LXIV. Systemic Geopolitical Modeling. Part 2: Subjectivity in Prediction of Geopolitical Events

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Abstract This paper studies subjective priorities for the data amounts in the processing of geopolitical data accoding to Mazis I. Th., theoretical paradigm of *Systemic Geopolitical Analysis*. After defining *geopolitical plans* and *geopolitical focus sets*, they are introduced *geopolitical preferences* and geopolitical management capacities. The geopolitical rational choice is studied, as well as the geopolitical preference-capacity distributions. Then, they are investigated *geopolitical contrasts* of subjective priorities by several geopolitical operators, and it is shown that there are cores and equilibriums of geopolitical contrasts, the study of which may provide useful information.

Keywords Systemic geopolitical analysis · Geopolitical operator · Numerical carrier of weighted geopolitical index · Geopolitical plan · Geopolitical focus set · Geopolitical preference · Geopolitical indifference · Geopolitical significance · Geopolitical management options · Mean geopolitical rational choice · Geopolitical preference distribution · Geopolitical capacity distribution · Geopolitical sector · Geopolitical contrasting of subjective priorities · Core of contrasting geopolitical subjective priorities · Equilibrium of contrasting geopolitical subjective priorities · Geopolitical equilibrium vectors

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Introduction

In a recent paper, it has been documented a holistic systemic geopolitical modeling by using two General mathematical methods predicting geopolitical events into a given geopolitical system (Daras and Mazis 2014). The starting point was to consider weighted geopolitical indices and their measurements. A weighted geopolitical index is a quantity which refers exclusively to a geopolitical index at any point of the space-time, endowed with an associated threshold above and below which it is marked a geopolitical change in the conduct of the geopolitical system. A geopolitical measurement gives the value of a geopolitical index measured at some discrete time moments and some geographic location points. If one is limited within a given region of space-time, then the corresponding set of weighted geopolitical indices over this region is a universality of weighted geopolitical indices. The distance between such a universality of weighted geopolitical indices and a parameterized surface which interpolates discrete points representing values of a geopolitical measurement can be considered as a measure for assessing the occurrence or not of a geopolitical event. They have been proposed two general frameworks for determining geographical location points and time moments where it is expected appearance of peculiar geopolitical events. The corresponding algorithmic formulations showed that the prediction problem is reduced to two respective classical nonlinear optimization problems.

A basic and reasonable question arises immediately and may be constitute the central subject of discussion in subsequent additional scientific studies. The question relates to the subjectivity of geopolitical choices and priorities: given that it is very doubtful whether the considered set of weighted geopolitical indices could be considered as exhaustive, one wonders if the above prediction is ultimately reliable. Equivalently, *if a geopolitical operator considers a set of weighted geopolitical indices and if another geopolitical operator considers a different set of weighted geopolitical indices, then how much the two predictions will differ or diverge?*

The purpose of this paper is to study in depth this question. The first part of the paper examines the case of a single geopolitical operator. Obviously, a geopolitical operator can choose or use only a numerical carrier (value or/and amount of numerical data) for each weighted geopolitical index. Thus, in "Geopolitical plans" section, we will describe how through its options, a geopolitical operator [processor] may prefer to focus only on some choices. A geopolitical plan for the geopolitical operator specifies the numerical carrier of each weighted geopolitical index that the geopolitical operator may take into account. Then, in "Geopolitical preferences " section, we will study the preferences of a geopolitical operator. A *geopolitical preference* is *the relation that determines the geopolitical selectivity of an operator*. In order to establish a well such preference, in "Weighted geopolitical systems and geopolitical management capacities" section, we will show how a geopolitical operator should associate a certain geopolitical significance in each weighted geopolitical index and we consider the corresponding geopolitical rational choice set, while in "Topology and neighboring geopolitical preferences" section we will study the topology of the space of geopolitical preferences and we shall describe neighboring preferences of a given geopolitical preference. Having regard to all these, in the next "The lower hemi-continuity of the geopolitical rational choice" section we will investigate the lower hemicontinuity of the relation defining the set of all geopolitical rational choices, and in "Mean geopolitical rational choice" section we will deal with the concept of the mean geopolitical rational choice for a set of geopolitical operators. The second part of the paper is devoted to the case of several geopolitical operators. In this case, each of the operators has its own priorities and preferences, and, after a brief introduction, we will see that there are cores and equilibriums of contrasts, the study of which may provide useful information ("The geopolitical contrast core and the geopolitical contrast equilibrium", "Determinateness of geopolitical equilibrium vectors" sections).

Rational choice of geopolitical sets

Geopolitical plans

Geo-politics (i.e., «geo-» (gaia [in Greek] = earth) and politics (politiki [in Greek] = politics)) is the investigation of actions and influences of the Geography (Human and Physical) to International Politics and International Relations (Devetak et al. 2007; Mazis 2001, 2013; Toncea 2006).

In consistency to this definition, the **Geopolitical analysis of a Geographical System** is characterized by an uneven distribution of power and is defined to be *the geographical method that studies, describes and predicts the attitudes and the consequences ensuing from relations between the opposing and distinct political practices for the re-distribution of power as well as their ideological metaphysics, within the framework of the geographical complexes where these practices apply* (Mazis 2015b: 1063).

From a methodological point of view, it is suggested the following approach (Mazis 2015a, b).

1st stage: Decoding the title of the topic

The title of the topic of a study of geopolitical analysis (should) define(s) the facts and the objectives of our problem. In particular it defines:

1. The boundaries of the Geographical Complex which constitutes the geographical area to be analyzed.
2. The (internal or external) space of the Complex under study as a field of distribution or

redistribution of power due to the activity of a specific geopolitical factor.

3. The above-mentioned geopolitical factor, the impact of which may affect the distribution of power, within or outside this Geographical Complex.

»Decoding the title of the Topic. Example

- *Topic: "The Geopolitics of the Islamic movement in the Greater Middle East"*
- *Analysis of the Title:*

  a. *Identification of the boundaries of the Geographical/Geopolitical Complex* The boundaries of the Geographical/Geopolitical Complex are defined by the term "Greater Middle East".
  b. *The precise identification of the Space under study* The Space under study of this specific Complex is the "interior" space of the Geographical/Geopolitical Complex of the Greater Middle East and this is evident by the use of "in", i.e., "in the inside of…", "within the boundaries of…".
  c. *Identification of the Geopolitical Factor* The designated Geopolitical Factor is the "Islamist movement".

2nd stage: Identifying the boundaries of the geopolitical systems under study

At this stage, we identify the boundaries of the Geopolitical Systems within which we are going to study the activity (or activities) of the Geopolitical Factor defined in the title. There are three levels of Systems defined according to the extent of the geographic area they refer to:

1. Sub-systems that are subsets of the Systems.
2. The System that is the Geographical Complex under investigation.
3. Supra-Systems, containing the main System under study—as a subset along with other Systems that may not concern the current analysis.

3rd stage: Defining the fields of influence of the "geopolitical factor"

Once we have defined the three levels of Systems, we should identify the fields of geopolitical influence of the "geopolitical factor" under study. In other words, we should determine which combination of the four "fields" or "geopolitical pillars" of the given "geopolitical factor" we are going to investigate, always within the framework of the chosen Systemic scale (e.g., on the level of "System" or on the level of "Subsystems").

In order to follow a rational order in the examination of the influences of the Geopolitical Factor (GP) we should start the investigation from the "Supra-systems" level and continue with the "System" level. Such a sequential order should prove that, in most cases, if the analysis of the influences of the GP on the level of the Sub-systems is completed, and if Sub-systems have been correctly identified, the respective analysis on the level of the whole System is also completed.

The Geopolitical pillars are as follows:

1. Defense/security
2. Economy
3. Politic
4. Culture and Information.

The aforementioned pillars are examined in terms of power, e.g., economic power, political power, etc.

»Identifying the function of the Geopolitical factor for the specific pillars of influence—Example

At this stage we are going to identify the geopolitical trends-dynamics for each designated Subsystem. These trends are identified only and exclusively in terms of "power". They answer the following questions:

a. The pillars (defense/security, economy, politics, culture/information) where the "geopolitical factor" under study prevails (in our case the GF "Islamist movement") and by consequence already determines or may determine their attitude within the framework of each Sub-system. This type of conclusion is defined as "positive sub-systemic component of the trend power" of the "geopolitical factor" in the "Interior of the System".
b. Which pillars absorb the influence of the "geopolitical factor", and by consequence, it does not influence the whole attitude of the Sub-system. This form of conclusion is defined as "zero sub-systemic component power trend" of the "geopolitical factor" in the "Interior of the System".

4th stage: Synthesis

The term "synthesis" refers to the procedure through which we can detect the Resultant Power Trend of the given Geopolitical factor on whichever final systemic scale (e.g., Sub-system, System or Supra-system level).We must distinguish two cases.

- *1st case* In case we have detected and defined the particular power components (of the geopolitical factor at hand) on the Sub-system level, and our objective is the Component of the System on the systemic level, then the stage of synthesis begins from the level of the System.
- *2nd case* In case the component in question is on the level of the Supra-system, then the stage of Synthesis starts after the conclusion of the Analysis of the components of the individual Systems. This means that the synthesis should start from the level of Sub-systems, and we should then shape the image of the components on the level of Systems, and finally conclude with the identification of the component on the level of Supra-system.

5th stage: Conclusions

The last stage of the geopolitical analysis is that of Conclusions. At this stage we must describe the geopolitical dynamics, to which the "Component of power" of the "geopolitical factor" under study, subject to the attitude of the System examined, in the context of the Supra-system.

We must stress that: At this stage of the study, as in any other stage of the aforementioned geopolitical analysis, we make no proposals.

1. At this stage, we discover: structures, actions, functions, influences, forms and dynamics of the geopolitical factor and we describe them.
2. We also describe how they affect the attitude of the System. Proposals do not form part of a Geopolitical Analysis. They are part of the Geostrategic approach which may be carried out, only if asked and by exploiting the results of the geopolitical analysis preceding (Mazis 2015a).

So, from an epistemological point of view, the above proposed methodological approach of the Systemic Geopolitical Analysis adopts the following Lacatosian structure (Mazis 2015b):

- Definition of the fundamental axiomatic assumptions (Elements) of the hard core of the geopolitical research program
- Definition of the auxiliary hypotheses (elements [e]) of the protective belt of the geopolitical research project
- The issue of the positive heuristics of the geopolitical research program
- The elements of the positive heuristics of the geopolitical research program.

According to the Lakatosian meta-theoretical approach, the hard core (fundamental assumptions) constitutes the basic premise of a research program. The hard core is protected by negative heuristics, in short, by the rule that prohibits researchers to contradict the fundamental ideas of a given research program, i.e., with the hard core of the program (as an attempt to address new empirical data which tend to invalidate the theory). That being said, we believe that:

I. *The first fundamental axiomatic assumption (Element 1)*, which constitutes the centre of the hard core of the geopolitical research program, is that all the characteristics of the above-mentioned subspaces of the geographical complex are countable or can be counted, through the countable results which they produce, e.g., the concept of "democracy" of a polity (according to western standards, since there are no other). This is a concept identified as a Geopolitical Index within the framework of the secondary causative "Political Space", as defined earlier, and can be countable by means of a multitude of specific results, which it produces in the society where this form of political governance is applied. Such are for example the number of printed and electronic media in the specific society, the number of political prisoners or their absence, the level of protection of children of single-parent families, the number of reception areas for immigrants and density of the latter per m^2 etc. These figures are classified, systematized and evaluated according to their specific gravity concerning the function of the figure to be quantified, and constitute the Geopolitical Indices that we are going to present and examine in detail below.

II. *»The second fundamental axiomatic assumption (Element 2)* of the hard core of the systemic geopolitical program is that, within the framework of the geographical area under study, there exist more than two consistent and homogeneous Poles which are also:

1. self-determined (as to "what" they consider "gain" and "loss" for themselves), and also in relation to their international environment;
2. hetero-determined, uniformly and identically to their international environment which is determined by the international actors that dwell within them and their common systemic relation is their characteristic. […], according to the Lakatosian meta-theoretical approach, a research program has the protective belt of complementary hypotheses, i.e., proposals that are subject to control, adaptation and re-adaptation, and that are replaced when new empirical data come to light.

Moreover, given Lakatos' dictum that "in the positive heuristic of a program there is, right at the start, a general outline of how to build the protective belts" and that "a research program [is defined] as degenerating even if it anticipates novel facts but does so in a patched-up development rather than by a coherent, pre-planned positive heuristic" (Lakatos 1970), we should proceed by formulating a (provisional) definition of that protective belt for our research program. Consequently, following the Lakatosian meta-theoretical paradigm, the protective belt of the geopolitical research program should be defined, complemented with the following auxiliary hypotheses-elements:

I. *(element [e1]): First auxiliary hypothesis of the protective belt of the geopolitical research program:* the size of the power is analyzed in four fundamental entities (Defense, Economy, Politics, Culture/Information), which in turn are analyzed in a number of geopolitical indices. These Geopolitical Indices, as already mentioned, are countable or can be counted and they are detected and counted in the internal structures of the those Poles that each time constitute the Sub-systems of the Geographical Complexes under geopolitical analysis.

II. *(element [e2]): Second auxiliary hypothesis of the protective belt of the geopolitical research program:* the above Poles constitute fundamental structural components of an international, and ever changing, unstable System.

III. *(element [e3]): Third auxiliary hypothesis of the protective belt of the geopolitical research program:* these Poles express social volitions or volitions of the deciding factors that characterize the international attitude of the Pole. Consequently, these poles can be national states, collective international institutions (e.g., international collective security systems, international development institutions, and international cultural institutions), economic organizations of an international scope (i.e., multinational companies, bank consortia) or combinations of the above which, however, present uniformity of action within the international framework concerning their systemic functioning.

IV. *(element [e4]): Fourth auxiliary hypothesis of the protective belt of the geopolitical research program:* consists of the above-mentioned "causal and causative" notions of the "Primary", "Secondary" and "Tertiary Space", as well as their combinations ("Complete" and «Special Composite Spaces").

V. *(element [e5] Fifth auxiliary hypothesis of the protective belt of the geopolitical research program* is the premise that the international system has a completely unsure, unstable and changing structure.

VI. *(element [e6] Sixth auxiliary hypothesis of the protective belt of the geopolitical research program:* systemic geopolitical analysis aims to conclusions of "practicality", shortly, of some "theory of practice" (Aron 1967), i.e., to the construction of a predictive model of the trends of power redistribution and in no case to "guidelines for action under some specific national or "polarized" perspective. The latter is nothing but the "geostrategic biased synthesis", not a "geopolitical analysis". This equals the use of the results (of the model of power redistribution) of the geopolitical analysis and follows the stage of geopolitical analysis. We must note that the "historicity" of the elements of the research program is represented by the cultural formations developing in the context of the fourth geopolitical pillar. Thus, their countability is possible in the same way as is for the rest of the geopolitical pillars that have a "qualitative nature", by means of the "geopolitical indices" of the Cultural pillar.

At this stage we should not forget that replacing a set of auxiliary assumptions by another set, is an intra-program problem shift, since only the protective belt and not the hard core is altered. The intra-program problem shifts should be made in accordance with the positive heuristics of the problem that is with a set of suggestions or advices that function as guidelines for the development of particular theories within the program.

Further, we should also emphasize that, a key concern of the Geopolitical Research Program is to describe the suggestions to the researcher that will determine the content of the positive heuristics of the Program in question. Without them, it is impossible to assess the progressivism of the Geopolitical analysis according to the necessary "novel empirical content" expected in our analytical spatial paradigm (model).

Given these necessary clarifications concerning the elements of the positive heuristics of the geopolitical research program, we define the following:

I. The methodology of each theoretical approach should remain stable until a possible detection of continuous degeneration.
II. The requirement of predictive ability and the expansion of the empirical basis of the theoretical approach should be maintained.
III. The empirical facts should constitute the final measure for assessing competitive theoretical approaches of the same set [research program].
IV. The facts that have been used to test a theoretical approach should not be the only ones used for verifying this approach but, with the progress of time of research, the testing of the theoretical approach should be referred also with facts that derive from the expansion of the empirical basis of the given approach». (Mazis 2015b).

We point out that a geopolitical analyst is a properly informed geographer who conducts a geopolitical analysis within the framework of a Geographical/ Geopolitical Complex. Below, without any risk of confusion and for obvious reasons of adopting assimilative generality and acceptable uniformity, we will prefer to use the appellation "**geopolitical operator**" instead of that of geopolitical analyst.

Hereafter, we are now in position to proceed to necessary mathematical foundations and several involved applications.

Let S be a geopolitical complex. A **weighted geopolitical index** of the complex S at date t and location (x, y, z)

$$g_S^{(j)} = g_S^{(j)}\left(P_{1/S}^{(j)}, \ldots, P_{N_j/S}^{(j)}; t, x, y, z\right)$$

is a numerical function of the values of its N_j intrinsic properties (physical characteristics) $\left(P_{1/S}^{(j)}, \ldots, P_{N_j/S}^{(j)}\right)$ into the system S, the date $t \in \mathbb{R}$ and the location $(x, y, z) \in \mathbb{R}^3$ at which it is studied. It is assumed that *there are a sufficiently great number of distinguishable weighted geopolitical indices of the complex S, say*

$$\begin{aligned} g_S^{(1)} &= g_S^{(1)}(t, x, y, z), \\ g_S^{(2)} &= g_S^{(2)}(t, x, y, z), \ldots, \\ g_S^{(\ell+1)} &= g_S^{(\ell+1)}(t, x, y, z), \quad \ell \gg 0 \end{aligned}$$

for any date t and any location (x, y, z).

Definition 2.1 Hereafter, for the value or/and the amount of numerical data of a given weighted geopolitical index $g_S^{(i)}$ over a geopolitical S at a date t and a location (x, y, z) we will use, without any distinction and risk of confusion, the single term **numerical carrier of** $g_S^{(i)}$.

Obviously, the *numerical carrier, say* d_i, *of a weighted geopolitical index* $g_S^{(i)}$ *over S at a date t and a location* (x, y, z) can be expressed by a real number. So, any unit vector

$$1_{d_i} = \left(\underbrace{0, \ldots, 0, 1, 0, \ldots, 0}_{i-\text{position}}\right) \text{ of } \mathbb{R}^{\ell+1}$$

is identified with one unit of numerical carrier of geopolitical weight index *over the complex S*, and the linear space $\mathbb{R}^{\ell+1}$, endowed with the corresponding product Euclidean topology, is a **continuous space of numerical carriers of weighted geopolitical indices** over S.

Definition 2.2 For a geopolitical operator $\mathcal{M}$, a **geopolitical plan** over the complex S specifies the numerical carrier for **each** weighted geopolitical index in S that $\mathcal{M}$ takes into account, as well as the numerical carrier for this weighted geopolitical index which he will make available. We shall use the convention that the numerical carrier of a weighted geopolitical index in S which is used **by** the geopolitical operator $\mathcal{M}$ is represented by a negative number, while the numerical carrier over S which has to be made available to the geopolitical operator $\mathcal{M}$ is represented by a positive number. Then, **every geopolitical plan can be represented by an element**

$x = (d_1, d_2, \ldots, d_{l+1})$

in the measurable space $\mathbb{R}^{\ell+1}$.

Remark 2.1 It is obvious that every element in $\mathbb{R}^{\ell+1}$ can be interpreted meaningfully as a geopolitical plan.

It is assumed that *for every geopolitical operator $\mathcal{M}$ there is a nonempty closed subset*

$\mathfrak{X}_{\mathcal{M}}$

in $\mathbb{R}^{\ell+1}$, which describes the set of all *a priori* possible geopolitical plans over the complex S. Here *a priori* possible means that, ignoring management acts, the geopolitical operator can carry out the geopolitical plan over the system S. More specifically, we have the following.

Definition 2.3

i A **geopolitical focus set** or simply a **focus set**

$\mathfrak{X}$

over the geopolitical complex S is a nonempty subset of the geopolitical plans over S which is closed, convex and bounded from below.

ii Given a vector $\boldsymbol{y} \in \mathbb{R}^{\ell+1}$ and a compact subset $E \subset \mathbb{R}^{\ell+1}$, we denote by

$\mathfrak{X}_{\boldsymbol{y};E}$

the compact set of all focus sets $\mathfrak{X}$ such that $\boldsymbol{y} \in \mathfrak{X}$ and $\mathfrak{X} \cap E \neq \emptyset$.

Remark 2.2 A focus set over the system S will typically belong to a *discrete* (not necessarily finite) set in $\mathbb{R}^{\ell+1}$.

Geopolitical preferences

Definition 2.4 We say that **a geopolitical operator $\mathcal{M}$ selects the geopolitical plan x instead of the geopolitical plan x'** if he wants to select x whenever he is offered the alternatives x and x'.

The binary relation "*selected*" becomes a powerful tool for modeling analysis if the behavior of the geopolitical operators *reveals* a certain 'consistency' of choices.

Definition 2.5 A **geopolitical selection preference**, or simply **geopolitical preference**, in the geopolitical complex S is a pair $(\mathfrak{X}, \succ)$ where

- *$\mathfrak{X}$ is a focus set over a system S and*
- *$\succ \subset \mathfrak{X} \times \mathfrak{X}$ is a transitive and non-reflexive binary relation on $\mathfrak{X}$ such that $\succ$ is open in $\mathfrak{X} \times \mathfrak{X}$.*

In what follows, instead of $(x, y) \in \succ$, we shall write

$x \succ y$.

Thus,

$x \not\succ y$ means $(x, y) \in \not\succ$.

Sometimes it is convenient to represent a geopolitical preference by an $\mathbb{R}$-valued function.

Definition 2.6 Given a $(\mathfrak{X}, \succ)$, a **geopolitical preference representation** in S is a continuous function $u : \mathfrak{X} \to \mathbb{R}$ such that

$x \succ y$ if and only if $u(x) > u(y)$.

Notation 2.1 It is well known that *if E is a compact subset of $\mathbb{R}^{\ell+1}$, then the set $\mathbb{P}(E)$ of all nonempty closed subsets of E together with the Hausdorff distance δ on E is a compact metric space*. So, in what follows we will always assume that

- *E is a compact subset of $\mathbb{R}^{\ell+1}$*
- *$\mathfrak{K}(\subset \mathbb{P}(E))$ is a compact subset of focus sets $\mathfrak{X} \subset E$.*
- *The set of all geopolitical preferences $(\mathfrak{X}, \succ)$ in S with $\mathfrak{X} \in \mathfrak{K}$ is denoted by*

  $\mathcal{P} = \mathcal{P}_{\mathfrak{K}}$.

- *The set of geopolitical preferences $(\mathfrak{X}, \succ)$ in S with $\mathfrak{X} \in \mathfrak{X}_{y;E}$ is denoted by*

$\mathcal{P}_{y;E}$.

Remark 2.3 Notice that the particular choice of E is immaterial. To restrict in this way the "universe" of focus sets X is no restriction for our analysis; however, it simplifies the mathematical presentation, since the set $\mathfrak{K}$ will turn out to be compact.

It is easy to verify the following basic properties.

Proposition 2.1 *To every geopolitical preference $(\mathfrak{X}, \succ) \in \mathcal{P}$ we associate the set*

$\mathcal{F} := \{(x, y) \in E \times E : x \in \mathfrak{X}, y \in \mathfrak{X} \text{ and } x \not\succ y\}$.

The set $\mathcal{F}$ is characterized by the following properties.

i. *$\mathcal{F}$ is a closed subset in $E \times E$.*
ii. *$\{x \in E : \text{there is a } y \text{ with } (x, y) \in \mathcal{F}\} \subset \mathfrak{K}$.*
iii. *$(x, y) \in \mathcal{F}$ implies $(x, x) \in \mathcal{F}$ and $(y, y) \in \mathcal{F}$.*
iv. *$(x, y) \notin \mathcal{F}$ and $(y, z) \notin \mathcal{F}$ implies $(x, z) \notin \mathcal{F}$.*

Conversely, given such a set $\mathcal{F}$, we obtain the corresponding geopolitical preference $(\mathfrak{X}, \succ) \in \mathcal{P}$ by setting

$\mathfrak{X} := \{x \in E : (x, x) \in \mathcal{F}\}$ and $\succ \; := (\mathfrak{X} \times \mathfrak{X}) \backslash \mathcal{F}$.

In order to investigate the behavior of the modeling process, it is often required additional properties of the geopolitical preferences. For this purpose, we will now define some useful auxiliary subsets of $\mathcal{P}$.

Definition 2.7 Let $(\mathfrak{X}, \succ) \in \mathcal{P}$ be a given geopolitical preference.

i $(\mathfrak{X}, \succ)$ is said to be **locally non-satiated** in the complex S if for each $x \in \mathfrak{X}$ and each neighborhood $U = U_x$ of x there exists a $x' \in \mathfrak{X} \bigcap U$ such that $x' \succ x$. The set of all locally non-satiated geopolitical preferences in $\mathcal{P}$ is denoted by

$\mathcal{P}_{lns}$.

ii $(\mathfrak{X}, \succ)$ is said to be **monotonic** in S if $0 \leq x \leq y$ and $x \neq y$ in $\mathfrak{X}$ imply $y \succ x$. The set of all monotonic geopolitical preferences in $\mathbb{P}$ is denoted by

$\mathcal{P}_{mo}$.

iii $(\mathfrak{X}, \succ)$ is said to be **negatively transitive** in S if for every $x, y, z \in \mathfrak{X}$ with $x \not\succ y$ and $y \not\succ z$ we have $x \not\succ z$. The set of all negatively transitive geopolitical preferences in $\mathcal{P}$ is denoted by

$\mathcal{P}^*$.

For a geopolitical preference in $\mathbb{P}^*$ one defines the ***geopolitical indifference*** in S by

$x \sim y$ if and only if $x \not\succ y$ and $y \not\succ x$.

The **indifference relation** $\sim$ on $\mathfrak{X}$ is reflexive, transitive and symmetric. The relation $\not\succ$ is then written as $\precsim$. Obviously, the ***geopolitical indifference*** $\precsim$ is reflexive, transitive and complete.

Definition 2.8 The geopolitical indifference $(\mathfrak{X}, \precsim) \in \mathcal{P}^*$ is called:

i **convex** in the complex S if for every $z \in \mathfrak{X}$, the set $\{x \in \mathfrak{X} : z \precsim x\}$ is convex and
ii **strongly convex** in S if for every $x \sim x'$, $x \neq x'$, and every $0 < \lambda < 1$ it follows that $\lambda x + (1 - \lambda)x' \succ x$.
iii The set of all convex (strongly convex) geopolitical preferences in $\mathcal{P}^*$ is denoted by

$\mathcal{P}^*_{co}$. $(\mathcal{P}^*_{sco})$.

Weighted geopolitical systems and geopolitical management capacities

To every weighted geopolitical index $g_S^{(i)}$ over S, there correspond two numerical values: its ***weight*** $\boldsymbol{w}_i$ and its ***balancing evaluation*** $\boldsymbol{b}_i$. The concept of the weight of the index $g_S^{(i)}$ has already been introduced in "Weighted geopolitical systems and geopolitical management capacities" section. Regarding the concept of balancing evaluation $\boldsymbol{b}_i$, this means that $\boldsymbol{b}_i/\boldsymbol{b}_j$ is the amount of available numerical carrier d_j for the weighted geopolitical index $g_S^{(j)}$ in order to obtain one unit of numerical carrier for the weighted geopolitical index $g_S^{(i)}$.

Definition 2.9 Hereafter, for the weight or/and the balancing evaluation of a given $g_S^{(i)}$ over a geopolitical complex S, we will use, without any distinction and risk of confusion, the single term **geopolitical significance of** $g_S^{(i)}$.

Obviously, a ***weighted geopolitical indexed system*** *in a complex S* associates to every weighted geopolitical index $g_S^{(i)}$ in S a real number $\boldsymbol{p}_i$, its ***geopolitical significance***. Thus $\boldsymbol{p}$ can be considered as an element of $\mathbb{R}^{\ell+1}$.

If a geopolitical operator $\mathcal{M}$ in S decides to consider and use the weighted geopolitical system with geopolitical significance $\boldsymbol{p} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_{\ell+1})$, then any $\mathcal{M}$'s choice of geopolitical plans $x = (d_1, \ldots, d_{\ell+1})$ in his geopolitical focus set $\mathfrak{X}_{\mathcal{M}}$ is further restricted. Indeed,

Definition 2.10 The weighted geopolitical system's value $\boldsymbol{p} \cdot x$ of x cannot exceed a certain number $\mathcal{C}_{\mathcal{M}}$, the **geopolitical management capacity** of $\mathcal{M}$ in S.

The real number $\mathcal{C}_{\mathcal{M}}$ represents the maximum weighted value of a potential geopolitical management by $\mathcal{M}$. Thus, a geopolitical management capacity $\mathcal{C}_{\mathcal{M}}$ in S is typically a function of prevailing weights for the weighted geopolitical indices. However, it will be convenient to treat the geopolitical management capacity as an independent argument.

Definition 2.11 Let $\mathcal{M}$ be a geopolitical operator, with geopolitical focus set $\mathfrak{X}$ and geopolitical management capacity $\mathcal{C}_{\mathcal{M}}$ in S. If $\mathcal{M}$ prefers a weighted geopolitical system with geopolitical significance $\boldsymbol{p} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_{\ell+1})$, we define the **set of geopolitical management options** of $\mathcal{M}$ in S by

$$\mathfrak{B}(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}) := \{x = (d_1, \ldots, d_{\ell+1}) \in \mathfrak{X} : (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_{\ell+1}) \cdot (d_1, d_2, \ldots, d_{\ell+1}) \leq \mathcal{C}_{\mathcal{M}}\}.$$

The geopolitical plan which actually is chosen in the set of management options $\mathfrak{B}(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{w})$ depends directly on the geopolitical selection preferences.

Definition 2.12 Let $\mathcal{M}$ be a geopolitical operator, with geopolitical selection preference $(\mathfrak{X}, \succ)$ and geopolitical management capacity $\mathcal{C}_{\mathcal{M}}$ in S. If $\mathcal{M}$ prefers a weighted geopolitical system with geopolitical significance $\boldsymbol{p}$ in S, we define his **geopolitical rational choice set** $\mathfrak{A} = \mathfrak{A}(\mathfrak{X}, \succ, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p})$ in S as the set of maximal elements in the set of geopolitical management options, i.e.

$$\mathfrak{A}(\mathfrak{X}, \succ, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}) = \{x^* = (x_1^*, \ldots, x_{\ell+1}^*) \in \mathfrak{B}(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}) : \textit{there is no } x = (d_1, \ldots, d_{\ell+1}) \in \mathfrak{B}(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}) \textit{ with } x \succ x^*\}.$$

Consequently,

$$x^* \in \mathfrak{A}(\mathfrak{X}, \succ, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}) \textit{ if and only if } x \succ x^* \textit{ implies } \boldsymbol{p} \cdot x > \mathcal{C}_{\mathcal{M}}.$$

Our next purpose will be to investigate how $\mathcal{M}$'s geopolitical rational choice set in S depends continuously on his geopolitical preference $(\mathfrak{X}, \succ)$, his geopolitical management capacity $\mathcal{C}_{\mathcal{M}}$ and weighted geopolitical system with geopolitical significance $\boldsymbol{p}$ in S.

Topology and neighboring geopolitical preferences

The tastes of geopolitical operators are described by geopolitical selection preferences. The intuitive concept of "similar" tastes is therefore made precise mathematically by a **topology on the set** $\mathcal{P}$, of all geopolitical selection preferences in the complex S. From the geopolitical management point of view, tastes can be qualified to be similar if they give rise to similar choices in similar "*geopolitical management capacity- weighted geopolitical system*" situations. This is a necessary condition for any meaningful and operational concept of similarity of tastes.

Surely, the discrete topology on $\mathcal{P}$ allows the correspondences on the set of geopolitical rational choices to have continuity properties. However, for clear reasons, we want a topology which is metrizable and separable or even compact.

Theorem 2.1

i *The set $\mathcal{P}$ of geopolitical selection preferences in the complex S endowed with the topology $\mathcal{T}_{\textit{closed}}$ of closed convergence is compact and metrizable.*

ii *A sequence $(\mathfrak{X}_n, \succ_n)_{n\in\mathbb{N}}$ of geopolitical selection preferences in S converges to $(\mathfrak{X}, \succ)$ in $(\mathcal{P}, \mathcal{T}_{\textit{closed}})$ if and only if*

$$\textit{liminf}_{n\to\infty}\{(x,y) \in \mathfrak{X}_n \times \mathfrak{X}_n : x \not\succ y\} = \textit{limsup}_{n\to\infty}\{(x,y) \in \mathfrak{X}_n \times \mathfrak{X}_n : x \not\succ y\} = \{(x,y) \in \mathfrak{X} \times \mathfrak{X} \;\; \mathfrak{x} \not\succ \mathfrak{y}\}.$$

iii *The topology $\mathcal{T}_{\textit{closed}}$ of closed convergence on the set $\mathcal{P}$ of geopolitical selection preferences in S is the coarsest topology on $\mathcal{P}$ which has the property that the set $\{(\mathfrak{X}, \succ, x, y) \in \mathcal{P} \times \mathbb{R}^{\ell+1} \times \mathbb{R}^{\ell+1} : x, y \in \mathfrak{X} \textit{ and } x \not\succ y\}$ is closed.*

Proof

i ***i*** It is well known that the set $\mathcal{F}\left(\mathbb{R}^{\ell+1} \times \mathbb{R}^{\ell+1}\right)$ of all closed subsets of $\mathbb{R}^{\ell+1} \times \mathbb{R}^{\ell+1}$ endowed with the topology $\mathcal{T}_{closed}$ of closed convergence is compact and metrizable. In order to show that $(\mathcal{P}, \mathcal{T}_{closed})$ is compact and metrizable, it suffices to show that

"$\mathcal{P}$ *is a closed subset of* $\left(\mathbb{R}^{\ell+1} \times \mathbb{R}^{\ell+1}, \mathcal{T}_{closed}\right)$".

In this direction, let us assume that

- $(\mathfrak{X}_n, \succ_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{P}$ and
- F is the closed limit of a sequence $(F_n)_{n\in\mathbb{N}}$ where $F_n := \{(x, y) \in \mathfrak{X}_n \times \mathfrak{X}_n : x \not\succ_n y\}$.

We have to show that "*the geopolitical selection preference* $(\mathfrak{X}, \succ)$ *belongs to* F", where $\mathfrak{X} := \left\{x \in \mathbb{R}^{\ell+1} : (x, x) \in F\right\}$ and $\succ := (\mathfrak{X} \times \mathfrak{X})\backslash F$. In other words, we have to show that

- $\mathfrak{X}$ is a geopolitical focus set over a system S (i.e., a nonempty subset of the geopolitical space $\mathbb{R}^{\ell+1}$ which is closed, convex and bounded from below) and
- $\succ \subset \mathfrak{X} \times \mathfrak{X}$ is a transitive and non-reflexive binary relation on $\mathfrak{X}$ such that $\succ$ is open in $\mathfrak{X} \times \mathfrak{X}$.

To do so, observe that, since $liminf_{n\to\infty} F_n = limsup_{n\to\infty} F_n = F$,

a. the set $\mathfrak{X}$ is the closed limit of the sequence $(\mathfrak{X}_n)_{n\in\mathbb{N}}$.

Further, the set $\mathfrak{X}$ is nonempty, since every set $\mathfrak{X}_n$ belongs to $\mathfrak{K}$. It follows that

b. the set $\mathfrak{X}$ intersects a given compact set.

On the other hand, since every geopolitical focus set $\mathfrak{X}_n$ is convex,

c. the closed limit $\mathfrak{X}$ is a convex set.

Indeed, let $x, y \in \mathfrak{X}$ and $0<\lambda<1$. Since $\mathfrak{X} = liminf_{n\to\infty}\mathfrak{X}_n$, there are sequences $(x_n \in \mathfrak{X}_n)_{n\in\mathbb{N}}$ and $(y_n \in \mathfrak{X}_n)_{n\in\mathbb{N}}$ converging to x and y respectively. Since $\mathfrak{X}_n$ is convex, we have $\lambda x_n + (1-\lambda)y_n \in \mathfrak{X}_n$. Consequently,
$\lambda x + (1-\lambda)y \in liminf_{n\to\infty}\mathfrak{X}_n = \mathfrak{X}$.

It is now easily seen that

d. $\mathfrak{X} \in \mathfrak{K}$.

We show now that the geopolitical selection preference $\succ$ on $\mathfrak{X}$ is non reflexive. Let $x \in \mathfrak{X}$. Then there is a sequence $(x_n \in \mathfrak{X}_n)_{n\in\mathbb{N}}$ converging to x. Since $\succ_n$ is non reflexive, we have $(x_n, x_n) \in F_n$. Hence $(x, x) \in F$, since $liminf_{n\to\infty}\mathfrak{X}_n = \mathfrak{X}$. Thus, we have $x \not\succ x$.

Next, we show that the geopolitical selection preference $\succ$ on $\mathfrak{X}$ is transitive. Let $x \succ y$ and $y \succ z$. To get a contradiction, let us assume that $x \not\succ z$, i.e., $(x, z) \in F$. Since $liminf_{n\to\infty} F_n = F$ there is a sequence $(x_n, z_n) \in F_n$ with $(x_n, z_n) \underset{n\to\infty}{\to} (x, z)$. For n large enough, we have $(x_n, y_n) \notin F_n$ and $(y_n, z_n) \notin F_n$, where $(y_n \in \mathfrak{X}_n)_{n\in\mathbb{N}}$ converging to y. Indeed, if this were not true, it would follow that $(x, y) \in limsup_{n\to\infty} F_n = F$ or $(y, z) \in F$, which contradicts $x \succ y$ and $y \succ z$. Hence, by transitivity of $\succ_n$ we obtain $(x_n, z_n) \notin F_n$ which constitutes a contradiction.

ii It is well known that, in a compact metrizable space M endowed with the topology of closed convergence, a sequence $(F_n \subset M)_{n\in\mathbb{N}}$ of closed subsets of M converges to a closed set $F \subset M$ with respect to the topology of closed convergence in M if and only if $liminf_{n\to\infty} F_n = limsup_{n\to\infty} F_n = F$. Application for $M = (\mathcal{P}, \mathcal{T}_{closed})$ proves the desired assertion.

iii Since $(\mathcal{P}, \mathcal{T}_{closed})$ is a compact space, every separated coarser topology on $\mathcal{P}$ coincides with $\mathcal{T}_{closed}$. Thus, it remains to show that the set $\{(\mathfrak{X}, \succ, x, y) : x, y \in \mathfrak{X} \text{ and } x \not\succ y\}$ is closed in $(\mathcal{P}, \mathcal{T}_{closed}) \times \mathbb{R}^{\ell+1} \times \mathbb{R}^{\ell+1}$. Let $(\mathfrak{X}_n, \succ_n, x_n, y_n) \underset{n\to\infty}{\to} (\mathfrak{X}, \succ, x, y)$, where $x_n, y_n \in \mathfrak{X}_n$ and $x_n \not\succ y_n$. Hence $(x_n, y_n) \in F_n$, which implies that

$(x, y) \in liminf_{n\to\infty} F_n = F$,

i.e., $x, y \in \mathfrak{X}$ and $x \not\succ y$.

For later easy reference we state three immediate consequences of Theorem 2.1.

Corollary 2.1 *The correspondence* $(\mathfrak{X}, \succ) \mapsto \mathfrak{X}$ *of* $\mathcal{P}$ *into* $\mathbb{R}^{\ell+1}$ *is closed and lower semi-continuous.*

Corollary 2.2 *The set*

$$\{(\mathfrak{X}, \succ, x, y) \in \mathcal{P} \times \mathbb{R}^{\ell+1} \times \mathbb{R}^{\ell+1} : x, y \in \mathfrak{X} \text{ and } x \succ y\}$$

is a Borel subset of $\mathcal{P} \times \mathbb{R}^{\ell+1} \times \mathbb{R}^{\ell+1}$.

Corollary 2.3 *Let* $(\mathfrak{X}, \succ) \in \mathcal{P}$, $x, y \in \mathfrak{X}$ *and* $x \succ y$. *Then there are neighborhoods* V, V_x *and* V_y *of* $(\mathfrak{X}, \succ)$ *in* $\mathcal{P}$, *x and y in* $\mathbb{R}^{\ell+1}$, *respectively, such that* $x' \succ' y'$, *for every* $(\mathfrak{X}', \succ) \in V$, $x' \in V_x \cap \mathfrak{X}'$ *and* $y' \in V_y \cap \mathfrak{X}'$.

In later chapters, it will—for technical measure theoretical reasons—be important to know that the four sets

- $\mathcal{P}_{mo}$(: *the set of all monotonic geopolitical preferences in* $\mathcal{P}$),
- $\mathcal{P}^{*}$ (: *the set of all negatively transitive geopolitical preferences in* $\mathcal{P}$),
- $\mathcal{P}^{*}_{co}$(: *the set of all convex (strongly convex) geopolitical preferences in* $\mathcal{P}^{*}$) and
- $\mathcal{P}^{*}_{sco}$(: *the set of all convex (strongly convex) geopolitical preferences in* $\mathcal{P}^{*}$)

are Borel subsets of the compact metrizable space $\mathcal{P}$. In that regard, it is easy to show the following result.

Proposition 2.2 *The sets* $\mathcal{P}_{mo}$, $\mathcal{P}^{*}$, $\mathcal{P}^{*}_{co}$ *and* $\mathcal{P}^{*}_{sco}$ *are not closed* G_δ*-sets in* $\mathcal{P}$, *with closures different from* $\mathcal{P}$.

The lower hemi-continuity of the geopolitical rational choice

Proposition 2.3 *Let* $\boldsymbol{p} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_{\ell+1})$ *be the geopolitical significance vector of the weighted geopolitical indexed system* $g_S = \left(g_S^{(1)}, g_S^{(2)}, \ldots, g_S^{(\ell+1)}\right)$ *over S. The defining relation* $\mathfrak{B}$ *of the set*

$$\mathfrak{B}(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}) := \{x = (d_1, \ldots, d_{\ell+1}) \in \mathfrak{X} : (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_{\ell+1}) \cdot (d_1, d_2, \ldots, d_{\ell+1}) \le \boldsymbol{\mathcal{C}_{\mathcal{M}}}\}$$

of geopolitical management options of a geopolitical operator $\mathcal{M}$ *in the complex S is closed in* $\mathcal{P} \times \mathbb{R} \times \mathbb{R}^{\ell+1}$ *and lower hemi-continuous at every point* $(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}) \in \mathcal{P} \times \mathbb{R} \times \mathbb{R}^{\ell+1}$.

Proof The defining relation $\mathfrak{B}$ is the intersection of the correspondence $(\mathfrak{X}, \succ) \mapsto \mathfrak{X}$ of $\mathcal{P}$ into $\mathbb{R}^{\ell+1}$ with the correspondence $(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}) \mapsto \{x \in \mathbb{R}^{\ell+1} : \boldsymbol{p} \cdot x \le \mathcal{C}_{\mathcal{M}}\}$ of $\mathcal{P} \times \mathbb{R} \times \mathbb{R}^{\ell+1}$ into $\mathbb{R}^{\ell+1}$. Since both correspondences are closed (Corollary 2.1), we infer that $\mathfrak{B}$ is closed. To show the lower hemi-continuity of $\mathfrak{B}$, let us consider the relation $\breve{\mathfrak{B}}$ defined by $\breve{\mathfrak{B}}(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}) := \{x \in \mathfrak{X} : \boldsymbol{p} \cdot x < \mathcal{C}_{\mathcal{M}}\}$. By assumption, there is a vector $x = (d_1, \ldots, d_\ell) \in \breve{\mathfrak{B}}(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p})$. Let $\left(\mathfrak{X}_n, \succ_n, \mathcal{C}_{\mathcal{M}}^{(n)}, \boldsymbol{p_n}\right)_{n \in \mathbb{N}}$ be a sequence converging to $(\mathfrak{X}, \succ, \mathcal{C}_{\mathcal{M}}, \boldsymbol{w})$ in $\mathcal{P}$. By Corollary 2.1, the correspondence $(\mathfrak{X}, \succ) \mapsto \mathfrak{X}$ of $\mathcal{P}$ into $\mathbb{R}^{\ell+1}$ is lower-semi-continuous. Thus that there is a sequence $(x_n \in \mathfrak{X})_{n \in \mathbb{N}}$ converging to $x \in \mathfrak{X}$. Evidently, the strict inequality $\boldsymbol{w} \cdot x < \mathcal{C}_{\mathcal{M}}$ implies $\boldsymbol{w_n} \cdot x_n < \mathcal{C}_{\mathcal{M}}^{(n)}$ for n large enough. Hence, $x_n \in \breve{\mathfrak{B}}\left(\mathfrak{X}_n, \mathcal{C}_{\mathcal{M}}^{(n)}, \boldsymbol{p_n}\right)$ for enough large n, which proves that the relation $\breve{\mathfrak{B}}$ is lower hemi-continuous at $(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p})$. The convexity of the geopolitical focus set $\mathfrak{X}$ implies that $\mathfrak{B}(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}) = \overline{\breve{\mathfrak{B}}(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p})}$. The desired assertion now follows, since the closure of a lower hemi-continuous correspondence is also lower hemi-continuous. □

Proposition 2.4 *The defining relation* $\mathfrak{A}$ *of the set*

$$\mathfrak{A}(\mathfrak{X}, \succ, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}) = \{x^* = (d_1, \ldots, d_{\ell+1}) \in \mathfrak{B}(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}) : \text{there is no } x = (d_1, \ldots, d_{\ell+1}) \in \mathfrak{B}(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}) \text{with } x \succ x^*\}$$

of geopolitical rational choices in the complex S is nonempty and compact in $\mathcal{P} \times \mathbb{R} \times \mathbb{R}^{\ell+1}$. *Further, it is lower hemi-continuous at every point* $(\mathfrak{X}, \succ, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}) \in \mathcal{P} \times \mathbb{R} \times \mathbb{R}^{\ell+1}$ *where the set* $\mathfrak{B}(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{w})$ *of management options of* $\mathcal{M}$ *is compact and* $\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}$ *satisfy the inequality*

$$\inf\{\boldsymbol{p} \cdot \mathfrak{X}\} < \boldsymbol{\mathcal{C}_{\mathcal{M}}}.$$

(Note that the assumption $\inf\{\boldsymbol{p} \cdot \mathfrak{X}\} < \mathcal{C}_{\mathcal{M}}$ cannot be weakened to $\inf\{\boldsymbol{p} \cdot \mathfrak{X}\} < \boldsymbol{\mathcal{C}_{\mathcal{M}}}$.)

Proof By Proposition 2.3, the defining relation $\mathfrak{B}$ of the set $\mathfrak{B}(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p})$ is closed and lower hemi-continuous at every point $(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{w}) \in \mathcal{P} \times \mathbb{R} \times \mathbb{R}^{\ell+1}$. Since the set $\mathfrak{B}(\mathfrak{X}, \boldsymbol{\mathcal{C}_{\mathcal{M}}}, \boldsymbol{p})$ of management options of $\mathcal{M}$ is compact and convex, the defining relation $\mathfrak{B}$ of $\mathfrak{B}(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p})$ is continuous at the point $(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p})$. Put

$$\mathfrak{S} = \mathcal{P} \times \mathbb{R} \times \mathbb{R}^{\ell+1}.$$

By Theorem 2.1.iii and Proposition 2.3, the set

$$\{(\mathfrak{s}, x, y) \in \mathfrak{S} \times \mathbb{R}^{\ell+1} \times \mathbb{R}^{\ell+1} : x, y \in \mathfrak{B}(\mathfrak{s}) \text{ and } x \not\succ_{\mathfrak{s}} y\}$$

is closed in $\mathfrak{S} \times \mathbb{R}^{\ell} \times \mathbb{R}^{\ell+1}$. The desired assertion follows as a direct application of the next well known result. □

Lemma 2.1 *Let β be a mapping of a metric space $\mathcal{S}$ into the metric space $\mathcal{T}$ and let $\succ_x$, $x \in \mathcal{S}$, be an irreflexive and transitive binary relation on $\beta(x)$ with the following property:*

> *The set*
> $\{(x,y,z) \in \mathcal{S} \times \mathcal{T} \times \mathcal{T} : y,z \in \beta(x) \text{ and } y \not\succ_x z\}$
> *is closed in $\mathcal{S} \times \mathcal{T} \times \mathcal{T}$*

If the set $\beta(x)$ is compact then the set $\mathfrak{M}(x)$ of maximal elements for $\succ_x$ in $\beta(x)$ is nonempty and compact and the mapping $\mathfrak{M}$ is lower hemi-continuous at every point x where β is continuous.

Mean geopolitical rational choice

Geopolitical sectors

We consider a finite set

$\mathbb{M}$

of geopolitical operators $\mathcal{M}$, each of whom is described by its geopolitical focus set $\mathfrak{X}_{\mathcal{M}}$ in the complex *S*, his geopolitical preference $\succ_{\mathcal{M}}$ *in S* and his geopolitical management capacity $\mathcal{C}_{\mathcal{M}}$ *in S*. We introduce the map

$$\mathfrak{s} : \mathbb{M} \to \mathcal{P} \times \mathbb{R} : \mathcal{M} \mapsto \mathfrak{s}(\mathcal{M}) = (\mathfrak{X}_{\mathcal{M}}, \succ_{\mathcal{M}}, \mathcal{C}_{\mathcal{M}}).$$

Notation 2.2 If $\mathcal{M}$ selects a weighted geopolitical system with geopolitical significance $\boldsymbol{p} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_{\ell+1})$ in *S*, then ***the geopolitical rational choice set of a geopolitical operator $\mathcal{M}$ with characteristics*** $\mathfrak{s}(\mathcal{M}) \in \mathcal{P} \times \mathbb{R}$ will be denoted by

$\mathfrak{A}(\mathfrak{s}(\mathcal{M}), \boldsymbol{p})$.

Thus, we are leaded to the following.

Definition 2.13 If each geopolitical operator $\mathcal{M}$ selects the weighted geopolitical system with geopolitical significance $\boldsymbol{p} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_{\ell+1})$ in the complex *S*, the **mean geopolitical rational choice** of the set $\mathbb{M}$ in *S* is given by

$$\bar{\mathfrak{A}}(\mathfrak{s}, \boldsymbol{p}) := \frac{1}{|\mathbb{M}|} \sum\nolimits_{\mathcal{M} \in \mathbb{M}} \mathfrak{A}(\mathfrak{s}(\mathcal{M}), \boldsymbol{p})$$

Here the notation $|\cdot|$ means cardinality of set.

If χ denotes the ***normalized counting measure*** on $\mathbb{M}$, i.e.,

$$\chi(\mathcal{E}) := |\mathcal{E}|/|\mathbb{M}|$$

for every subset $\mathcal{E}$ of $\mathbb{M}$, it is immediately verified that

$$\bar{\mathfrak{A}}(\mathfrak{s}, \boldsymbol{p}) := \int_{\mathbb{M}} \mathfrak{A}(\mathfrak{s}(\cdot), \boldsymbol{p}) d\chi.$$

Clearly, the integral is defined for more general mappings $\mathfrak{s}$ and measures χ. Indeed, we shall define later “mean geopolitical rational choice” by this formula in a more general situation. However, let us first prepare and motivate this step of abstraction.

Definition 2.14 The image measure ϱ of χ with respect to the mapping $\mathfrak{s}$ is called the **geopolitical preference-capacity distribution** of the set $\mathbb{M}$ of geopolitical operators in the complex *S*.

Thus,

$$\varrho(B) = \chi(\mathfrak{s}^{-1}(B))$$

denotes the fraction of geopolitical operators in $\mathbb{M}$ whose geopolitical characteristics belong to $B \subset \mathcal{P} \times \mathbb{R}$.

Definition 2.15 The marginal distributions

$\varrho^{\mathcal{P}}$ *on* $\mathcal{P}$ *and* $\varrho^{\mathbb{R}}$ *on* $\mathbb{R}$

are called the **geopolitical preference distribution** and **geopolitical capacity distribution** in the complex *S*, respectively.

Remark 2.5 The geopolitical preference-capacity distribution in the complex *S* may or may not to be the product of its marginal distributions, i.e., it *is not assumed that the geopolitical capacity distribution is independent of the geopolitical preference distribution.*

Notation 2.3 In some general cases, one is not primarily concerned about the total geopolitical rational choice of a small number of geopolitical operators. Typically, one is interested in the **total geopolitical rational choice** in *S* of all geopolitical operators in a large society. In this case, it seems natural and convenient (for analytical reasons) to view the geopolitical preference-capacity distribution ϱ as an ***atomless distribution***[1] over the space of all geopolitical characteristics $\mathcal{P} \times \mathbb{R}$, that is as a distribution satisfying

$$\varrho(\mathfrak{X}, \succ, \mathcal{C}_{\mathcal{M}}) = 0 \text{ for every } (\mathfrak{X}, \succ, \mathcal{C}_{\mathcal{M}}) \in \mathcal{P} \times \mathbb{R}.$$

To view the distribution of operator's characteristics of a finite set $\mathbb{M}$ of geopolitical operators as an atomless distribution means, strictly speaking, that t*he "actual" distribution is considered as a distribution of a sample of size $|\mathbb{M}|$ drawn from a "hypothetical" population*. This statistical point of view is based on the well-known fact that the sample distributions converge with increasing sample size to the hypothetical distribution. Naturally, it remains to show that the geopolitical management aspects derived from the "hypothetical" distribution (e.g., mean geopolitical rational choice) do not differ essentially from the one derived from the "actual" distribution.

There is another, probably deeper, reason why one should consider atomless distributions of geopolitical operator's characteristics: *the very fact that geopolitical operators are not alike*—which means in our framework that *the support of the geopolitical preference-capacity distribution is "spread over" the set $\mathcal{P} \times \mathbb{R}$ can give rise to properties*, for example of the mean geopolitical rational choice in S, which would not hold without the diversification of geopolitical operator's characteristics. To be more specific, if geopolitical selection preferences, say in $\mathcal{P}^{*}$, are not strongly convex in S, the mean geopolitical rational choice in S for a set of geopolitical operators is, in general, not unique. However, given a weighted geopolitical system with geopolitical significance vector $\boldsymbol{p} \gg 0$, Proposition 2.4 guarantees that generically (i.e., for a topologically large subset of characteristics $(\precsim, \mathcal{C}_{\mathcal{M}})$) the geopolitical rational choice set in S has a small diameter. Thus, ***for a "widely spread" distribution of geopolitical characteristics one can hope that for "most" geopolitical operators the geopolitical rational choice set in the complex S is small***. Of course, we do not claim here that a subset which is large from a topological point of view (i.e., open and dense) is also large from a measure theoretic point of view (i.e., the measure is concentrated on it). Still, it may be possible—by properly restricting the space of geopolitical preferences and strengthening its topology—to characterize a class of atomless distributions ϱ such that the geopolitical rational choice set is small or even unique for ϱ—almost all characteristics. These arguments, indeed, are extremely vague and need clarification.

Just as an illustration, the reader will have no difficulty in giving examples of "hypothetical" distributions ϱ on $\mathcal{P}^{*} \times \mathbb{R}$ such that for every weighted geopolitical system with geopolitical significance vector $\boldsymbol{p} \gg 0$, one has

$$\varrho\{t \in \mathcal{P} \times \mathbb{R} : \mathfrak{A}(t, \boldsymbol{p}) \textit{is unique}\} = 1$$

but for every weighted geopolitical system with geopolitical significance vector $\boldsymbol{p} \gg 0$, there is a $t \in$ *supp*(ϱ) such that $\mathfrak{A}(t, \boldsymbol{p})$ contains more than one element. Consequently, given the weighted geopolitical system with geopolitical significance vector $\boldsymbol{p}$, the mean geopolitical rational choice of a sample distribution in S is unique, with probability one.

The assumption of atomless distributions of geopolitical operators' characteristics (or of geopolitical analyst's characteristics), in particular, requires that "many" geopolitical operators be involved. One may focus attention to this aspect alone without assuming that the geopolitical operators' characteristics are diversified. In the next Chapter, we shall particularly emphasize this point. Therefore, we shall be quite brief here. If there are "many" geopolitical operators, then the geopolitical focus decision in the complex S of a typical geopolitical operator will have only a "small" influence on the total geopolitical rational choice in S. It is clear that if we want to describe only this aspect, namely, that the influence of an individual geopolitical operator on collective actions is negligible, we do not need that the distribution of geopolitical operators' characteristics is atomless, but this distribution is induced from a very "large" set of geopolitical operators

The above discussion motivates the following

Definition 2.16 Let $\mathbb{M}$ be the set of all geopolitical operators in the complex S.

i A **geopolitical sector** in the complex S is a measurable mapping

[1] A distribution μ on $\mathbb{P} \times \mathbb{K}$ is atomless if $\mu(\mathfrak{X}, \succ, \mathcal{C}_{\mathcal{M}}) = 0$ for every $(\mathfrak{X}, \succ, \mathcal{C}_{\mathcal{M}}) \in \mathcal{P} \times \mathbb{R}$.

$\mathfrak{s} : (\mathbb{M}, \boldsymbol{\mathcal{A}}, v) \to \mathcal{P} \times \mathbb{R}$

of a measure space $(\mathbb{M}, \boldsymbol{\mathcal{A}}, v)$, consisting of the set $\mathbb{M}$, a σ-algebra $\mathcal{A}$ of subsets of $\mathbb{M}$ and a (probability) measure v on $\boldsymbol{\mathcal{A}}$, into the space $\mathcal{P} \times \mathbb{R}$ of geopolitical characteristics such that the mean geopolitical management capacity

$$\int_{\mathbb{M}} \mathcal{C}_{\mathcal{M}} \circ \mathfrak{s} dv$$

is finite.

ii A geopolitical sector in S is called

- **simple**, if the measure space $(\mathbb{M}, \boldsymbol{\mathcal{A}}, v)$ is simple, i.e., $\mathbb{M}$ is a finite set, $\boldsymbol{\mathcal{A}}$ is the set of all subsets of $\mathbb{M}$, and $v(\boldsymbol{\mathcal{E}}) = (|\boldsymbol{\mathcal{E}}|/|\mathbb{M}|)$ whenever $\boldsymbol{\mathcal{E}} \subset \mathbb{M}$;
- **atomless**, if the measure space $(\mathbb{M}, \mathcal{A}, v)$ is atomless, i.e., for every $\boldsymbol{\mathcal{E}} \in \boldsymbol{\mathcal{A}}$ with $v(\boldsymbol{\mathcal{E}}) > 0$ there is a set $\boldsymbol{\mathcal{K}} \subset \boldsymbol{\mathcal{E}}$ with $0 < v(\boldsymbol{\mathcal{K}}) < v(\boldsymbol{\mathcal{E}})$;
- **convex**, if almost all geopolitical operators of every atom of the measure space $(\mathbb{M}, \boldsymbol{\mathcal{A}}, v)$ have convex geopolitical preferences.

Remark 2.6 According to this definition, *an atomless geopolitical sector in S is always convex in S.*

Notation 2.3 i The generic element in the set $\mathbb{M}$ of a geopolitical sector in S is a **geopolitical operator** $\mathcal{M}$ in S.

ii The geopolitical selection preference and geopolitical management capacity of a geopolitical operator in S are denoted by

$$\mathfrak{s}(\mathcal{M}) = \left(\mathfrak{X}_{\mathfrak{s}(\mathcal{M})}, \succ_{\mathfrak{s}(\mathcal{M})}, \mathcal{C}_{\mathfrak{s}(\mathcal{M})}\right).$$

iii If it is clear which mapping $\mathfrak{s}$ is considered, we shall write, as usually, shorter

$$(\mathfrak{X}_{\mathcal{M}}, \succ_{\mathcal{M}}, \mathcal{C}_{\mathcal{M}}).$$

iv The image measure

$$v \circ \mathfrak{s}^{-1}$$

is called the **preference-capacity distribution of the geopolitical sector** $\mathfrak{s} : (\mathbb{M}, \boldsymbol{\mathcal{A}}, v) \to \mathcal{P} \times \mathbb{R}$ in S and is denoted by

$\varrho_{\mathfrak{s}}$, or simply ϱ.

v Given a weighted geopolitical vector $\boldsymbol{w} \in \mathbb{R}^{\ell+1}$, the integral

$$\int_{\mathbb{M}} \mathfrak{A}(\mathfrak{s}(\cdot), \boldsymbol{p}) dv$$

is called the **mean geopolitical rational choice of the geopolitical sector** $\mathfrak{s} : (\mathbb{M}, \boldsymbol{\mathcal{A}}, v) \to \mathcal{P} \times \mathbb{R}$ in S. It is denoted by

$$\bar{\mathfrak{A}}(\mathfrak{s}, \boldsymbol{p}).$$

The meaning and interpretation of a simple geopolitical sector and its derived concepts are clear and need no comment.

An atomless geopolitical sector in S is, in fact, a more abstract concept. Its interpretation relies on the analogy to the case of a simple geopolitical sector. It describes a geopolitical sector in S with a very large set of geopolitical operators—an uncountable infinite set—where every individual geopolitical operator has strictly no influence on the mean geopolitical rational choice.

The σ-algebra $\boldsymbol{\mathcal{A}}$ has only been introduced for technical reasons. Conceptually $\boldsymbol{\mathcal{A}}$ should be considered—as in the case of a simple geopolitical sector—as the set of all subsets of $\mathbb{M}$.

Geopolitical preference-capacity distribution

One easily verifies (we shall prove a more general result in Theorem 2.2 below) that the mean geopolitical rational choice $\overline{\mathfrak{A}}(s, \boldsymbol{p})$ in the complex S only depends on the geopolitical preference-capacity distribution $\varrho = \chi \circ s^{-1}$ in S, provided the geopolitical rational choice sets $\mathfrak{A}(s(\mathcal{M}), \boldsymbol{p})$ are convex in S. More precisely, we obtain

$$\overline{\mathfrak{A}}(s, \boldsymbol{p}) = \int_{\mathcal{P} \times \mathbb{R}} \mathfrak{A}(\cdot, \boldsymbol{p}) d\varrho.$$

However, in general, the mean rational geopolitical choice in S depends on the geopolitical preference-capacity distribution in S and on the number $|\mathbb{M}|$ of geopolitical operators in $\mathbb{M}$. We shall now show in which situation the mean geopolitical rational choice in S is determined by the geopolitical preference-capacity distribution in S.

To prove the first result of this paragraph, we may quote some auxiliary material with necessary background.

Lemma 2.2 *Let* (Ω, A, m) *be a measure space consisting of a set* Ω, *a* σ-*algebra* A *of subsets of* Ω *and a (probability) measure* m *on* A.

i Let φ *be a mapping with a measurable graph of a measurable space* T *into* $\mathbb{R}^n$.

  a. If h *is a measurable function of* T *into* $\mathbb{R}^n$, *then the mapping* $\omega \mapsto \varphi(\omega) + h(\omega)$ *has a measurable graph.*

  b. *If h is a measurable function of* (Ω, A, v) *into* T, *then the composition* $\varphi \circ h : \omega \mapsto \varphi(h(\omega))$ *has a measurable graph.*

ii *Let* φ *be a mapping with a measurable graph of* (Ω, A, m) *into a complete separable metric space* Υ *and* h *a measurable mapping of* Υ *into a separable metric space* M.

  a. *The mapping* $h \circ \varphi : \omega \mapsto h(\varphi(\omega))$ *has an* $(A \times \mathcal{B}(M))$-*analytic graph.*[2]

  b. *If* φ *is a mapping with a measurable graph of* (Ω, A, v) *into* $\mathbb{R}^n$, *then the mapping*

  $$conv(\varphi) : \omega \mapsto conv(\varphi)(\omega)$$

  has an $(A \times \mathcal{B}^n)$-*analytic graph.*

iii *Suppose* Ω *is a topological complete space and* Υ *is another complete separable metric space. If* φ *is a mapping with a measurable graph of* Ω *into* Υ *and u is a measurable function of* Υ *into* $\mathbb{R}$, *then*

  a. the function

  $$\sup u(\varphi(\cdot)) : \Omega \to \mathbb{R} : \ \omega \mapsto \sup u(\varphi(\omega)) := \ \sup\{u(x) : x \in \varphi(\omega)\}$$

  is measurable and

  b. the mapping

  $$\varphi^u : \Omega \to \Upsilon : \omega \mapsto \{x \in \varphi(\omega) : u(x) = \sup u(\varphi(\omega))\}$$

  has a measurable (analytic) graph.

  c. *If, in particular,* $\Upsilon = \mathbb{R}^n$, *the graph of the mapping* $conv(\varphi)$ *is measurable.*

iv *If* (Ω, A, m) *is an atomless measure space and* φ *is a mapping with a measurable graph of* (Ω, A, m) *into* $\mathbb{R}^n$, *then the following properties hold.*

  a. *The integral*

  $$\int_\Omega \varphi \, dm$$

  is a convex set in $\mathbb{R}^n$.

  b. *Let* Υ *be a set in* $\mathbb{R}^n$. *If* $\varphi(\omega) := S$ *for every* $\omega \in \Omega$, *then*

  $$\int_\Omega \varphi \, dm = conv(\Upsilon).$$

v *If the mapping* φ *of* (Ω, A, m) *into* $\mathbb{R}^n$ *is closed-valued and integrably bounded, then the integral*

$$\int_\Omega \varphi \, dm$$

is a compact subset of $\mathbb{R}^n$.

vi *Let* φ *be a mapping with a measurable graph of the measurable space* (Ω, A, m) *into* $\mathbb{R}^n$. *If* $\int\varphi \neq \varphi(\omega)$, $\omega \in A$, *then*

$$sup\{p \cdot x : x \in \textstyle\int\varphi\} = \textstyle\int sup\{p \cdot x : x \in \varphi(\cdot)\}$$

for every vector $p \in \mathbb{R}^n$.

vii *Let* φ *be a mapping with a measurable graph of the measurable space* (Ω, A, m) *into* $\mathbb{R}^n_+$. *The following hold.*

$$conv\left(\int_\Omega \varphi \, dm\right) = \int_\Omega conv\,(\varphi) \, dm.$$

In particular, if the measure space is atomless, then

$$\int_\Omega \varphi \, dm = \int_\Omega conv\,(\varphi) \, dm.$$

viii *Let* φ *be a mapping with a measurable graph of a measurable space* $(T, \mathfrak{J})$ *into* $\mathbb{R}^n$ *such that* $\varphi(t)$ *is closed convex and contains no straight line whenever* $t \in T$. *If* h *is a measurable function of* (Ω, A, v) *into* T, *then*

[2] $\mathcal{B}(M)$ *denotes the Borel* σ-*algebra generated by the open subsets of* M.

$$\int_{\Omega} \varphi \circ h \, dm = \int_{T} \varphi \, d\left(m \circ h^{-1}\right).$$

Proof

i

a. The mapping $f : (\omega, x) \mapsto (\omega, x - h(\omega))$ of $\Omega \times \mathbb{R}^n$ into $\Omega \times \mathbb{R}^n$ is $(A \otimes \mathcal{B}^n)$-measurable. Here $\mathcal{B}$ denotes the Borel σ -algebra on $\mathbb{R}$ generated by the open subsets of $\mathbb{R}$. Consequently, if $G_{\varphi+h}$ and G_{φ} are the graphs of $\varphi + h$ and φ, respectively then $G_{\varphi+h} = f^{-1}\left(G_{\varphi}\right) \in A \otimes \mathcal{B}^n$.
b. Similarly, since the mapping $g : (\omega, x) \mapsto (h(\omega), x)$ of $\Omega \times \mathbb{R}^n$ into $T \times \mathbb{R}^n$ is measurable, the graph $G_{\varphi \circ h} = g^{-1}\left(G_{\varphi}\right)$ is measurable.

ii

a. The set $G = \{(\omega, x, z) \in \Omega \times \Upsilon \times M : x \in \varphi(\omega) \, and \, z = h(x)\}$ belongs to $A \otimes \mathcal{B}(\Upsilon) \otimes \mathcal{B}(M)$. Since the graph $G_{h \circ \varphi}$ is obtained by projecting the set G on $\Omega \times M$, and since Υ is complete separable metric space, we infer that $G_{h \circ \varphi}$ is an $(A \times \mathcal{B}(M))$-analytic graph (Meyer 1966, p. 34).
b. Let

$$\Delta = \{(\xi_1, \ldots, \xi_{n+1}) : \xi_i \geq 0 \\ and \sum\nolimits_{i=1}^{n+1} \xi_i = 1\Big\}.$$

The mapping $\psi : \omega \mapsto \varphi(\omega) \times \ldots \times \varphi(\omega) \times \{(\xi_1, \ldots, \xi_{n+1})\}$ of (Ω, A, m) into $\mathbb{R}^{n(n+1)} \times \Delta$ has a measurable graph. The function $h : (x_1, \ldots, x_{n+1}, \xi_1, \ldots, \xi_{n+1}) \mapsto \sum_{i=1}^{n+1} \xi_i \cdot x_i$ of $\mathbb{R}^{n(n+1)} \times \Delta$ into $\mathbb{R}^n$ is continuous. Since $conv(\varphi) = h \circ \varphi$, part ii.a implies that the graph $G_{conv(\varphi)}$ is analytic.

iii

a. We have to show that for every $c \in \mathbb{R}$ the set

$$\Omega_c^{(\varphi)} := \{\omega \in \Omega : \sup u(\varphi(\omega)) > c\}$$

belongs to A. Since

$$\Omega_c^{(\varphi)} = proj_{\Omega}\left\{(\omega, x) \in G_{\varphi} : u(x) > c\right\}$$

(G_{φ} is the graph of φ) and since the assumptions on φ and u imply that

$$\left\{(\omega, x) \in G_{\varphi} : u(x) > c\right\} \in A \otimes \mathcal{B}(\Upsilon)$$

it follows from the Projection Theorem that $\Omega_c^{(\varphi)} \in A$.
b. The second assertion now follows readily. The function

$$(\omega, x) \mapsto u(x) - \sup u(\varphi(\omega))$$

is $A \otimes \mathcal{B}(\Upsilon)$-measurable. Hence

$$V = \{(\omega, x) \in \Omega \times \Upsilon : \\ u(x) = \sup u(\varphi(\omega))\} \in A \otimes \mathcal{B}(\Upsilon)$$

and consequently,

$$G_{\varphi^u} = G_{\varphi} \bigcap V \in A \otimes \mathcal{B}(\Upsilon).$$

c. Let

$$\varphi^k(\omega) = \{x \in \varphi(\omega) : |x| \leq k\}(k \\ = 1, 2, \ldots).$$

One easily verifies that $conv(\ phi(\omega)) = \bigcup_{k=1}^{\infty} conv\left(\varphi^k(\omega)\right)$. For every $v \in \mathbb{R}^n$, consider the mapping H_v of Ω into $\mathbb{R}^n$:

$$H_v(\omega) := \left\{x \in \mathbb{R}^n : v \cdot x \leq \sup v \cdot \varphi^k(\omega)\right\}.$$

By part iii.b, the function $\omega \mapsto \sup v \cdot \varphi^k(\omega)$ is measurable, and hence the graph of H_v is measurable. Since $conv\left(\varphi^k(\omega)\right) = \bigcap_{v \in D} H_v(\omega)$, where D denotes a countable dense subset in $\mathbb{R}^n$, the graph of $conv\left(\varphi^k\right)$ $(k = 1, 2, \ldots)$ is measurable, and hence the graph of $conv(\varphi)$ is measurable.

iv

a. Let $x_1, \ x_2 \in \int \varphi \, dm$ and $0 < \lambda < 1$. We denote by $\mathfrak{L}_{\varphi}$ the set of m-integrable functions $f : \Omega \to \mathbb{R}^n$ such that $f(\omega) \in \varphi(\omega)$ almost everywhere in Ω. There are integrable functions $f_1, f_2 \in \mathfrak{L}_\varphi$ such that $x_1 = \int f_1 dm$ and $x_2 = \int f_2 dm$. From Liapunov's Theorem, it follows that the set

$$\left\{ \left(\int_E f_1 dm, \int_E f_2 dm \right) \in \mathbb{R}^{2n} : E \in A \right\}$$

is convex. Since $(0,0)$ and (x_1, x_2) belong to this set, there exists a set $E \in A$ such that

$$(\lambda x_1, \lambda x_2) = \left(\int_E f_1 dm, \int_E f_2 dm \right).$$

Define the function $f \in \mathfrak{L}_\varphi$ by

$$f(\omega) = \begin{cases} f_1(\omega), & if\, \omega \in E \\ f_2(\omega), & if\, \omega \notin E. \end{cases}$$

Then, one easily verifies that

$$\int f = \lambda x_1 + (1 - \lambda) x_2.$$

This shows that *the integral* $\int_\Omega \varphi \; dm$ *is a convex set in* $\mathbb{R}^n$.

b. From part (iv).a, it follows that

$$conv(\Upsilon) \subset \int_\Omega \varphi \; dm.$$

On the other hand, it is easily proved, by induction on the dimension of $\mathbb{R}^n$, that $\int_\Omega f \; dm \in \mathrm{conv}(\Upsilon)$ for every f with $f(\omega) \in \Upsilon$, almost everywhere on Ω. In particular,

$$\int_\Omega \varphi \; dm \subset conv(\Upsilon).$$

v Observe that, by Fatou's lemma in n-dimension, *if* $(\varphi_v)_{v \in \mathbb{N}}$ *is a sequence of mappings of* (Ω, A, m) *into* $\mathbb{R}^n_+$ *such that there exists a sequence* $(g_v)_{v \in \mathbb{N}}$ *of functions of* Ω *into* $\mathbb{R}^n_+$ *with the properties:*

(i) $\varphi_v(\omega) \leq g_v(\omega)$, *almost everywhere in* Ω,

(ii) *the sequence* $(g_v)_{v \in \mathbb{N}}$ *is uniformly integrable and the set* $\{g_v(\omega) : v \in \mathbb{N}\}$ *is bounded almost everywhere in* Ω,

then

$$limsup_{v \in \mathbb{N}} \left(\int \varphi_v \right) \subset \int limsup_{v \in \mathbb{N}} (\varphi_v).$$

Letting $\varphi_v = \varphi \quad (v \in \mathbb{N})$, we have $limsup_{v \in \mathbb{N}} \left(\int \varphi_v \right) = \varphi(\omega)$ since $\varphi(\omega)$ is closed. Thus, by the above remark, every adherent point of $\int \varphi$ belongs to $\int \varphi$. This proves assertion v.

vi The left-hand side is clearly at most equal to the right-hand side. From part (iv), it follows that the function $\omega \mapsto sup\{p\, \varphi(\omega)\}$ is A-measurable. Since there is by assumption an integrable selection for φ, the right-hand side is well defined (it may be $+\infty$).
Consider a real number $\alpha < \int s$, where

$$s(\omega) = sup\{p \cdot \varphi(\omega)\}.$$

We have to show that there is a function $f \in \mathfrak{L}_\varphi$ such that $\alpha \leq p \cdot \int f$. For this we choose an integrable selection $h \in \mathfrak{L}_\varphi$ and consider for every integer v the truncated mapping

$$\varphi_v(\omega) = \{x \in \varphi(\omega) : |x - h(\omega)| \leq v\}.$$

Clearly, the graph of φ_v is measurable. Hence, by part (c), the function

$$s_v(\omega) = sup\{p \cdot \varphi_v(\omega)\}$$

is measurable. It is also integrable, since h is integrable. Since

$$(s_v(\omega))_{v \in \mathbb{N}} \nearrow s(\omega),$$

we obtain, by the Monotone Convergence Theorem, that

$$\int s_v \to \int s.$$

Consequently, for v large enough, we have

$$\alpha < \int s_v.$$

Thus, there is an integrable function g of Ω into $\mathbb{R}$ such that

$$\alpha < \int g \text{ and } g(\omega) < s_v(\omega), \quad \omega \in \Omega.$$

Let

$\psi(\omega) := \{x \in \varphi_v(\omega) : p \cdot x > g(\omega)\}.$

Clearly $\psi(\omega) \neq \varphi(\omega)$, $\omega \in A$, and the graph of the mapping ψ is measurable. Consequently, by the Measurable Selection Theorem, there exists a measurable selection f of ψ, and hence of φ, which is even integrable. Since $g(\omega) < p \cdot f(\omega)$ we obtain $\int g < p \cdot \int f$ and consequently

$$\alpha < p \cdot \int f.$$

vii We prove the assertion by induction on the dimension n of $\mathbb{K}^n$. Clearly the theorem holds for $n = 0$. First we show that

$$\int conv(\varphi) = \int \varphi \quad \text{if and only if } conv\left(\int \varphi\right) = \int \varphi. \tag{1}$$

If $\int conv(\varphi) = \phi$, then $\int \varphi = \phi$ and $conv\left(\int \varphi\right) = \phi$. To show the converse, we assume that

$$\int conv(\varphi) \neq \phi.$$

Let f be an integrable selection of $conv(\varphi)$. Let $v \in \mathbb{R}^n$ and $v \gg 0$. Consider the set

$\psi(\omega) = \{x \in \varphi(\omega) : v \cdot x \leq v \cdot f(\omega)\}.$

Since $f(\omega) \in conv(\varphi(\omega))$, we have $\psi(\omega) \neq \phi$ almost everywhere in Ω. The graph of the mapping ψ is measurable. Therefore, by the measurable selection theorem, there exists a measurable selection h of ψ. Since f is integrable, $v \gg 0$ and φ is positive, the selection h is integrable, and hence

$$\int \varphi \neq \phi.$$

In the remainder of the proof of (vii), we shall assume that $\int \varphi \neq \phi$. Next, we show that:

$$conv\left(\int \varphi\right) \text{ and } \int conv(\varphi) \quad \text{have the same closure.} \tag{2}$$

For every $v \in \mathbb{R}^n$, one obtains

$$sup\left(v \cdot \int \varphi\right) \leq sup\left(v \cdot \int conv(\varphi)\right) \leq \int sup(v \cdot \varphi) = sup\left(v \cdot \int \varphi\right).$$

Indeed, the two inequalities are trivial and the equality follows from (v). Hence, for every $v \in \mathbb{R}^n$, we have $sup\left(v \cdot \int conv(\varphi)\right) = sup\left(v \cdot conv\left(\int \varphi\right)\right)$, which proves property (2), since the two sets are convex.
For every subset X of $\mathbb{R}^n$ and every $v \in \mathbb{R}^n$ we define

$X^v := \{x \in X : v \cdot x = \sup v \cdot X\}.$

It remains to show that for every $v \in \mathbb{R}^n$ we have

$$\left(\int conv(\varphi)\right)^v = \left(conv \int \varphi\right)^v.$$

Using the Measurable Selection Theorem, one easily shows that if ψ is a mapping of (Ω, A, m) into $\mathbb{R}^n$ whose graph is analytic and $\int \psi \neq \phi$, then for every $v \in \mathbb{R}^n$ we have $\left(\int \psi\right)^v \neq \phi$ if and only if

$\psi^v(\omega) \neq \phi$ almost everywhere in Ω and

$$\int \psi^v \neq \left(\int \psi\right)^v.$$

We want to apply this to the mappings φ and $conv(\varphi)$. Since the graph of $conv(\varphi)$ is analytic, but may not be measurable, we have to use here the Measurable Selection Theorem for analytic sets. (: if φ is closed-valued, then $conv(\varphi)$ has a measurable graph, by part (iii)(c)). Also one easily verifies that for every nonempty subset X of $\mathbb{R}^n$ one has $(convX)^v = conv(X^v)$. Consequently

$$\left(\int conv(\varphi)\right)^v = \int (conv(\varphi))^v = \int conv(\varphi^v)$$

and analogously

$$conv\left(\int \varphi\right)^v = conv\left(\int \varphi\right)^v = conv\left(\int \varphi^v\right).$$

Thus, it remains to show that:

$$\int conv(\varphi^v) = conv\Big(\int \varphi^v\Big). \qquad (3)$$

Since, by part (iii), the graph of the relation φ^v is measurable, it follows, from (1) that $\int conv(\varphi^v) = \phi$ if and only if $\int \varphi^v = \phi$. Thus we may assume in the remainder of the proof that

$$\int \varphi^v \neq \phi.$$

Consider the hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n : v \cdot x = 0\}$. There exists a coordinate axis $\mathcal{L}$, say the first, which is not contained in $\mathcal{H}$. We now consider the projection $\mathcal{Q}$ parallel to $\mathcal{L}$ into $\mathcal{H}$. Let h be the function of Ω into $\mathbb{R}^n$ defined by $\omega \mapsto h(\omega) := x - \mathcal{Q}x$ for some $x \in \varphi^v(\omega)$. The function h is well-defined and measurable. Clearly $\varphi^v(\omega) = \mathcal{Q}\varphi^v(\omega) + h(\omega)$. One easily verifies that

$$\begin{aligned} conv\Big(\int \varphi^v\Big) &= conv\Big(\int (\mathcal{Q}\varphi^v + h)\Big) \\ &= conv\Big(\int \mathcal{Q}\varphi^v\Big) + \int h \end{aligned}$$

and

$$\begin{aligned} \int conv(\varphi^v) &= \int conv(\mathcal{Q}\varphi^v + h) \\ &= \int conv(\mathcal{Q}\varphi^v) + \int h. \end{aligned}$$

Hence, in order to prove (3) it suffices to prove that

$$conv\Big(\int \mathcal{Q}\varphi^v\Big) = \int conv(\mathcal{Q}\varphi^v). \qquad (4)$$

This follows from the induction hypothesis. Indeed, the vectors $\mathcal{Q}e_2,\ldots,\mathcal{Q}e_n$ form a basis for the hyperplane $\mathcal{H}$ (e_j denotes the jth unit vector in $\mathbb{R}^n$). With respect to this basis, the mapping $\mathcal{Q}\varphi^v$ becomes a mapping φ^v of Ω into $\mathbb{K}^{n-1}$. The mapping φ^v is positive, since φ is positive. Moreover, φ^v has a measurable graph, since $\mathcal{Q}\varphi^v$ has a measurable graph. Therefore, by induction hypothesis, we obtain $conv\big(\int \varphi^v\big) = \int conv(\varphi^v)$. Let $\mathcal{T}$ denote the linear and injective mapping of $\mathbb{R}^{n-1}$ into $\mathbb{R}^n$, defined by $\mathcal{T}(\zeta_2,\ldots,\zeta_n) = \sum_j \zeta_j \mathcal{Q}e_j$. Clearly $\mathcal{Q}\varphi^v(\omega) = \mathcal{T}\varphi^v(\omega)$. One easily verifies that

$$\begin{aligned} \int conv(\mathcal{T}\circ\varphi^v) &= \mathcal{T}\int conv(\varphi^v) \\ &= \mathcal{T} conv\Big(\int \varphi^v\Big) \\ &= conv\Big(\int \mathcal{T}\circ\varphi^v\Big). \end{aligned}$$

Thus, we obtain (4).

viii It is well known that if f is a measurable selection for φ, then $f \circ h$ is a measurable selection for $\varphi \circ h$. Therefore, the "change-of-variable formula"[3] implies that $\int \varphi d(m \circ h^{-1}) \subset \int \varphi \circ h\, dm$. In order to prove the converse inclusion we have to show that for every $x \in \int \varphi \circ h\, dm$ we can find an integrable selection $g \in \mathcal{L}_{\varphi\circ h}$ with $x \in \int g$ which is of the form: $g = f \circ h$, where f is a measurable function of T into $\mathbb{R}^n$. There exists a measurable function $f : T \to \mathbb{R}^n$ such that $g = f \circ h$ if and only if g is $h^{-1}(\mathfrak{J})$-measurable. Hence it remains to show that for every $g \in \mathcal{L}_{\varphi\circ h}$ there exists a $h^{-1}(\mathfrak{J})$-measurable selection of $\varphi \circ h$ with the same integral. But such a selection is easily found. Let $\mathcal{K} = h^{-1}(\mathfrak{J})$. Consider the conditional expectation $\mathbb{E}^{\mathcal{K}}g$ of g given the σ-algebra $\mathcal{K}$ (the conditional expectation is taken coordinatewise). By definition, $\mathbb{E}^{\mathcal{K}}g$ is a $\mathcal{K}$-measurable function of Ω into $\mathbb{R}^n$ and one has $\int \mathbb{E}^{\mathcal{K}}g\, dm = \int g\, dm$. Thus we have only to show that the function $\mathbb{E}^{\mathcal{K}}g$ is a selection for $\varphi \circ h$. Let $\psi = \varphi \circ h$. Since $g \in \mathcal{L}_\psi$, we obtain for every $v \in \mathbb{K}^n$ that $\inf v \cdot \psi(\omega) \le v \cdot g(\omega)$ almost everywhere in Ω.

By part i.b, the graph of ψ belongs to $\mathcal{K} \otimes \mathcal{B}^n$. Recall that $\mathcal{B}$ denotes the Borel σ-algebra on $\mathbb{R}$ generated by the open subsets of $\mathbb{R}$. Hence, part ii implies that $\inf v \cdot \psi(\cdot)$ is $\mathcal{K}_v$-

[3] If M is a metric space, f is a measurable mapping of Ω into M and h is a measurable mapping of M into $\mathbb{K}$, then h is $m \circ f^{-1}$-integrable if and only if $h \circ f$ is m-integrable and $\int_M h\, dm = \int_\Omega h \circ f\, dm$.

measurable, and thus is almost everywhere equal to a $\mathcal{K}$-measurable function. Consequently, almost everywhere in Ω, depending on v, one obtains that

$$\begin{aligned}\inf v\cdot\psi(\omega) &= \left(\mathbb{E}^{\mathcal{K}}\inf v\cdot\psi\right)(\omega)\\ &\le \left(\mathbb{E}^{\mathcal{K}}v\cdot\psi\right)(\omega) = v\cdot\left(\mathbb{E}^{\mathcal{K}}g\right)(\omega).\end{aligned}$$

Thus, if Q denotes a countable dense subset of $\mathbb{R}^n$, we have shown that, almost everywhere in Ω, one has $\inf v\cdot\psi(\omega)\le v\cdot\left(\mathbb{E}^{\mathcal{K}}g\right)(\omega)$, for every $v\in Q$. Since $\psi(\omega)$ is closed, convex and contains no straight line, this implies that almost everywhere in Ω, it holds

$$\left(\mathbb{E}^{\mathcal{K}}g\right)(\omega)\in\psi(\omega).$$

Theorem 2.2 *For every geopolitical sector $\mathfrak{s}:(\mathbb{M},\boldsymbol{\mathcal{A}},v)\to\mathcal{P}\times\mathbb{R}$ in the complex S and every weighted geopolitical system with geopolitical significance vector $\boldsymbol{p}\gg 0$ in S, one has*

- i *$conv\left(\bar{\mathfrak{A}}(\mathfrak{s},\boldsymbol{p})\right)=conv\left(\int_{\mathcal{P}\times\mathbb{R}}\mathfrak{A}(\cdot,\boldsymbol{p})d\varrho\right)$, where $\varrho=v\circ\mathfrak{s}^{-1}$.*
- ii *If the geopolitical sector $\mathfrak{s}$ is convex in S, then the mean geopolitical rational choice set $\bar{\mathfrak{A}}(\mathfrak{s},\boldsymbol{p})$ in S is convex.*
- iii *If $\inf\{\boldsymbol{p}\cdot\mathfrak{X}_{\mathcal{M}}\}\le\mathcal{C}_{\mathcal{M}}$, a.e. in $\mathbb{M}$, then the mean geopolitical rational choice set $\bar{\mathfrak{A}}(\mathfrak{s},\boldsymbol{p})$ in S is nonempty and compact.*

Proof

i Since, by Lemma 2.3.ii.a and Lemma 2.3.i.a, the mappings $\mathfrak{A}(\cdot,\boldsymbol{p})$ and $\mathfrak{A}(\mathfrak{s}(\cdot),\boldsymbol{p})$ have both measurable graph and since they are bounded from below, we have

$$\begin{aligned}conv\left(\int\mathfrak{A}(\mathfrak{s}(\cdot),\boldsymbol{p})dv\right) &= \int conv(\mathfrak{A}(\mathfrak{s}(\cdot),\boldsymbol{p}))dv\\ &= \int conv(\mathfrak{A}(\cdot,\boldsymbol{p}))d\varrho\\ &= conv\int\mathfrak{A}(\cdot,\boldsymbol{p})d\varrho.\end{aligned}$$

Indeed, the first and third equality follows from Lemma 2.3.vii. The second equality follows from the transformation formula of Lemma 2.3.viii, since, by Lemma 2.3.iii.c, the graph of the mapping

$$conv(\mathfrak{A}(\cdot,\boldsymbol{p}))$$

belongs to $\mathcal{B}_{\varrho}(\mathcal{P}\times\mathbb{R})\times\mathcal{B}^{\ell+1}$.

ii It is easily seen that the measure space $(\mathbb{M},\boldsymbol{\mathcal{A}},v)$ can be decomposed into a countable union of atoms and an atomless part. Since on atoms the geopolitical selection preferences are convex, the ***geopolitical rational choice set*** is also convex. Therefore, by Lemma 2.3.iv.a, the mean geopolitical rational choice set $\bar{\mathfrak{A}}(\mathfrak{s},\boldsymbol{p})$ is convex.

iii It remains to show that $\int\mathfrak{A}(\cdot,\boldsymbol{p})d\varrho$ is nonempty and compact. Since

$$\boldsymbol{p}\gg 0 \text{ and } \mathfrak{X}_{\mathcal{M}}\le\mathcal{C}_{\mathcal{M}},$$

ϱ-almost everywhere on $\mathcal{P}\times\mathbb{R}$. The geopolitical rational choice set $\mathfrak{A}(\mathfrak{X},\succ,\mathcal{C}_{\mathcal{M}},\boldsymbol{p})$ is nonempty almost everywhere. Thus, by the Measurable Selection Theorem and Lemma 2.3.v, the integral $\int\mathfrak{A}(\cdot,\boldsymbol{p})d\varrho$ is nonempty and compact if the mapping $\mathfrak{A}(\cdot,\boldsymbol{p})$ is integrably bounded. To show this, one can assume without loss of generality that $\mathfrak{A}$ takes values only in $\mathbb{R}^{\ell+1}_{+}$. Then consider the function

$$\begin{aligned}&\mathbb{V}:\mathcal{P}\times\mathbb{R}\to\mathbb{R}^{\ell+1}:(\mathfrak{X},\succ,\mathcal{C}_{\mathcal{M}})\mapsto\\ &\mathbb{V}(\mathfrak{X},\succ,\mathcal{C}_{\mathcal{M}}):=\left(\frac{\mathcal{C}_{\mathcal{M}}}{\boldsymbol{p}_1},\ldots,\frac{\mathcal{C}_{\mathcal{M}}}{\boldsymbol{p}_{\ell+1}}\right).\end{aligned}$$

Clearly $\mathfrak{A}(\mathfrak{X},\succ,\mathcal{C}_{\mathcal{M}},\boldsymbol{p})\le\mathbb{V}(\mathfrak{X},\succ,\mathcal{C}_{\mathcal{M}})$. Since, by assumption, $\int\mathcal{C}_{\mathcal{M}}d\varrho<\infty$, we conclude that the function $\mathbb{V}$ is ϱ-integrable. □

Geopolitical contrasting of subjective priorities

Introduction

In this chapter we study a simple form of geopolitical management activity: the evaluation of subjective priorities by several geopolitical operators. To this end, let us consider a set $\mathbb{M}$ of geopolitical operators $\mathcal{M}$, each of whom is described by its geopolitical focus set $\mathfrak{X}_{\mathcal{M}}$ over the complex S, his corresponding geopolitical preference $\succ_{\mathcal{M}}$ over S and his available geopolitical carrier over S. Hence each geopolitical operator is characterized by an element in the space $\mathcal{P} \times \mathbb{R}^{\ell+1}$.

A **geopolitical contrasting of subjective priorities** is defined by a mapping

$$\mathfrak{S} : \mathbb{M} \to \mathcal{P} \times \mathbb{R}^{\ell+1}.$$

For reasons which will become clear later, we shall also consider sets $\mathbb{M}$ of geopolitical operators which are infinite. In the latter case, of course, the "totally available geopolitical numerical carrier over S" is infinitely large. To overcome the problems that this creates, we shall replace then the concept of "*totally available geopolitical numerical carrier in S*" by that of "*mean available geopolitical numerical carrier over S*".

The outcome of any geopolitical contrasting of subjective priorities is a ***redistribution*** of the of initially available geopolitical numerical carriers over S. The analysis of a geopolitical contrasting, as presented here, consists of specifying a certain class of redistributions as possible outcomes. The actual process by which the redistribution is accomplished is not considered explicitly. Two equilibrium concepts are analyzed: *the cooperative concept* and *the non-cooperative concept*.

Let us consider first the *cooperative concept*. *The **core of contrasting geopolitical subjective priorities** or the contrast core of the subjective choices for the geopolitical priorities,* which, in essence, consists in all reallocations of numerical carriers over a complex S, that cannot lead to improved conclusions originated from reallocations of numerical carriers of other groups.

Let us now turn to the *non-cooperative* concept. *An **equilibrium of contrasting geopolitical subjective priorities**, or a contrast equilibrium of the subjective choices for the geopolitical priorities, consists of a redistribution of the available geopolitical numerical carriers over S and a vector of geopolitical weights such that no individual geopolitical operator acting independently can improve upon his conclusions when these geopolitical weights prevail*. To say that ***certain geopolitical weights prevail*** means that every geopolitical operator takes this geopolitical weighted data as given (beyond his influence) and that there is a "program" where the geopolitical operators can use any amount of data of every weighted geopolitical index by using these geopolitical weights.

Below, in Proposition 3.1, it will be shown that *any geopolitical contrast equilibrium belongs to the geopolitical contrast core*. In order to study the converse inclusion, we have to give a precise meaning to the geopolitical concept of "*pure contrasting*", that is to say, a geopolitics where the influence of every individual geopolitical operator is negligible. Or, in other words, a set of geopolitical operators each of whom cannot influence the outcome of their collective activity but certain interplays of whom can influence that outcome. This leads logically to the concept of an ***"atomless" geopolitics***, also called ***geopolitics with a "continuum of geopolitical operators."***

The essential result of this section, Theorem 3.1, is *the identity of the core of contrasting geopolitical subjective priorities and the set of equilibrium allocations for such an "atomless" geopolitics*.

In order to understand more clearly the idea of an "atomless" geopolitics, we shall show how such geopolitics may be considered as a "limit" of a sequence of finite geopolitics (see Proposition 3.2). This fact, namely that one can treat an "atomless" geopolitics as a limit, plays an essential role throughout the paper.

Main definitions

In the context of pure contrasting, a geopolitical operator is described by a point in the space $\mathcal{P} \times \mathbb{R}^{\ell+1}$ the space of operators' characteristics. In order to simplify the presentation we shall often assume in this chapter that

- *the geopolitical focus set* $\mathfrak{X}$ *over a complex S is equal to the positive orthant* $\mathbb{R}^{\ell+1}_{+}$ and
- *the vector*

  δ

  of available geopolitical numerical carrier is ≥ 0.

With this formalism we are in position to give a more rigorous definition for the concept of geopolitical contrasting of subjective priorities.

Definition 3.2

i A **geopolitical contrasting of subjective priorities** $\Im$ over the complex S is a measurable mapping

$$\Im : (\mathbb{M}, \boldsymbol{\mathcal{A}}, v) \to \mathcal{P} \times \mathbb{R}^{\ell+1}$$

of a measure space $(\mathbb{M}, \boldsymbol{\mathcal{A}}, v)$, consisting of the set $\mathbb{M}$, a σ-algebra $\boldsymbol{\mathcal{A}}$ of subsets of $\mathbb{M}$ and a (probability) measure v on $\boldsymbol{\mathcal{A}}$, into the space $\mathcal{P} \times \mathbb{R}^{\ell+1}$ of geopolitical operators' characteristics such that the mean available geopolitical numerical carrier over S

$$\int_{\mathbb{M}} \delta \circ \Im \, dv$$

is finite (compare with Definition 2.16).

ii An **allocation for the geopolitical contrasting of subjective priorities** $\Im$ over S is an integrable function

$$f : (\mathbb{M}, \boldsymbol{\mathcal{A}}, v) \to \mathbb{R}^{\ell+1}$$

such that almost everywhere in $\mathbb{M}$, the focus vector $f(\mathcal{M})$ belongs to the geopolitical focus set of the geopolitical operator $\mathcal{M}$.

iii An allocation f for the geopolitical contrasting of subjective priorities $\Im$ over S is called **attainable** or a **state** of the contrasting $\Im$ if

$$\int_{f(\mathbb{M})} fdv = \int_{\mathbb{M}} \delta \circ \Im \, dv.$$

iv A geopolitical contrasting of subjective priorities $\Im$ over S is called

a. **simple** if the measure space $(\mathbb{M}, \boldsymbol{\mathcal{A}}, v)$ is simple, i.e.
   - $\mathbb{M}$ is a finite set,
   - $\boldsymbol{\mathcal{A}}$ is the set of all subsets of $\mathbb{M}$ and
   - $v(\boldsymbol{\mathcal{E}}) = (|\boldsymbol{\mathcal{E}}|/|\mathbb{M}|)$ whenever $\boldsymbol{\mathcal{E}} \subset \mathbb{M}$;

b. **atomless** if the measure space $(\mathbb{M}, \boldsymbol{\mathcal{A}}, v)$ is atomless, i.e., for every $\mathcal{E} \in \boldsymbol{\mathcal{A}}$ with $v(\boldsymbol{\mathcal{E}})$ there is a $\boldsymbol{\mathcal{K}} \subset \boldsymbol{\mathcal{E}}$ such that $\mathcal{K} \in \boldsymbol{\mathcal{A}}$ *and* $0 < v(\boldsymbol{\mathcal{K}}) < v(\boldsymbol{\mathcal{E}})$;

c. **convex** if almost all geopolitical operators of every atom of the measure space $(\mathbb{M}, \boldsymbol{\mathcal{A}}, v)$ have convex geopolitical preferences.

The geopolitical focus set, geopolitical selection preference and *totally available geopolitical numerical carrier S* of an operator $\mathcal{M}$ in $\mathbb{M}$ are denoted by

$$\Im(\mathcal{M}) = \big(\mathfrak{X}(\Im(\mathcal{M})), \succ_{\Im(\mathcal{M})}, \delta(\Im(\mathcal{M}))\big).$$

It is clear which geopolitical contrasting $\Im$ is considered, we will shorten this

$(\mathfrak{X}(\mathcal{M}), \succ_{\mathcal{M}}, \delta(\mathcal{M}))$ or $(\mathfrak{X}_{\mathcal{M}}, \succ_{\mathcal{M}}, \delta_{\mathcal{M}})$.

Definition 3.3

i.
Subsets of $\mathbb{M}$ belonging to $\boldsymbol{\mathcal{A}}$ are called **geopolitical interplays**.

ii.
The distribution of $\Im$, i.e., *the measure*

$v \circ \Im^{-1}$ *on* $\mathcal{P} \times \mathbb{R}^{\ell+1}$

is called the **geopolitical preference-availability distribution of the geopolitical contrasting** $\Im$ and is denoted by

$\mu_{\Im}$ or simply by μ.

The meaning and interpretation of a *simple* geopolitical contrasting of subjective priorities and an allocation for such a geopolitical contrasting are clear and need no comment. If f is an allocation for the simple geopolitical contrasting $\Im$, then $f(\mathcal{M})$ denotes the vector of geopolitical weights allocated to the operator $\mathcal{M}$ and

$$\int \delta \, dv = \frac{1}{|\mathbb{M}|} \sum\nolimits_{\mathcal{M} \in \mathbb{M}} \delta_{\mathcal{M}}$$

is the mean available geopolitical numerical carrier δ of the geopolitical contrasting $\Im$ *over S.*

We emphasize that $\int_{\mathcal{E}} fdv$ does not mean the vector of geopolitical weights allocated to an interplay $\boldsymbol{\mathcal{E}}$, indeed, if $\Im$ is a *simple* geopolitical contrasting of subjective priorities, then

$$\int_{\mathcal{E}} f \, dv = \frac{1}{|\mathbb{M}|} \sum\nolimits_{\boldsymbol{\mathcal{E}} \in \boldsymbol{\mathcal{A}}} f.$$

An *atomless geopolitical contrasting of subjective priorities* is, in fact, a quite abstract concept. The interpretation relies—for the time being—on analogy to the case of a simple contrasting. The key to a rigorous interpretation will be given in Proposition 3.2 at the end of this section. As in the case of a simple contrasting, $f(\mathcal{M})$ denotes the vector of geopolitical weights allocated to the geopolitical operator $\mathcal{M}$. The number $v(\boldsymbol{\mathcal{E}})$ is interpreted as the fraction of the totality of geopolitical operators belonging to $\boldsymbol{\mathcal{E}}$. The integral $\int \delta\, dv$ is the mean available geopolitical numerical carrier δ of the geopolitical contrasting $\Im$ over S. The σ-algebra $\boldsymbol{\mathcal{A}}$ of interplays is introduced for technical measure theoretic reasons. As in the case of a simple contrasting, there is no a priori restriction on possible interplays. Since for an atomless measure space $(\mathbb{M}, \boldsymbol{\mathcal{A}}, v)$ the set $\mathbb{M}$ must be uncountably infinite, we shall speak of a "*continuum of geopolitical operators*" as the set of participants. The results of this chapter provide a strong justification for considering the atomless geopolitical contrasting of subjective priorities as the proper mathematical formulation of the traditional geopolitical concept of "pure contrasting", that is to say, a set of geopolitical operators, each of whom cannot influence the outcome of their collective activity but certain interplays of whom can influence that outcome. The later concept is, in fact, as abstract as the former, which has the decisive advantage of being mathematically well defined.

A *convex* geopolitical contrasting of subjective priorities has been defined in order to have a concise way of referring to a geopolitical contrasting that is either atomless or simple with convex geopolitical preferences. From a formal point of view one might also consider a measure space $(\mathbb{M}, \boldsymbol{\mathcal{A}}, v)$ with atoms and an atomless part. In terms of the interpretation given earlier, one could consider a **geopolitical atom** as a group of geopolitical operators which cannot split up; either all of them join a interplay or none does so. Note, however, that the mapping $\Im$, and also every allocation f, must be constant on an atom. This means that all geopolitical operators in the atom must have identical characteristics and must receive the same bundle in an allocation. This is so special a case that it makes the interpretation of atoms as syndicates of little geopolitical contrasting significance.

An alternative approach is to consider an atom as a "*big*" geopolitical operator. In the framework of the model under consideration "big" can only mean "big" in terms of the available geopolitical numerical carriers. Thus, *a geopolitical atom would be a geopolitical operator who has infinitely more available geopolitical numerical carriers than any geopolitical operator in the atomless part*. Now, the measure $v(\boldsymbol{\mathcal{K}})$ has a different interpretation. Formerly it expressed the relative number of geopolitical operators in the geopolitical interplay $\boldsymbol{\mathcal{K}}$, here it expresses something like the relative size of the amount of available geopolitical data of $\boldsymbol{\mathcal{K}}$. Moreover the allocation $f(\mathcal{M})$ and the preferences $\succ_{\mathcal{M}}$ must also be reinterpreted. The net result is far from clear.

The geopolitical contrast core and the geopolitical contrast equilibrium

A state of geopolitical contrasting is clearly not in equilibrium if one geopolitical operator or a group of geopolitical operators could carry out decisions under the current circumstances and arrive at a position which is more advantageous to all members of the group than the current state. The underlying notion of geopolitical equilibrium is based on the behavioral assumption that geopolitical operators want to improve their position, and that to achieve a preferred situation they are willing to cooperate. This notion of equilibrium leads to the basic concept of the geopolitical contrast core of subjective priorities. In such a case, the interplay of geopolitical operators can improve upon a redistribution of the available geopolitical numerical carriers over S if the interplay, by using the amount of geopolitical numerical carriers available to it, can make each member better off. The geopolitical contrast core of subjective priorities is defined as the set of all redistributions that no interplay can improve upon. Formally:

Definition 3.4 Let

$\Im : (\mathbb{M}, \boldsymbol{\mathcal{A}}, v) \to \mathcal{P} \times \mathbb{R}^{\ell+1}$

be a geopolitical contrasting of subjective priorities over the complex S. Let also f be an allocation for $\Im$. The geopolitical interplay $\boldsymbol{\mathcal{C}} \in \boldsymbol{\mathcal{A}}$ can **improve upon** the allocation f if there exists another allocation g for $\Im$ such that

(i) $g(\mathcal{M}) \succ_{\mathcal{M}} f(\mathcal{M})$, almost everywhere in the interplay $\boldsymbol{\mathcal{C}}$,

(ii) $v(\boldsymbol{\mathcal{C}}) > 0$ and $\int_{\mathcal{C}} g dv = \int_{\mathcal{C}} \delta dv$.

The set of all attainable allocations for the geopolitical contrasting $\Im$ that no interplay in $\boldsymbol{\mathcal{A}}$ can improve upon is called the **core of contrasting geopolitical subjective priorities** for $\Im$, or simply the geopolitical contrast core for $\Im$, and is denoted by

$\mathfrak{C}(\Im)$.

The meaning of the definition of "improve" and core of contrasting geopolitical subjective priorities is clear for simple geopolitical contrasts. In the framework presented here, all externalities of geopolitical are excluded (i.e., geopolitical preferences do not depend on the numerical carriers of the other geopolitical operators); the utility level of the members in an interplay does not depend on actions taken by operators outside the interplay. The core of geopolitical contrasting expresses what interplays can or cannot do for them, not what they can or cannot do to their opponents. Therefore, we used the term "to improve upon" and not "to block."

We now introduce a different concept of equilibrium. Suppose the following hold.

- Every weighted geopolitical index $g_S^{(i)}$ over S has a **geopolitical significance** $\boldsymbol{p}_i$.
- Every geopolitical operator in a geopolitical contrasting considers this **geopolitical significance** as given.

Then, the geopolitical operator $\mathcal{M}$ with characteristics $(\mathfrak{X}_{\mathcal{M}}, \succ_{\mathcal{M}}, \delta_{\mathcal{M}})$ considers only vectors of numerical carriers in his set of geopolitical options $\{x \in \mathfrak{X}_{\mathcal{M}} : \boldsymbol{p} \cdot x \leq \boldsymbol{p} \cdot \delta_{\mathcal{M}}\}$ and chooses a most desired vector in that set. If all these individually taken decisions—decentralized through the ***geopolitical significance*** system $\boldsymbol{p}$—yield a situation where the geopolitical rational choice set equals the total supply we call that state of the geopolitical contrast equilibrium. This concept of equilibrium is based on the behavioral assumption that geopolitical operators consider the ***geopolitical significance*** system as given and make their decisions independently of each other. The only link between these individual decisions is the ***geopolitical significance*** system. Formally:

Definition 3.5 An allocation f for the geopolitical contrasting of subjective priorities $\Im : (\mathbb{M}, \boldsymbol{\mathcal{A}}, v) \to \mathcal{P} \times \mathbb{R}^{\ell+1}$ and a **geopolitical significance** system $\boldsymbol{p} \in \mathbb{R}^{\ell+1}$ is said to be an **equilibrium of contrasting geopolitical subjective priorities** for $\Im$, or simply a geopolitical contrast equilibrium for $\Im$, if the following two conditions are satisfied.

i. $f(\mathcal{M}) \in \mathfrak{A}(\mathfrak{X}_{\mathcal{M}}, \succ_{\mathcal{M}}, \underbrace{\boldsymbol{p} \cdot \delta_{\mathcal{M}}}_{\mathcal{C}_{\mathcal{M}}}, \boldsymbol{p})$ almost everywhere in $\mathbb{M}$,
i.e., $f(\mathcal{M})$ is a maximal element for $\succ_{\mathcal{M}}$ in the set
$$\mathfrak{B} = \mathfrak{B}(\mathfrak{X}, \mathcal{C}_{\mathcal{M}}, \boldsymbol{p}) := \{x = (d_1, \ldots, d_{\ell+1}) \in \mathfrak{X} : (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_{\ell+1}) \cdot (d_1, d_2, \ldots, d_{\ell+1}) \leq \mathcal{C}_{\mathcal{M}}\}$$
of geopolitical options of $\mathcal{M}$.

ii. $\int_{f(\mathbb{M})} f dv = \int_{\mathbb{M}} \delta dv$,
i.e., mean geopolitical rational choice equals mean geopolitical data availability.

The allocation f for the geopolitical contrasting $\Im$ is called a **contrasting allocation** if there exists a weight vector $\boldsymbol{p} \in \mathbb{R}^{\ell+1}$ such that $(f, \boldsymbol{p})$ is an equilibrium of contrasting geopolitical subjective priorities for $\Im$. The set of all equilibriums of geopolitical subjective priorities' contrasting for $\Im$ is denoted by

$\mathfrak{W}(\Im)$.

A **geopolitical significance** system $\boldsymbol{p} \in \mathbb{R}^{\ell+1}$ is said to be an **geopolitical equilibrium of significance levels** or simply a **geopolitical equilibrium vector** for the geopolitical contrasting $\Im$ if there exists an allocation f for $\Im$ such that $(f, \boldsymbol{p})$ is an equilibrium of contrasting geopolitical subjective priorities for $\Im$. The set of all **geopolitical equilibrium vectors** for $\Im$ which are normalized, i.e.$|\boldsymbol{p}| = 1$, is denoted by

$\mathfrak{e}(\Im)$.

Under what conditions is the behavioral assumption that geopolitical operators adapt themselves to the prevailing ***geopolitical significance*** system justified? The obvious answer is the following: *geopolitical operators take weights as given if they have no influence on them*. This leads to an atomless geopolitical contrasting of subjective priorities. In that case, one could ask what is special about the contrasting allocations among the allocations that cannot be improved upon. We shall show that there is nothing special; *every deviation from a contrasting allocation can be improved upon*. That is to say, *contrasting allocations, and only they, belong to the core of contrasting geopolitical subjective priorities*, i.e.

$\mathfrak{W}(\Im) = \mathfrak{C}(\Im)$.

One part of the identity is trivial:

Proposition 3.1 *For every geopolitical contrasting of subjective priorities $\Im$ in S, we have*

$\mathfrak{W}(\Im) \subset \mathfrak{C}(\Im)$.

Proof Let $f \in \mathfrak{W}(\Im)$ but $f \notin \mathfrak{C}(\Im)$. Thus, there is a geopolitical interplay $\boldsymbol{\mathcal{C}} \in \boldsymbol{\mathcal{A}}$, $v(\boldsymbol{\mathcal{C}}) > 0$, and there is an allocation g such that

i. $g(\mathcal{M}) \succ_{\mathcal{M}} f(\mathcal{M})$, almost everywhere in the interplay $\boldsymbol{\mathcal{C}}$
ii. $v(\boldsymbol{\mathcal{C}}) > 0$ and $\int_{g(\boldsymbol{\mathcal{C}})} g dv = \int_{\boldsymbol{\mathcal{C}}} \delta dv$.

By i and the definition of a geopolitical contrasting allocation, we obtain $\boldsymbol{p} \cdot \delta(\mathcal{M}) < \boldsymbol{p} \cdot g(\mathcal{M})$ almost everywhere in $\boldsymbol{\mathcal{C}}$, where $\boldsymbol{p}$ denotes an equilibrium weight associated with f. Hence

$$\boldsymbol{p} \cdot \int_{\boldsymbol{\mathcal{C}}} \delta dv < \boldsymbol{p} \cdot \int_{\boldsymbol{\mathcal{C}}} g dv,$$

which contradicts ii.

The central result of this section is proved in the following.

Theorem 3.1 *Let $\Im : (\mathbb{M}, \boldsymbol{\mathcal{A}}, v) \to \mathcal{P}_{mo} \times \mathbb{R}_{+}^{\ell+1}$ be an atomless geopolitical contrasting of subjective priorities in S with $\int \delta dv \gg 0$. Then*

$\mathfrak{W}(\Im) = \mathfrak{C}(\Im)$.

Proof By Proposition 3.1, it is enough to show that $f \in \mathfrak{C}(\Im)$ implies $f \in \mathfrak{W}(\Im)$. Consider for every geopolitical operator $\mathcal{M} \in \mathbb{M}$, the sets

$$\prec_{\mathcal{M}}(f) := \{x \in \mathfrak{X}_{\mathcal{M}} : x \succ_{\mathcal{M}} f(\mathcal{M})\} \text{ and } \mathfrak{h}(\mathcal{M}) := \{\prec_{\mathcal{M}}(f) - \delta(\mathcal{M})\} \bigcup \{0\}.$$

Since the measure space $(\mathbb{M}, \boldsymbol{\mathcal{A}}, v)$ is atomless, the integral $\int \mathfrak{h} dv$ is a convex subset in $\mathbb{R}^{\ell+1}$. Since $0 \in \int \mathfrak{h} dv$, it is clear that

$$\int \mathfrak{h} dv \neq \mathfrak{A}.$$

We now claim that $\int \mathfrak{h} dv \bigcap \mathbb{K}_{-}^{\ell+1} = \{0\}$. Assume to the contrary that there is an integrable function h in the set $\mathcal{L}_{\mathfrak{h}}$ of all integrable selections of $\mathfrak{h}$, (that is of the set of all v-integrable $h : \mathbb{M} \to \mathbb{R}^{\ell+1}$ which have the property that $h(\mathcal{M}) \in \mathfrak{h}(\mathcal{M})$ almost everywhere in $\mathbb{M}$.), with $\int \mathfrak{h} dv < 0$. Then the interplay $\mathcal{C} = \{\mathcal{M} \in \mathbb{M} : \mathfrak{h}(\mathcal{M}) \neq 0\}$ can improve the allocation f with the allocation

$$g(\mathcal{M}) = \mathfrak{h}(\mathcal{M}) + \delta(\mathcal{M}) - \frac{\int \mathfrak{h} dv}{v(\mathcal{C})}.$$

Indeed, $v(\mathcal{C}) > 0$, $g(\mathcal{M}) \succ_{\mathcal{M}} f(\mathcal{M})$ for every $\mathcal{M} \in \mathcal{C}$ and $\int_{\mathcal{C}} g dv = \int_{\mathcal{C}} \delta dv$. Consequeny, there exists a hyperplane separating the two convex sets $\int \mathfrak{h} dv$ and $\mathbb{R}_{-}^{\ell+1}$, i.e., there is a vector $\mathfrak{p} \in \mathbb{R}^{\ell+1}$, $\mathfrak{p} \geq 0$, $\mathfrak{p} \neq 0$, such that

$$0 \leq \mathfrak{p} \cdot z \text{ for every } z \in \int \mathfrak{h} dv. \tag{5}$$

The graph of the mapping $\mathfrak{h}$ is measurable. Indeed the set

$$G := \left\{(\mathfrak{X}, \succ, x, y) \in \mathcal{P}_{mo} \times \mathbb{R}_{+}^{\ell+1} \times \mathbb{R}_{+}^{\ell+1} : x \succ y\right\}$$

is a Borel set in $\mathcal{P}_{mo} \times \mathbb{R}_{+}^{\ell+1} \times \mathbb{R}_{+}^{\ell+1}$. Now the graph of the mapping $\mathcal{M} \mapsto \mathfrak{h}(\mathcal{M}) \backslash \{0\}$, i.e., the set

$$\left\{(\mathcal{M}, x) \in \mathbb{M} \times \mathbb{R}^{\ell+1} : x + \delta(\mathcal{M}) \succ_{\mathcal{M}} f(\mathcal{M})\right\}$$

is equal to $\mathfrak{h}^{-1}(G)$ where $\mathfrak{h}$ is a mapping of $\mathbb{M} \times \mathbb{R}^{\ell+1}$ into $\mathbb{M} \times \mathbb{R}^{\ell+1} \times \mathbb{R}^{\ell+1}$ defined by

$$\mathfrak{h}(\mathcal{M}, x) := (\mathfrak{X}_{\mathcal{M}}, \succ_{\mathcal{M}}, x + \delta(\mathcal{M}), f(\mathcal{M})).$$

Clearly the mapping $\mathfrak{h}$ is measurable, and hence the graph of $\mathfrak{h}$ is measurable. Therefore it follows

$$inf_{z \in \int \mathfrak{h} dv} \mathfrak{p} \cdot z = \int inf_{x \in \mathfrak{h}(\cdot)} \mathfrak{p} \cdot x dv.$$

Consequently, we obtain from (5) that $0 \leq \int \inf \mathfrak{p} \cdot \mathfrak{h} dv$. Since by definition the set $\mathfrak{h}(\mathcal{M})$ contains 0, we clearly have $\inf p \cdot \mathfrak{h}(\mathcal{M}) \leq 0$. Hence, it follows that, almost everywhere in $\mathbb{M}$, $\inf \mathfrak{p} \cdot \mathfrak{h}(\mathcal{M}) = 0$. Thus, we have shown that

$$\text{almost everywhere in } \mathbb{M},\ \mathfrak{p} \cdot \delta(\mathcal{M}) \leq \mathfrak{p} \cdot x \text{ for every } x \succ_{\mathcal{M}} f(\mathcal{M}). \tag{6}$$

It follows from (6) that almost everywhere in $\mathbb{M}$,

$$\mathfrak{p} \cdot \delta(\mathcal{M}) = \mathfrak{p} \cdot f(\mathcal{M}).$$

Indeed, first we obtain from (6) that $\mathfrak{p} \cdot \delta(\mathcal{M}) \leq \mathfrak{p} \cdot f(\mathcal{M})$ almost everywhere in $\mathbb{M}$. Now, if $\mathfrak{p} \cdot \delta(\mathcal{M}) < \mathfrak{p} \cdot f(\mathcal{M})$ for a set of geopolitical operators with positive measure, then we obtain

$$\mathfrak{p} \cdot \int \delta dv < \mathfrak{p} \cdot \int f dv,$$

which contradicts $\int \delta dv = \int f dv$. Since by assumption $\int \delta dv \gg 0$ and since $\mathfrak{p} \geq 0$, $\mathfrak{p} \neq 0$, we surely have $v\{\mathcal{M} \in \mathbb{M} : \mathfrak{p} \cdot \delta(\mathcal{M}) > 0\} > 0$. But for a geopolitical operator $\mathcal{M}$ with positive income, i.e., $\mathfrak{p} \cdot \delta(\mathcal{M}) > 0$, property (6) implies that $f(\mathcal{M}) \in \mathfrak{A}(\mathfrak{X}, \succ_{\mathcal{M}}, \mathfrak{p} \cdot \delta_{\mathcal{M}}(\mathcal{M}), \mathfrak{p})$. Indeed, for $x \in \mathbb{R}_{+}^{\ell+1}$ with $\mathfrak{p} \cdot x < \mathfrak{p} \cdot \delta(\mathcal{M})$, it follows from (6) that $x \not\succ_{\mathcal{M}} f(\mathcal{M})$. Since in the case $\mathfrak{p} \cdot \delta(\mathcal{M}) > 0$ for every $x \in \mathbb{R}_{+}^{\ell+1}$ with $\mathfrak{p} \cdot x = \mathfrak{p} \cdot \delta(\mathcal{M})$ is limit of a sequence (x_n) with $\mathfrak{p} \cdot x_n < \mathfrak{p} \cdot \delta(\mathcal{M})$, the continuity of the preference relation $\succ_{\mathcal{M}}$ implies $x \not\succ_{\mathcal{M}} f(\mathcal{M})$. Thus $f(\mathcal{M})$ is a maximal element for $\not\succ_{\mathcal{M}}$ in the set of geopolitical management options $\{x \in \mathbb{R}_{+}^{\ell+1} : \mathfrak{p} \cdot x \leq \mathfrak{p} \cdot \delta(\mathcal{M})\}$. This, together with the monotony of preferences, implies that $\mathfrak{p} \gg 0$. Hence $f(\mathcal{M})$ belongs to the geopolitical rational choice set $\mathfrak{A}(\mathfrak{X}, \succ_{\mathcal{M}}, \mathfrak{p} \cdot \delta_{\mathcal{M}}(\mathcal{M}), \mathfrak{p})$ even in the case $\mathfrak{p} \cdot \delta(\mathcal{M}) = 0$ since, by (6), the vector $\delta(\mathcal{M})$ belongs to the set of geopolitical management options which in this case is equal to $\{0\}$. This proves that $(f, \mathfrak{p})$ is an equilibrium of contrasting geopolitical subjective priorities for $\Im$. □

Determinateness of geopolitical equilibrium vectors

In this section we will investigate the existence of geopolitical equilibrium vectors for *a* ***geopolitical contrasting of subjective priorities*** *$\Im$ over the complex S* with particular emphasis on the case where the geopolitical selection preferences are not assumed to be convex. Clearly, the classical assumption of convex preferences cannot simply be dropped. Indeed, in a geopolitical contrasting of subjective priorities, where the influence of a certain individual operator cannot be neglected, the convexity of his preferences is essential in proving the existence of ***geopolitical equilibrium vectors***. The extreme case, where the geopolitical contrasting of subjective priorities is atomless, is particularly simple (see Theorem 3.2 below). For an atomless geopolitical contrasting of subjective priorities, mean geopolitical rational choice set is convex (Proposition 2.5), and therefore a standard fixed-point argument can be applied. In the case of a simple geopolitical contrasting of subjective priorities with non-convex preferences, the mean geopolitical rational choice set may be non-convex.

The results of this section will show very clearly *the important role which the number of participants in a geopolitical contrasting of subjective priorities plays for the existence of* ***geopolitical*** *equilibrium vectors if preferences are not convex*. Proposition 2.2 and its corollaries show that the ***geopolitical equilibrium vectors*** *depend in a continuous way on the data which define the geopolitical contrasting of subjective priorities.*

Let us introduce some notation. We shall write

$$\mathfrak{A}(t, \boldsymbol{p}) \equiv \mathfrak{A}\left(\underbrace{\mathfrak{X}, \succ, \boldsymbol{p} \cdot \delta}_{t}, \boldsymbol{p}\right)$$

$$\left(\text{instead of } = (\mathfrak{X}, \succ, \boldsymbol{p} \cdot \delta) \in \mathcal{P} \times \mathbb{R}^{\ell+1} \text{and} \boldsymbol{p} \in \mathbb{R}^{\ell+1}\right).$$

Consequently, the mean geopolitical rational choice set of a geopolitical contrasting of subjective priorities $\Im : (\mathbb{M}, \boldsymbol{\mathcal{A}}, v) \to \mathcal{P} \times \mathbb{R}^{\ell+1}$, given the geopolitical equilibrium vector $\boldsymbol{p}$, is denoted by $\mathfrak{u}(\Im, \boldsymbol{p}) := \int \mathfrak{A}(\Im(\cdot), \boldsymbol{p}) dv$. Given $\Im$ and $\boldsymbol{p}$, the mapping

$$\begin{aligned} \Im^{\boldsymbol{p}} &: (\mathbb{M}, \boldsymbol{\mathcal{A}}, v) \to \mathcal{P} \times \mathbb{R} \\ &: \Im^{\boldsymbol{p}}(\mathcal{M}) := \left(\mathfrak{X}_{\Im(\mathcal{M})}, \succ_{\mathcal{M}}, \boldsymbol{p} \cdot \delta_{\Im(\mathcal{M})}\right) \end{aligned}$$

defines a geopolitical sector in the complex *S*. With the Notation 2.3(v) of "Mean geopolitical rational choice" section, we clearly have $\mathfrak{u}(\Im, \boldsymbol{p}) = \bar{\mathfrak{A}}(\Im^{\boldsymbol{p}}, \boldsymbol{p})$. However, the sets

$$\int_{\mathcal{P} \times \mathbb{R}^{\ell+1}} \mathfrak{u}(t, \boldsymbol{p}) d\varrho_{\Im} \text{ and } \int_{\mathcal{P} \times \mathbb{R}} \mathfrak{u}(\cdot, \cdot, \boldsymbol{p}) d\varrho_{\Im^{\boldsymbol{p}}}$$

may well be defined; only their convex hulls are identical.

It is easy to prove the following.

Proposition 3.2 *If the sequence $(\Im_n)_{n\in\mathbb{N}}$ of geopolitical contrasting of subjective priorities converges in distribution to the geopolitical contrasting of subjective priorities $\Im$ and if $lim_{n\to\infty}\boldsymbol{p_n} = \boldsymbol{p}$, then the sequence $\left(\Im_n^{\boldsymbol{p_n}}\right)_{n\in\mathbb{N}}$ of geopolitical sectors in the complex S converges in distribution to the geopolitical sector $\Im^{\boldsymbol{p}}$ in the complex S.*

Before giving the main result on the existence of geopolitical equilibrium vectors for *a* geopolitical contrasting of subjective priorities $\Im$ over the complex *S*, we need some preparatory material.

Definition 3.6 *Let $\Im$ be a geopolitical contrasting of subjective priorities $\Im$ over the complex S. For every weighted geopolitical system with geopolitical significance vector $\boldsymbol{p} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_{\ell+1})$, we define the* ***mean excess geopolitical rational choice*** *$\mathcal{Z}(\boldsymbol{p})$ by*

$$\mathcal{Z}(\boldsymbol{p}) := \mathfrak{u}(\Im, \boldsymbol{p}) - \int \delta dv.$$

As it is readily seen, *a geopolitical significance vector $\boldsymbol{p}^* = (\boldsymbol{p}_1^*, \boldsymbol{p}_2^*, \ldots, \boldsymbol{p}_{\ell+1}^*)$ is a geopolitical equilibrium vector for a geopolitical contrasting of subjective priorities $\Im$ over the complex S* if $0 \in \mathcal{Z}(\boldsymbol{p}^*)$. The existence of geopolitical significances for the geopolitical contrasting $\Im$ therefore depends on properties of the mean excess geopolitical rational choice relation $\mathcal{Z}$. The relevant properties of $\mathcal{Z}$ are summarized in the following.

Proposition 3.3 *Let $\Im : (\mathbb{M}, \boldsymbol{\mathcal{A}}, v) \to \mathcal{P}_{mo} \times \mathbb{R}_+^{\ell+1}$ be a* ***geopolitical contrasting of subjective priorities*** *over the complex S with $\int \delta dv \gg 0$. Then the mean* ***excess geopolitical rational choice mapping*** *$\mathcal{Z}$ has the following properties.*

i. *$\mathcal{Z}$ is* ***homogeneous of degree zero*** *(i.e., for every $\boldsymbol{p} \gg 0$ and $\lambda > 0$ one has*

   $\mathcal{Z}(\boldsymbol{p}) = \mathcal{Z}(\lambda\boldsymbol{p})$).

ii. *For every weighted geopolitical system with* ***geopolitical significance vector*** *$\boldsymbol{p} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_{\ell+1}) \gg 0$ and $z \in \mathcal{Z}(\boldsymbol{p})$ one has* $\boldsymbol{p} \cdot z = 0$.

iii *The mapping $\mathcal{Z}$ is compact-valued, bounded from below and upper hemi-continuous.*[4]

iv. *If the sequence $\left(\boldsymbol{p}^{(n)} = \left(\boldsymbol{p}_1^{(n)} \boldsymbol{p}_2^{(n)}, \ldots, \boldsymbol{p}_{\ell+1}^{(n)}\right)\right)_{n\in\mathbb{N}}$ of strictly positive geopolitical significance vectors converges to $\boldsymbol{p}$ which is not strictly positive, then*

   $$inf_{n\in\mathbb{N}}\left\{\sum_{i=1}^{\ell+1} z_i : z \in \mathcal{Z}\left(\boldsymbol{p}^{(n)}\right)\right\} > 0$$

   for n large enough.

Proof Property i follows immediately from the definition of the set $\mathcal{Z}(\boldsymbol{p})$. Since geopolitical preferences are monotonic, we have $\boldsymbol{p} \cdot x = \boldsymbol{p} \cdot \delta(\mathcal{M})$ for every $x \in \mathfrak{A}(\Im(\mathcal{M}), \boldsymbol{p})$. This clearly implies property ii. Let now $\bar{\boldsymbol{p}} \gg 0$. Then there is a neighborhood $U_{\bar{\boldsymbol{p}}}$ of $\bar{\boldsymbol{p}}$ consisting of strictly positive vectors. For any fixed $\mathcal{M} \in \mathbb{M}$, the mapping $\boldsymbol{p} \mapsto \mathfrak{u}(\Im(\mathcal{M}), \boldsymbol{p})$ is closed at $\bar{\boldsymbol{p}}$. Further, there is an integrable real function h of $\mathbb{M}$ such that

$$|\mathfrak{u}(\Im(\mathcal{M}), \boldsymbol{p})| \le h(\mathcal{M}) \text{ whenever } \mathcal{M} \in \mathbb{M} \text{ and } \boldsymbol{p} \in U_{\bar{\boldsymbol{p}}},$$

e.g.,

$$h(\mathcal{M}) = \frac{1}{min\{\boldsymbol{p}_i : \boldsymbol{p} \in U_{\bar{\boldsymbol{p}}}, i = 1, 2, \ldots, \ell + 1\}} |\delta(\mathcal{M})|.$$

Thus, the mapping $\boldsymbol{p} \mapsto \mathfrak{u}(\Im, \boldsymbol{p})$ is closed at $\bar{\boldsymbol{p}}$. Since the correspondence $\mathfrak{u}(\Im, \cdot)$ is bounded on the neighborhood $U_{\bar{\boldsymbol{p}}}$ of $\bar{\boldsymbol{p}}$, it follows that $\mathfrak{u}(\Im, \cdot)$ is compact-valued and upper hemi-continuous at $\bar{\boldsymbol{p}}$. This clearly implies property iii. Finally, property iv follows from the fact that the geopolitical preferences are assumed to be monotone and $\int \delta dv \gg 0$. □

The following Proposition is the fundamental mathematical result in geopolitical contrasting equilibrium analysis.

Proposition 3.4 *Let $\mathcal{Z}$ be a mapping of*

[4] A relation φ of the metric space M into the metric space N is said to be upper hemi-continuous at $x \in$ M if $\varphi(x) \not\equiv \varphi$ and if for every neighborhood $U_{\varphi(x)}$ of $\varphi(x)$ there exists a neighborhood U_x of x such that $\varphi(U_x) \subset U_{\varphi(x)}$. A relation ψ of the metric space M into the metric space N is said to be lower hemi-continuous at $x \in$ M if $\psi(x) \not\equiv \psi$ and if for every open set G in N with $\psi(x) \bigcap G \neq \psi$ there exists a neighborhood U_x of x such that $\psi(U_x) \bigcap G \neq \psi$.

$$int\Delta = int\left\{\boldsymbol{p} = \left(\boldsymbol{p}_1,\boldsymbol{p}_2,\ldots,\boldsymbol{p}_{\ell+1}\right) \in \mathbb{R}_+^{\ell+1} : \sum_{i=1}^{\ell+1}\boldsymbol{p}_i = 1\right\}$$

into $\mathbb{R}_+^{\ell+1}$ *which has the properties ii, iii and iv of Proposition 3.3. Then there exists a vector* $\boldsymbol{p}^* \gg 0$ *such that*

$0 \in conv\mathcal{Z}(\boldsymbol{p}^*)(=$ the convex hull of $\mathcal{Z}(\boldsymbol{p}^*))$.

Proof For any $n \geq \ell$, we set

$$\Delta_{\boldsymbol{n}} := \left\{\boldsymbol{p} = (\boldsymbol{p}_1,\boldsymbol{p}_2,\ldots,\boldsymbol{p}_\ell) \in \mathbb{R}_+^\ell : \sum_{i=1}^{\ell}\boldsymbol{p}_i = 1 \, and \, \boldsymbol{p}_i \geq \frac{1}{n} \forall i = 1,2,\ldots,\ell\right\}.$$

Applying the fixed point theorem to the map $\Delta_{\boldsymbol{n}} \to \mathbb{R}^\ell : \boldsymbol{p} \mapsto conv\mathcal{Z}(\boldsymbol{p})$, we infer the existence of vectors

$$\boldsymbol{p}^{(n)} = \left(\boldsymbol{p}_1^{(n)}\boldsymbol{p}_2^{(n)},\ldots,\boldsymbol{p}_\ell^{(n)}\right) \in \Delta_{\boldsymbol{n}} \text{ and}$$
$$\boldsymbol{z}^{(n)} = \left(\boldsymbol{z}_1^{(n)}\boldsymbol{z}_2^{(n)},\ldots,\boldsymbol{z}_\ell^{(n)}\right) \in \mathbb{R}^\ell$$

such that

$$\boldsymbol{z}^{(n)} \in conv\mathcal{Z}\left(\boldsymbol{p}^{(n)}\right) \text{ and} \tag{7}$$

$$\boldsymbol{p} \cdot \boldsymbol{z}^{(n)} \leq 0 \text{ for every } \boldsymbol{p} \in \Delta_{\boldsymbol{n}} \, (n \geq \ell). \tag{8}$$

It suffices to show that $\boldsymbol{z}^{(n)} = 0$ for some n. Without of generality, we can assume that the sequence $\left(\boldsymbol{p}^{(n)} = \left(\boldsymbol{p}_1^{(n)}\boldsymbol{p}_2^{(n)},\ldots,\boldsymbol{p}_\ell^{(n)}\right)\right)_{n\in\mathbb{N}}$ is convergent, say $lim_{n\to\infty}\boldsymbol{p}^{(n)} = \boldsymbol{p} \in \Delta$. One may claim that $\boldsymbol{p} \gg 0$. Otherwise, it would follow that $\sum_{i=1}^{\ell} z_i^{(n)} > 0$ for n large enough, which contradicts (8). Since $\boldsymbol{p} \gg 0$, it follows that $\boldsymbol{z}^{(n)} = 0$ for n large enough.

Indeed, let $\bar{n}$ be such that $int\Delta_{\bar{n}}$ contains $\boldsymbol{p}$. Clearly, we have $\boldsymbol{p}^{(n)} \cdot \boldsymbol{z}^{(n)} = 0$, and, since for n large enough, $\boldsymbol{p}^{(n)} \in int\Delta_{\bar{n}}$, it follows, from (8), that $\boldsymbol{z}^{(n)} = 0$. □

As an immediate consequence of Propositions 3.3 and 3.4, we have the following result.

Theorem 3.2 *Let* $\Im : (\mathbb{M}, \boldsymbol{\mathcal{A}}, v) \to \mathcal{P}_{mo} \times \mathbb{R}_+^\ell$ *be a geopolitical contrasting of subjective priorities over the complex S with* $\int \delta \, dv \gg 0$. *Then there exists an equilibrium* (f, p^*) *of geopolitical contrasts for* $\Im$, *with* $p^* \gg 0$.

Proof It suffices only to note that the mean geopolitical rational choice set $\bar{\mathfrak{A}}(\Im, p^*)$ is convex (Theorem 2.2). □

Corollary 3.1 *The geopolitical contrast core is nonempty for every convex* ***geopolitical contrasting of subjective priorities*** $\Im : (\mathbb{M}, \boldsymbol{\mathcal{A}}, v) \to \mathcal{P}_{mo} \times \mathbb{R}_+^\ell$ *over the complex S with* $\int \delta \, dv \gg 0$.

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