\ e_4 = ((t,x,y,z); 0,0,0,1)\},$$

we have

$$\nabla_{e_1} \boldsymbol{f} = \left(\overline{\overline{\mathfrak{F}}}(t,x,y,z);\ \frac{\partial f}{\partial t}(t,x,y,z)\right) = \left(\overline{\overline{\mathfrak{F}}}(t,x,y,z); \frac{\partial f_1}{\partial t}(t,x,y,z), \ldots, \frac{\partial f_{\ell+5}}{\partial t}(t,x,y,z)\right),$$

$$\nabla_{e_2} \boldsymbol{f} = \left(\overline{\overline{\mathfrak{F}}}(t,x,y,z);\ \frac{\partial f}{\partial x}(t,x,y,z)\right) = \left(\overline{\overline{\mathfrak{F}}}(t,x,y,z); \frac{\partial f_1}{\partial x}(t,x,y,z), \ldots, \frac{\partial f_{\ell+5}}{\partial x}(t,x,y,z)\right),$$

$$\nabla_{e_3} \boldsymbol{f} = \left(\overline{\overline{\mathfrak{F}}}(t,x,y,z);\ \frac{\partial f}{\partial y}(t,x,y,z)\right) = \left(\overline{\overline{\mathfrak{F}}}(t,x,y,z); \frac{\partial f_1}{\partial y}(t,x,y,z), \ldots, \frac{\partial f_{\ell+5}}{\partial y}(t,x,y,z)\right),$$

$$\nabla_{e_4} \boldsymbol{f} = \left(\overline{\overline{\mathfrak{F}}}(t,x,y,z);\ \frac{\partial f}{\partial z}(t,x,y,z)\right) = \left(\overline{\overline{\mathfrak{F}}}(t,x,y,z); \frac{\partial f_1}{\partial z}(t,x,y,z), \ldots, \frac{\partial f_{\ell+5}}{\partial z}(t,x,y,z)\right). \blacksquare$$

VI Distance between a universality of weighted geopolitical indices and a section of geopolitical measurements

In this section, we will use the results of measurement to predict dates and places where there will be future geopolitical events. To do so, we will construct the parameterized surface interpolating geopolitical measurement results at the points of a given set $E \subset U$ and will consider the geopolitical deviation of this surface from the universality of weighted geopolitical indices at each point (t,x,y,z) of $U \subset\subset \mathbb{R}^4$.

Definition 6.1. *If there is a point* $(t^*,x^*,y^*,z^*) \in U$ *on which the geopolitical deviation* $\boldsymbol{\mathcal{V}}(t^*,x^*,y^*,z^*)$ *exceeds a certain critical tolerance value* $\epsilon_{critical} = \epsilon_{critical}(U)$*, we will say that this point is* ***a point of expected specific geopolitical event****.* ■

Remark 6.1. The exact or approximate determination of tolerances, and the study of their properties, and is a key quality issue which affects predominantly the scientific concern of geopolitical analysts. ■

Remark 6.2. Possibly, it is interesting to find an increasing sequence of many critical tolerance values $\epsilon^{(j)}_{critical}$, each of which has its own importance. ■

We begin by recalling some well known interpolation methods.

6.1. The case of a fixed geographical location

First, we will assume that the geopolitical study concerns to a fixed geographical location, say

$$x = x_0 = const, y = y_0 = const\ z = z_0 = const.$$

Let

$$\{t_\nu \in [T_0, T_1]: \nu = 1,2,\dots, M+1\}$$

be a given finite set of time moments into a fixed time interval $[T_0, T_1]$, such that $(t_\nu, x_0, y_0, z_0) \in U$ and $t_\nu \neq t_{\nu'}$, whenever $\nu, \nu' \in \{1,2,\dots, M+1\}$ with $\nu \neq \nu'$.

Assume that, for any $j = 1,2,\dots,\ell+1$, we know the values

$$f_j(t,x,y,z) := \bar{\mathfrak{F}}\left(g_S^{(j)}\left(P_{1/S}^{(j)},\dots,P_{N_j/S}^{(j)}; t,x,y,z\right)\right)$$

of the j^{th} geopolitical index $g_S^{(j)}\left(P_{1/S}^{(j)},\dots,P_{N_j/S}^{(j)}; t,x,y,z\right)$ accordingly to a geopolitical measurement $\mathfrak{M}_{\mathfrak{F}}$ at the $M+1$ discrete points (t_ν, x_0, y_0, z_0), $\nu = 1,2,\dots, M+1$. For each $j = 1,2,\dots,\ell+1$, consider the ***unique*** Lagrange polynomial of degree at most M

$$\mathcal{L}_M^{(j)}(t,x_0,y_0,z_0) = \sum_{\nu=1}^{M+1}\left[f_j(t_\nu,x_0,y_0,z_0)\prod_{\nu'=1,\nu'\neq\nu}^{M+1}\frac{t-t_{\nu'}}{t_\nu-t_{\nu'}}\right]$$

interpolating the $M+1$ values $f_j(t_\nu,x_0,y_0,z_0)$ of the j^{th} geopolitical index $g_S^{(j)}\left(P_{1/S}^{(j)},\dots,P_{N_j/S}^{(j)}; t,x,y,z\right)$.

Proposition 6.1. *The parameterized surface*

$$\mathfrak{L}_M: \mathbb{R}^4 \to \mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}: (t,x,y,z) \mapsto$$

$$\mathfrak{L}_M(t,x,y,z) := \left(t,x,y,z,\mathcal{L}_M^{(1)}(t,x_0,y_0,z_0),\dots \dots,\mathcal{L}_M^{(\ell+1)}(t,x_0,y_0,z_0)\right)$$

interpolates the section $\bar{\mathfrak{F}}: U \to \mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}$ of the geopolitical measurement $\mathfrak{M}_{\mathfrak{F}}$ for the system S at the $M+1$ points (t_ν,x_0,y_0,z_0), in the sense that

$$\mathfrak{L}_M(t_\nu,x_0,y_0,z_0) = \bar{\mathfrak{F}}(t_\nu,x_0,y_0,z_0),$$

whenever $\nu = 1,2,\dots,M+1$. ∎

From well known results of classical approximation theory, we have the following result.

Proposition 6.2.i. *The optimal choice for the moments $t_\nu \in [T_0,T_1]$ minimizing the error of the interpolation is*

$$t_\nu = \frac{T_1-T_0}{2}\cos\left(\frac{2\nu+1}{2\nu+2}\pi\right)+\frac{T_1+T_0}{2}, \nu = 1,2,\dots,M+1.$$

ii. *Let*

$$\mathbb{T} = \left(t_\nu^{(M+1)}\right)_{M+1\geq 0, 0\leq\nu\leq M+1} =$$

$$\begin{pmatrix} t_0^{(000)} & & & & & \\ t_0^{(111)} & t_1^{(111)} & & & & \\ t_0^{(222)} & t_1^{(222)} & t_2^{(222)} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ t_0^{(M+1)} & t_1^{(M+1)} & t_2^{(M+1)} & \dots & t_{M+1}^{(M+1)} & \\ \vdots & \vdots & \vdots & & \vdots & \ddots \end{pmatrix}$$

be an infinite lower triangular matrix whose entries are interpolation time moments $t_\nu^{(M+1)}$ in the interval $[T_0,T_1]$. If the functions $f_j(t,x,y,z)$ $(j = 1,2\dots,\ell+1)$ are ***analytic*** *in $[T_0,T_1]$, then each component $\mathcal{L}_M^{(j)}$ of the sequence of interpolation polynomials induced by $\mathbb{T}$ converges uniformly on $[T_0,T_1]$ to the corresponding component f_j of the section $\bar{\mathfrak{F}}$:*

$$lim_{M\to\infty}sup_{t\in[T_0,T_1]}\left|\mathcal{L}_M^{(j)}(t,x_0,y_0,z_0) - f_j(t,x_0,y_0,z_0)\right| = 0$$

$(j = 1,2\dots,\ell+1)$. ∎

With this terminology, the well-defined and unique interpolant $\mathfrak{L}_M: \mathbb{R}^4 \to \mathbb{R}^{\ell+5}$ can approximate entirely, and thus replace completely the unknown geopolitical measurements' parameterized surface $\mathfrak{M}_{\mathfrak{F}}$ in U, determined by a section $\bar{\mathfrak{F}}: U \to \mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}$ of the geopolitical measurement $\mathfrak{M}_{\mathfrak{F}}$ for the system S over U.

We are thus in position to formulate our first theoretical method for solving the problem of predicting the signs of space-time in which there will be a geopolitical event.

Frame Work of the 1st algorithm for determining points of expected specific geopolitical events

Assume that the geopolitical study concerns to a fixed geographical location, say

$x = x_0 = const, y = y_0 = const$ and $z = z_0 = const.$

Let

$$\{t_\nu \in [T_0, T_1]: \nu = 1,2,\dots, M+1\}$$

be a given finite set of time moments into a fixed time interval $[T_0, T_1]$, such that $(t_\nu, x_0, y_0, z_0) \in U$ and $t_\nu \neq t_{\nu'}$, whenever $\nu, \nu' \in \{1,2,\dots, M+1\}$ with $\nu \neq \nu'$.[4]

[4] According to Proposition 6.2.i, a suitable selection of the $M+1$ time moments t_ν is given by the formula

Assume that, for any $j = 1,2,\dots,\ell+1$, we know the values

$$f_j(t,x,y,z) = \bar{\mathfrak{F}}\left(g_S^{(j)}\left(P_{1/S}^{(j)},\dots,P_{N_j/S}^{(j)};t,x,y,z\right)\right)$$

of the j^{th} geopolitical index $g_S^{(j)}\left(P_{1/S}^{(j)},\dots,P_{N_j/S}^{(j)};t,x,y,z\right)$ accordingly to a geopolitical measurement $\mathfrak{M}_{\mathfrak{F}}$ at the $M+1$ discrete points (t_ν,x_0,y_0,z_0), $\nu = 1,2,\dots,M+1$.

1. For each $j = 1,2,\dots,\ell+1$, consider the unique Lagrange polynomial of degree at most M
$$\mathcal{L}_M^{(j)}(t,x_0,y_0,z_0) = \sum_{\nu=1}^{M+1}\left[f_j(t_\nu,x_0,y_0,z_0)\prod_{\nu'=1,\nu'\neq\nu}^{M+1}\frac{t-t_{\nu'}}{t_\nu-t_{\nu'}}\right]$$
interpolating the $M+1$ measurement values $f_j(t_\nu,x_0,y_0,z_0)$ of the j^{th} geopolitical index
$$g_S^{(j)}\left(P_{1/S}^{(j)},\dots,P_{N_j/S}^{(j)};t,x,y,z\right).$$
2. Construct the *parameterized surface*
$$\mathfrak{L}_M:\mathbb{R}^4\to\mathbb{G}(S)\equiv\mathbb{R}^{\ell+5}:(t,x,y,z)\mapsto$$
$$\mathfrak{L}_M(t,x,y,z) := \left(t,x,y,z,\mathcal{L}_M^{(1)}(t,x_0,y_0,z_0),\dots\right.$$
$$\left.\dots,\mathcal{L}_M^{(\ell+1)}(t,x_0,y_0,z_0)\right)$$
3. Choose a *critical tolerance value* $\epsilon_{critical}$.
4. Find points $(t^*,x^*,y^*,z^*)\in U$ *on which the geopolitical deviation*
$$\bar{\boldsymbol{\mathcal{V}}}(t^*,x^*,y^*,z^*) = \bar{\boldsymbol{\mathcal{V}}}_{\mathfrak{L}_M}(t^*,x^*,y^*,z^*)$$
$$= \mathbb{d}\left(\mathfrak{D}(t^*,x^*,y^*,z^*),\mathfrak{L}_M(t^*,x^*,y^*,z^*)\right)$$
exceeds $\epsilon_{critical}$.
5. *The points* (t^*,x^*,y^*,z^*) *are points of expected specific geopolitical events*.

Remark 6.3. In the fourth step, the solution of the inequality can be derived by solving the following nonlinear optimization problem $(\mathbf{OP_1})$ with constraints

$$(\mathbf{OP_1})\quad\begin{cases} min_{(t,x,y,z)}\ 1, \\ subject\ to: \\ -\boldsymbol{\mathcal{V}}_{\mathfrak{L}_M}(t,x,y,z)+\epsilon_{critical}\le 0 \\ (t,x,y,z)\in U. \end{cases}\ \blacksquare$$

6.2. The case of varying geographic locations

Next, we will assume that the geopolitical study concerns varying geographical locations. To this end, we suppose *the spatiotemporal historical phase U is an open convex set* in $\mathbb{R}^4$ and

$$\{(t_\nu,x_\nu,y_\nu,z_\nu):\nu=1,2,\dots,M+1\}$$

is a given finite set of points in U, such that $t_\nu\neq t_{\nu'}$, whenever $\nu,\nu'\in\{1,2,\dots,M+1\}$ with $\nu\neq\nu'$. Below we will construct a polynomial interpolation, which can be regarded as a generalization of the Lagrange polynomial.

Let n be the unique integer such that

$$M+1=\binom{n+4}{n}=\binom{n+4}{4}.$$

The choice of n is very restrictive. Indeed,

- for $n=1$, the number
$$M+1=\binom{n+4}{n}=\binom{1+4}{1}$$
must be equal to 5;
- for $n=2$, the number
$$M+1=\binom{n+4}{n}=\binom{2+4}{2}$$
must be equal to 15;
- for $n=3$, the number
$$M+1=\binom{n+4}{n}=\binom{3+4}{3}$$
must be equal to 35 etc.

The following result holds.

Proposition 6.3 (Gasca & Sauer 2000) *For any* $j=1,2,\dots,\ell+1$, *consider the polynomial of degree n defined by*

$$p_M^{(j)}(t,x,y,z) = \sum_{\nu=1}^{M+1}\left[f_j(t_\nu,x_\nu,y_\nu,z_\nu)\prod_{\mu=1}^{n}\frac{G_{\nu,\mu}(t,x,y,z)}{G_{\nu,\mu}(t_\nu,x_\nu,y_\nu,z_\nu)}\right]$$

where we have used the notation

$$f_j(t,x,y,z) := \bar{\mathfrak{F}}\left(g_S^{(j)}\left(P_{1/S}^{(j)},\dots,P_{N_j/S}^{(j)};t,x,y,z\right)\right).$$

The parameterized surface

$$\boldsymbol{P}_M:U\to\mathbb{G}(S)\equiv\mathbb{R}^{\ell+5}:(t,x,y,z)\mapsto\boldsymbol{P}_M(t,x,y,z):=$$
$$\left(t,x,y,z,p_M^{(1)}(t,x,y,z),\dots,p_M^{(\ell+1)}(t,x,y,z)\right)$$

interpolates the section $\bar{\mathfrak{F}}:U\to\mathbb{G}(S)\equiv\mathbb{R}^{\ell+5}$ *of the geopolitical measurement* $\mathfrak{M}_{\mathfrak{F}}$ *for the system S over U at the* $M+1$ *points* $(t_\nu,x_\nu,y_\nu,z_\nu)$, *in the sense that*

$$\boldsymbol{P}_M(t_\nu,x_\nu,y_\nu,z_\nu)=\bar{\mathfrak{F}}(t_\nu,x_\nu,y_\nu,z_\nu)$$

whenever $\nu=1,2,\dots,M+1$. $\blacksquare$

There are expressions giving explicit representations for the multivariate interpolation error. For further information, interested readers are referred to Gasca and Sauer (2000) and references cited therein. However, unlike the fixed geographical location case, such an interpolation

---

$$t_\nu=\frac{T_1-T_0}{2}\cos\left(\frac{2\nu+1}{2\nu+2}\pi\right)+\frac{T_1+T_0}{2}.$$

parameterized surface is not defined uniquely. Moreover, there are no general formulas describing convergence of a interpolation parameterized surface sequence $\left(\boldsymbol{P}_M(t,x,y,z)\right)_{M\in\mathbb{N}}$. For these reasons, we will prefer to give bellow a very different approach to the form of multivariate interpolation providing a canonical polynomial of total degree $\leq M$ which interpolates each sufficiently differentiable function $f_j(t,x,y,z)$ at $M+1$ points in the open convex set U in $\mathbb{R}^4$. This approach has been taken by P. Kergin in 1980. More specifically, we have the following.

Proposition 6.4(Kergin 1980).*Given $M+1$ not necessarily distinct points in U, $(t_\nu,x_\nu,y_\nu,z_\nu)$, suppose f_j is $M-$times continuously differentiable function on the convex hull of*

$$\mathfrak{T}_M = \{(t_\nu,x_\nu,y_\nu,z_\nu) : \nu = 1,2,\dots,M+1\}$$

$j = 1,2,\dots,\ell+1$. There is a canonical choice of a ***unique*** *interpolating polynomial $\mathcal{K}_M(f_j)$,of total degree $d_j \leq M$ that satisfies:*

1) *$\mathcal{K}_M(f_j)(t_\nu,x_\nu,y_\nu,z_\nu) = f_j(t_\nu,x_\nu,y_\nu,z_\nu)$ for $\nu = 1,2,\dots,M+1$;*
   if a point $(t_\nu,x_\nu,y_\nu,z_\nu)$ is repeated $s\geq 2$ times, then $\mathcal{K}_{\mathfrak{T}_M}(f_j)$ and f_j have the same Taylor series up to order $s-1$ at $(t_\nu,x_\nu,y_\nu,z_\nu)$;
2) *the mapping $f_j \to \mathcal{K}_M(f_j)$ is linear and continuous.* ■

The parameterized surface

$\boldsymbol{\mathcal{K}}_M : U \to \mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}$:

$$(t,x,y,z) \mapsto \boldsymbol{\mathcal{K}}_M(t,x,y,z) :=$$
$$= \left(t,x,y,z,\mathcal{K}_M(f_1)(t,x,y,z),\dots,\mathcal{K}_M(f_{\ell+1})(t,x,y,z)\right)$$

interpolates the parameterized surface $\bar{\mathfrak{F}} : U \to \mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}$ of the geopolitical measurement $\mathfrak{M}_{\mathfrak{F}}$ for the system S at the $M+1$ points $(t_\nu,x_\nu,y_\nu,z_\nu)$, in the sense that

$$\boldsymbol{\mathcal{K}}_M(t_\nu,x_\nu,y_\nu,z_\nu) = \bar{\mathfrak{F}}(t_\nu,x_\nu,y_\nu,z_\nu),$$

whenever $(t_\nu,x_\nu,y_\nu,z_\nu) \in \mathfrak{T}$. ■

An explicit formula for $\mathcal{K}_M(f_j)$ was given in (P. Milman & C. Micchelli 1999). The formula shows that the coefficients of $\mathcal{K}_M(f_j)$ are given by integrating derivatives of f_j over faces in the convex hull of $\mathfrak{T}$. To see this, let us denote by Σ_r the simplex

$$\Sigma_r = \{(\sigma_1,\sigma_2,\dots,\sigma_{r+1}) \in \mathbb{R}^{r+1} : \sigma_k \geq 0 \ and \ \textstyle\sum_{\nu=1}^{r+1}\sigma_\nu = 1\}$$

and use the notation

$$\int_{[(t_1,x_1,y_1,z_1),\dots,(t_{r+1},x_{r+1},y_{r+1},z_{r+1})]} g = \int_{\Sigma_r} g\big(\sigma_1(t_1,x_1,y_1,z_1)+\cdots+ \sigma_{r+1}(t_{r+1},x_{r+1},y_{r+1},z_{r+1})\big)\ d\sigma_1 \dots d\sigma_{k+1}.$$

Then

$$\mathcal{K}_M(f_j)(t,x,y,z) = \sum_{\nu=1}^{M} \int_{[(t_1,x_1,y_1,z_1),\dots,(t_{r+1},x_{r+1},y_{r+1},z_{r+1})]} [D_{(t,x,y,z)-(t_1,x_1,y_1,z_1)} \dots D_{(t,x,y,z)-(t_r,x_r,y_r,z_r)} f_j$$

where $D_{\boldsymbol{u}} f_j$ denotes the directional derivative of f_j in the direction $u \in U \subset \mathbb{R}^4$. The error of the interpolation takes the form

$$\left(\mathcal{K}_M(f_j) - f_j\right)(t,x,y,z) = \int_{[(t,x,y,z),(t_1,x_1,y_1,z_1),\dots,(t_{M+1},x_{M+1},y_{M+1},z_{M+1})]} D_{(t,x,y,z)-(t_1,x_1,y_1,z_1)} \dots D_{(t,x,y,z)-(t_{M+1},x_{M+1},y_{M+1},z_{M+1})} f_j.$$

■

Remark 6.5. One fundamental problem lies in the *convergence of a sequence of such interpolation polynomials*. The results could be obtained more easily if we had the ability to generalize and resort to the use of methods and types of Complex Analysis. To this end, we have to consider the vector space $\mathbb{R}^4$ as a subspace of $\mathbb{C}^4$ ($\mathbb{R}^4 \equiv \mathbb{R}^4 + i\{0\}$) and, thereby, we have every interest to consider extending the definition of Kergin interpolation to the complex context. Towards this direction, suppose that each f_j is analytic in an open cube $I^{(j)} \subset U$. By the root test, f_j extends continuously (and analytically) to a function $\tilde{f}_j$ into an open $\mathbb{C}-$convex domain $\Omega^{(j)}$ in $\mathbb{C}^4$ (i.e. every intersection of $\Omega^{(j)}$ with a complex affine line is connected and simply connected). Let

$$\mathfrak{Q}_M = \Big\{p_\nu = \Big(t_\nu^{(1)} + it_\nu^{(2)}, x_\nu^{(1)} + ix_\nu^{(2)}, y_\nu^{(1)} + iy_\nu^{(2)}, z_\nu^{(1)} + iz_\nu^{(2)}\Big) : \nu = 1,2,\dots,M+1\Big\}$$

be $M+1$ points in $I := \bigcap_{j=1}^{\ell+1} I^{(j)}$. As in the real context, Kergin interpolation provides a canonical choice of a *unique* interpolating polynomial $\mathcal{k}_M(\tilde{f}_j)$, of total degree $\leq M$ that satisfies:

1) $\mathcal{k}_M(\tilde{f}_j)(p_\nu) = \tilde{f}_j(p_\nu)$ for $\nu = 1,2,\dots,M+1$;
2) the mapping $\tilde{f}_j \to \mathcal{k}_M(\tilde{f}_j)$ is linear and continuous.

In contrast to the real case, in this complex context, there are several theorems describing the convergence behavior of a Kergin interpolation sequence $\left(\mathscr{k}_M(\tilde{f}_j)\right)_{M=1,2,\dots}$ to the function $\tilde{f}_j$. Below, we give only two indicative results. For more information the interested reader can consult the relevant references.

– *Suppose* $(\mathfrak{Q}_M)_{M=1,2,\dots} \subset \bar{B}(0;r)$*=the closed ball in* $\mathbb{C}^4$ *with center at* 0 *and radius* r. *Suppose each* $\tilde{f}_j$ *extends to an entire function on* $\mathbb{C}^4$ *satisfying the inequality*

$$\left|\tilde{f}_j(z)\right| \leq C \cdot \rho^{-(1/2+\varepsilon)} \cdot 2^{\rho/kr}, |z| = \rho$$

where C *is a constant and* $\varepsilon > 0$ *is as small as we like. Then*

$$lim_{M\to\infty}\mathscr{k}_M(\tilde{f}_j)(z) = \tilde{f}_j(z)$$

uniformly on bounded subsets of $\mathbb{C}^4$ (Goodman & Sharma 1984).

– *Suppose* $(\mathfrak{Q}_M)_{M=1,2,\dots}$ *is an increasing sequence of distinct points in* $\mathbb{C}^4$. *We let* $D(R)$ *denote the number of points of the sequence in the polydisk* $\Delta(0;R)$ *with centre at* 0 *and multiradius* R. *Suppose each* $\tilde{f}_j$ *extends to an entire function on* $\mathbb{C}^4$ *of order* $\leq \mu$, *type* $\leq \tau$. *If*

$$\frac{\tau}{\delta} < 2^{1-\mu}\int_0^1 \frac{x^{\mu-1}}{2-x}\,dx \text{ and } liminf_{R\to\infty}\frac{D(R)}{R^\mu} \geq \delta$$

for a positive constant δ, *then*

$$lim_{M\to\infty}\mathscr{k}_M(\tilde{f}_j)(z) = \tilde{f}_j(z)$$

uniformly on compact subsets of $\mathbb{C}^4$ (Bloom 1984).

With this terminology, the well-defined and unique interpolant $\boldsymbol{\mathcal{K}}_M: U \to \mathbb{R}^{\ell+5}$ can approximate entirely, and thus replace completely the unknown geopolitical measurements' parameterized surface $\mathfrak{M}_{\mathfrak{F}}$ in U, determined by a section $\bar{\mathfrak{F}}: U \to \mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}$ of the geopolitical measurement $\boldsymbol{\mathfrak{M}}_{\mathfrak{F}}$ for the system S.

We are thus in position to formulate a second theoretical method for solving the problem of predicting the signs of space-time in which there will be a geopolitical event.

Frame Work of the 2nd algorithm for determining points of expected specific geopolitical events

Assume that the geopolitical study concerns varying geographical locations in the spatiotemporal historical phase U:

$$\{(t_v, x_v, y_v, z_v) \in U: v = 1,2,\dots,M+1\}$$

$(t_v \neq t_{v'}$, with $v \neq v')$.

Assume that, for any $j = 1,2,\dots,\ell+1$, we know the values

$$f_j(t,x,y,z) := \bar{\mathfrak{F}}\left(g_S^{(j)}\left(P_{1/S}^{(j)},\dots,P_{N_j/S}^{(j)};t,x,y,z\right)\right)$$

of the j^{th} geopolitical index $g_S^{(j)}\left(P_{1/S}^{(j)},\dots,P_{N_j/S}^{(j)};t,x,y,z\right)$ accordingly to a geopolitical measurement $\mathfrak{M}_{\mathfrak{F}}$ at the $M+1$ discrete points (t_v, x_0, y_0, z_0), $v = 1,2,\dots,M+1$.

1. For each $j = 1,2,\dots,\ell+1$, consider the unique Kergin polynomial of degree at most M:

$$\mathcal{K}_M(f_j)(t,x,y,z) = \sum_{v=1}^{M}\int_{[(t_1,x_1,y_1,z_1),\dots,(t_{r+1},x_{r+1},y_{r+1},z_{r+1})]} \left[D_{(t,x,y,z)-(t_1,x_1,y_1,z_1)}\dots D_{(t,x,y,z)-(t_r,x_r,y_r,z_r)}f_j\right.$$

interpolating the $M+1$ measurement values $f_j(t_v,x_v,y_v,z_v)$ of the j^{th} geopolitical index $g_S^{(j)}\left(P_{1/S}^{(j)},\dots,P_{N_j/S}^{(j)};t,x,y,z\right)$.

2. Construct the *parameterized surface*

$$\boldsymbol{\mathcal{K}}_M: \mathbb{R}^4 \to \mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}:$$
$$(t,x,y,z) \mapsto \boldsymbol{\mathcal{K}}_M(t,x,y,z) := \left(t,x,y,z,\mathcal{K}_M(f_1)(t,x,y,z),\dots,\mathcal{K}_M(f_{\ell+1})(t,x,y,z)\right)$$

3. Choose a *critical tolerance value* $\delta_{critical}$.

4. Find points $(t^*,x^*,y^*,z^*) \in U$ on *which the geopolitical deviation*

$$\bar{\boldsymbol{\mathcal{V}}}(t^*,x^*,y^*,z^*) = \bar{\boldsymbol{\mathcal{V}}}_{\boldsymbol{\mathcal{K}}_M}(t^*,x^*,y^*,z^*) = \mathbb{d}\left(\boldsymbol{\mathfrak{D}}(t^*,x^*,y^*,z^*),\boldsymbol{\mathcal{K}}_M(t^*,x^*,y^*,z^*)\right)$$

exceeds $\delta_{critical}$.

5. *The points* (t^*,x^*,y^*,z^*) *are points of expected specific geopolitical events.*

Remark 6.6. As in the frame work of the 1st algorithm, in the fourth step, the solution of the inequality $\boldsymbol{\mathcal{V}}(t^*, x^*, y^*, z^*) > \delta_{critical}$ can be derived by solving the following nonlinear optimization problem $(\mathbf{OP_2})$ with constraints

$$(\mathbf{OP_2}) \quad \begin{cases} min_{(t,x,y,z)} \ 1, \\ subject\ to: \\ -\boldsymbol{\mathcal{V}}_{\mathcal{K}_M}(t,x,y,z) + \delta_{critical} \leq 0 \\ (t,x,y,z) \in U. \ \blacksquare \end{cases}$$

References

Aron, R. (1967). Qu'est-ce qu'une théorie des relations internationales? *Revue française de science politique*, vol. 17, no 5.

Avriel, M. (2003). *Nonlinear Programming: Analysis and Methods*, Dover Publishing, ISBN 0-486-43227-0

Bazaraa, M.S. & Shetty, C. M. (1979). *Nonlinear programming. Theory and algorithms*, John Wiley & Sons, ISBN 0-471-78610-1

Bertsekas, D. P. (1999). *Nonlinear Programming (Second ed.). Cambridge*, MA.: Athena Scientific, ISBN 1-886529-00-0

Bloom, T. (1984). On the Convergence of Interpolating Polynomials for Entire Functions, Analyse Complexe, Proceedings, Toulouse 1983, *Lecture Notes in Mathematics 1094*, Springer-Verlag

Bonnans, F. J., Gilbert, C. J., Lemaréchal, C. & Sagastizábal, C. A. (2006). *Numerical optimization: Theoretical and practical aspects*, Universitext (Second revised ed. of translation of 1997 French ed.). Berlin: Springer-Verlag. pp. xiv+490, doi:10.1007/978-3-540-35447-5, ISBN 3-540-35445-X. MR 2265882,

Brinkhuis, J. & Tikhomirov, V. (2005). *Optimization: Insights and Applications*, Princeton University Press

Cederman, L. E. (2002). Endogenizing geopolitical boundaries with agent-based modeling. *Proceedings of the National Academy of Sciences of the United States of America* (PNAS) 99(3), pp. 7796-7303

Cederman, L. E. (2003). Explaining State Sizes: A Geopolitical Model. *Proceedings of Agent 2003: Challenges in Social Simulation*, edited by Macal, Charles, Michael North, and David Sallach. Argonne, IL: Argonne National Laboratory

Cederman, L. E. (2004). Agent-Based Models of Geopolitical Processes. *Bildarchiv der ETH-Bibliothek Prod*. http://dx.doi.org/10.3929/ethz-a-004743173

Cederman, L.E & Girardin, L. (2005). *Exploring geopolitics with Agent-Based Modeling*, prepared for presentation at University of Tokyo, September 24-25, 2005, http://citrus.c.u-tokyo.ac.jp/mas/achievement/ws2005/ws2005_ced erma_paper.pdf

Cooper, C. S., Beyeler W. E., Hobbs, J. A., Mitchell, M., Copley, Z. R., & Antognoli, M. (2012). Behaviors of actors in a resource-exchange model of geopolitics, *Complex*, pp. 70-82

Daras, N.J. & Mazis, J. Th. (in preparation). Geopolitical Systemic Modeling. Part 2. Subjectivity in the Prediction of Geopolitical Events. In preparation

Elman, C. & Elman, M. F. (eds.) (2003). *Progress in International Relations Theory: Appraising the Field*, MIT Press, Cambridge, Massachusets, London, England (Belfer Center for Science and International Affairs, John F. Kennedy School of Government, Harvard University)

Gasca, M. & Sauer, T. (2000). Polynomial Interpolation in Several Variables, *Advances in Computational Mathematics*, 12 (4), pp. 377-410

Gawlik, R. (2010). Managerial Decision Making in Geopolitically Turbulent Environments. in: *Generating Innovative Solutions to Recurring Problems in the Global Business Environment: A Multi-, Inter-, and Trans Disciplinary Approach to Formulating and Maintaining a Competitive Organizational Edge*, pp. 214-221, http://mpra.ub.uni-muenchen.de/45408/1/MPRA_paper_45408.pdf

Goodman, T. N. T. & Sharma, A. (1984). Convergence of Multivariate Polynomials Interpolating on a Triangular Array. *Transactions of the American Mathematical Society*, 285(1), pp. 141-157

Hoff P. D. & Ward M. D. (2004). Modeling Dependencies in International Relations Networks. *Political Analysis* 12, pp. 160-175, http://www.csss.washington.edu/Papers/wp35.pdf

Kergin, P. (1980). A natural interpolation of C^K functions, *J. Approx. Th.* 29, pp. 278–293

Kuperman, M. N.(2010). A model for the emergence of geopolitical division. *Phil. Trans. R. Soc. A*. 368, pp. 5695-5706

Levin, R. I. (2006). *Statistics for Management*, Prentice-Hall, 3th edition, March 1984, Digitized: 30 March 2006, **ISBN-10:** 0138452482, **ISBN-13**: 978-0138452483

Luenberger, D. G. & Ye, Y. (2008). *Linear and nonlinear programming*, International Series in Operations Research & Management Science **116** (Third ed.), New York: Springer. pp. xiv+546, ISBN 978-0-387-74502-2. MR 2423726

Mazis, I. Th. (2002). La geopolitica contemporanea: basi e definizioni di metodo, DADAT, Università degli Studi di Napoli-Federico II, Dipartimento delle Dinamiche Ambientali e Territoriali, *Saggi di Geopolitica*, Napoli, pp. 1-11.

Mazis, I. Th. (2008). [China-Bei Jing], *Writing Methodology of a Geopolitical Analysis [Structure, Concepts and Terms],* C.I.I.S.S. / I.A.A.: China Institute for International Strategic Studies (C.I.I.S.S.)/ Defence Analyses Institute (D.A.I.), Cooperation on Defence Diplomacy, Athens/ Beijing at May 2008, *Defensor Pacis* (Special Issue I.A.A./C.I.I.S.S.), Special Issue, Vol 23, pp. 53-59

Mazis, I. Th. (2013). L'Analyse Géopolitique Systémique: Propositions Terminologiques et Définitions Métathéoriques selon l'exigence métathéorique lakatienne, *Géographies, Géopolitiques et Géostratégies Régionales*, 1(1), pp. 21-32

Mazis, I. Th. (2014). Methodology for Systemic Geopolitical Analysis according to the Lakatosian model, (Post graduate Colloquium) *Avrasya Enstitüsü*, İstanbul Üniversitesi, Turkey

Mazis, I. Th. (in preparation). *Critique Métathéorique de la Théorie des Relations Internationales et de la Géopolitique. Le Cadre Néopositiviste*, book in preparation.

Micchelli, C.A. and Milman, P. (1980). A formula for Kergin interpolation in $\mathbb{R}^K$, *J. Approx. Th.*, 29, pp. 294–296

Nocedal J. & Wright S. J. (2006) *Numerical Optimization, Springer*, 2nd edition, ISBN 0-387-98793-2

Parsopoulos, K.E. & Vrahatis, M.N. (2002). Recent Approaches to Global Optimization Problems through Particle Swarm Optimization, *Natural Computing*, 1, pp. 235-306

Parsopoulos, K.E. & Vrahatis, M.N. (2010). *Particle Swarm Optimization and Intelligence: Advances and Applications*, Information Science Reference, Hershey-New York, ISBN 978-1-61520-666-7 (hardcover)- ISBN 978-1-61520-667-4 (e-book)

Robertson, G. H. (1993). *Quality through Statistical Thinking: Improving Process Control and Capability*, Amer Supplier Inst., 2nd edition, **ISBN-10:** 0941243087, **ISBN-13:** 978-0941243087

Ruszczyński A. P. (2006). *Nonlinear Optimization*, Princeton, NJ: Princeton University Press, pp.xii+454, ISBN 978-0691119151. MR 2199043